Essay on the potential of charged objects PhL 5.2.10

In this essay I will discuss the potentials of various charged metal objects, as well as the Green's functions for those same metal objects, meaning the potential of said objects in the presence of a point Green's charge when they are grounded. By "metal" I of course mean "conducting". I intent to concentrate more on qualitative aspects of things than quantitative. I have done plenty of the latter in other docs. I might also mention Dirichlet, Neumann and mixed BC problems.

1. Charged objects.

Sphere.

The simplest object of interest is a charged (by Q) metal sphere of radius r_0, with potential $V = Q(r_0/r)$. This potential is certainly an "elementary function" and has no summations, no special functions, it is very simple, and the potential outside the sphere is the same of that of a point charge Q at the sphere origin.

Distorted sphere.

As the sphere is distorted "gracefully" into similar shapes, the potential is similarly distorted. The most general distortion we consider here is the charged sphere morphing into a general charged **ellipsoid** whose three semi-major axes are all different. When this problem is solved in ellipsoidal coordinates, one finds that the potential is given by a function of the form $V = F(\sin^{-1}(\text{stuff}))$, where $F = sn^{-1}$ is a first kind elliptical integral function. The "stuff" is a simple function of only one variable ξ which is the variable that labels ellipsoids in ellipsoidal coordinates. Just as the charged sphere was labeled by $r = r_0$, the charged ellipsoid is labeled by $\xi = \xi_0$. So the metal surface in this case is both localized, and is described by one of the three 3D coordinates being a constant, just as with the sphere. We have now involved a "special function" F, but no summations are required in the answer, so even for an ellipsoid, the result is pretty simple. There are several special cases of interest. When two of the ellipsoidal axes become equal and are longer than the third axis, the potential becomes just $V = \sin^{-1}(\text{stuff})$, and we no longer have a special function involved. This is the situation of **oblate spheroidal** coordinates. Special cases of this special case include the **flat elliptical disk**, and of course then the **circular disk**. The prolate spherical limit of ellipsoidal still involves the $F(\sin^{-1}(\text{stuff}))$ form.

Localized vs non-localized objects.

In **Cartesian** coordinates, an object defined by coordinate = constant is a plane, such as $z = 0$. We know the potential of such a charged plane to be $V = \text{constant} * \sigma z$. This is an example where all surfaces for coordinate = constant are planes and are thus non-localized objects, they have infinite extent.

In **spherical** coordinates, the surface φ = constant and θ = constant define non-localized surfaces (half-plane and cone apexed at origin), whereas r = constant does define a localized object. The same is true in all the coordinate systems alluded to in the previous paragraph: there is always one coordinate ξ, which we often call "the radial coordinate", such that ξ = constant defines a localized surface, such as an ellipsoid with its limits like elliptical or circular flat disk. This coordinate is somehow just a distortion of the r coordinate of the spherical system.

In **cylindrical** coordinates, the surfaces z = constant (plane), ρ = constant (cylinder), φ = constant (half plane), we are again like Cartesian coordinates in that none of these equations describes a localized surface. The new infinite surface of interest here is the infinitely long cylinder, and if we charge it, we know the potential is that of the 2D charged circle problem, whose solution is $V = -2 \ln(\rho/\rho_0)$.

The **toroidal** coordinate system brings something new to this discussion. In this system, the azimuth φ = constant again defines the usual half plane, but both μ = const and η = const (the other two coordinates) define localized objects. The former surfaces are toroids, while the latter are spherical bowls. So we are interested in the potential of both object types when they are charged. The potential of the charged toroid is given in M&F and has the form $V = \sqrt{\text{ch}\mu - \cos\eta}\ \Sigma_n\ A_n(\mu_0)\ P_{n-1/2}(\cosh\mu)\cos(n\eta)$, where the toroid is labeled by $\mu = \mu_0$. I don't think there is a simpler way to express this result, so we have both the involvement of special functions and we now have a sum, a single sum in this case because we have azimuthal symmetry. Perhaps the complexity of the answer reflects the fact that a toroid is a more complicated surface than a sphere or its distorted sisters. I have not computed the potential of a charged spherical bowl in toroidal coordinates, but have seen it in my "canonical" doc in sphericals and the result has the form $V = \Sigma_n\ a_n\ P_n(\cos\theta)$ so we still have a sum of special functions, so the complexity level is about the same.

In **oblate spheroidal** coordinates, the localized objects are spheroids, and the non-localized surfaces are hyperboloids of one and two sheets. I think the situation is similar for prolate spheroidals.

In **ellipsoidals** the situation is about the same, but the hyperboloids are distorted so as not to be surfaces of revolution, recall the famous Q-bomb (Mouse that Roared).



2. Green's functions.

For general placement of the Green's point charge, the potential is no longer azimuthally symmetric, and always involves (I think) a double sum, where one index is the azimuthal one "m" (if there is an azimuth) and the other is the quantum number (often "n") of a Sturm Liouville problem in one of the oscillatory coordinates. For the sphere, the result can be expressed in this manner, but reduces to the potential of the Green's charge plus an image charge, an immense simplification which is worth reviewing. I think in all other cases, you don't get such simplification and you are stuck with the double sum involving special functions. This includes for the spheroid and its limit the disk, as well as for the one-sheeted hyperboloid and its limit the iris. When the location of the Green's charge is restricted in some manner, it is possible that the Green's function might simplify into something that is not a double sum.