Green's function for an iris: Problem 38 the easy way PhL 7.8.10

This probably is how Smythe intended his student to do **Problem 38.** This doc is only 7 pp and has the whole thing, soup to nuts. Instead of doing disk to bowl to iris, here we just go directly from disk to iris.

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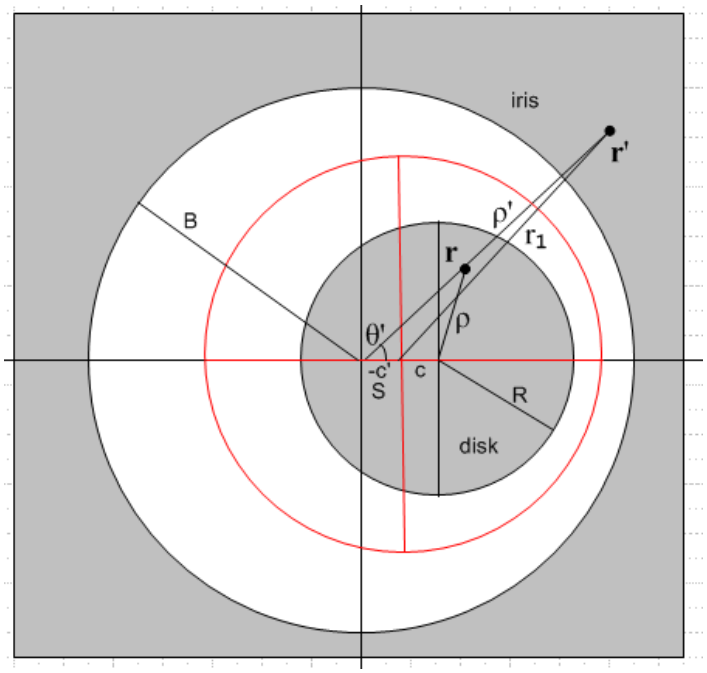
Overview (2 pages, written 12.3.10)

The picture. Just absorbing the picture below takes some doing. In this problem, we have a standard inversion mapping between two spheres, one lying inside the other as shown. The equatorial plane of the inner sphere of radius R is our charged metal disk. The equatorial plane *outside* the larger sphere of radius B is our "iris". The inversion circle (radius a) is red and is centered at the red cross hairs -- the inversion origin. R-space for this problem contains the charged disk (charge Q), to which we add a constant potential which (1) gets the disk to V = 0; (2) creates a point charge in R' space at the inversion origin. Once the disk is at V = 0, so is the iris.

R'-space contains the iris plus this point charge. The point charge appears at the inversion origin (red cross hairs) which is distance S from the center of the iris. The distance from the point charge to an observation point on the iris **r'** has two equivalent names: r' and r_1. The displacement of the point charge from the iris origin is -c' = S. The symbols B and S and r_1 were used in my double inversion solution to this problem, so I have used the same names here to facilitate comparison of the results.



In **Section 1** we roll out the Jackson Φ and σ for the charged disk:

$\Phi_{\mathbf{disk}}(x,y,z) = (Q/R)\ \sin^{-1}\left[\ 2R \,/\, \left(\sqrt{(\rho-R)^2 + z^2}\ + \sqrt{(\rho+R)^2 + z^2}\ \right)\right] \qquad \rho = \sqrt{(x-c)^2 + y^2}$

$\sigma_{\mathbf{disk}}(x,y,z=0) = (Q/2\pi R) \,/\sqrt{R^2-\rho^2}$ // sum of both sides, green Jackson page 93

Then in **Section 2** we add the offsetting constant potential -Q/C which creates a point charge in R' space and lowers the disk to zero potential in R space. We know C = (2/π)R for the disk, so added constant V = - (π/2) (Q/R), and we just add this to $\Phi_{\mathbf{disk}}$ shown above. Then we combine this into the first term which changes it from $\sin^{-1}$ to $\cos^{-1}$. In section 4 we will show that the point charge size is q' = - a(π/2)(Q/R).

In **Section 3** we roll out the Jackson inversion formulas for σ and Φ

$\Phi'(\mathbf{r'}) = (a/r')\ \ \Phi(\mathbf{r})$

$\sigma'(\mathbf{r'}) = (a/r')^3\ \sigma(\mathbf{r})$

where $\mathbf{r'} = (a^2/r^2)\mathbf{r}$ which implies $r' = a^2/r$ so that $(r'/a) = (a/r)$

In **Section 4** we compute that the point charge is q' = - a(π/2)(Q/R). We then multiply everything in the problem by q_1/q' to rescale things so we have a point charge of size q_1 . At this point, we have the following potential and charge density in R' space

$\Phi'(\mathbf{r'}) = \ 2q_1/(\pi r') * \ \cos^{-1}\left[\ 2R \,/\, \left(\sqrt{(\rho-R)^2 + z^2}\ + \sqrt{(\rho+R)^2 + z^2}\ \right)\right] \qquad \rho = \sqrt{(x-c)^2 + y^2}$

$\sigma'(\mathbf{r'}) = \ - a^2 q_1 \,/\, (\pi^2 r'^3) * 1 \,/\sqrt{R^2-\rho^2}$ // both sides

where $\mathbf{r} = (a^2/r'^2)\mathbf{r'}$ which implies $r = a^2/r'$ so that $(r/a) = (a/r')$

In **Section 5** we develop some equations which relate the disk variables and parameters ρ, R, c (properties in R space) to the corresponding iris quantities in R' space ρ', R', c' .

In **Section 6** we use these equations to rewrite the R'-space potential in terms of R'-space quantities. This takes a lot of algebra (this is always the case with this representation of the potential) and we use some Maple assist. When the dust clears, we have this result:

$$\Phi'(\mathbf{r'}) = 2q_1/(\pi r_1) \cos^{-1}\left[\sqrt{2}\, B\, r_1 \,/\, \{ \sqrt{m + (B^2-S^2)\sqrt{[(\rho'-B)^2+z'^2]\,[(\rho'+B)^2+z'^2]}} \} \right]$$

where

$$m = (\rho'^2+z'^2)(S^2-B^2) + B^2(B^2- S^2 + 2r_1^2)$$

$$r_1^2 = \rho'^2 + S^2 - 2S\rho'\cos\theta + z'^2 \qquad \theta = 0 \text{ in direction of the point charge}$$

In **Section 7** we use these same equations to rewrite the R'-space σ in terms of R'-space quantities. More Maple is required, and we end up with this result:

$$\sigma'(\mathbf{r'}) = -\, q_1 / (\pi^2 r_1^2) * \sqrt{B^2-S^2} \,/ \sqrt{\rho'^2-B^2} \qquad \text{// both sides}$$

This is the amazingly simple Smythe result that held me at bay for several months.

In **Section 8** I just summarize the results already stated. The results agree exactly with those I got doing Problem 38 "the hard way" using a double inversion involving a bowl (and it *was* the hard way!)

I am sure there is some way to do this problem where the "miraculously simple" Maple algebra results are not needed, but I decided not to spend more time trying to find that way. There is some kind of group theory going on with the disk to disk transformation, I just know it.

In **Section 9** I try but fail to show why we get the amazingly simple result that the quantity $(R^2-\rho^2)$ is a multiple of the quantity $(B^2-\rho'^2)$. This remains on my mystery list.

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1. Jackson Disk Start. We start with the potential and charge on the charged disk in R space. We can translate Jackson's 3.178 as follows (our radius is R instead of a)

$$V_{\mathbf{disk}}(x,y,z) = (Q/R) \sin^{-1}\left[2R / \left(\sqrt{(\rho-R)^2 + z^2} + \sqrt{(\rho+R)^2 + z^2} \right) \right] \qquad \rho = \sqrt{(x-c)^2 + y^2}$$

$$\sigma_{\mathbf{disk}}(x,y,z=0) = (Q/2\pi R) / \sqrt{R^2-\rho^2} \qquad \text{// sum of both sides, green Jackson page 93}$$

2. The constant potential. Now we know that the potential on this charged disk in R space is (Q = sum on both sides)

$$V = Q/C = Q/\left[(2/\pi)R \right] = (\pi/2)(Q/R) \qquad \text{ie} \qquad C = (2/\pi)R$$

so we want to add, in R space, the constant potential -V to bring the disk down to 0 potential. Therefore, this shall be our total potential in R space

$$\Phi(x,y,z) = (Q/R)\ \sin^{-1}\left[\ 2R \,/\, \left(\sqrt{(\rho-R)^2 + z^2}\ + \sqrt{(\rho+R)^2 + z^2}\ \right)\right] - (\pi/2)\ (Q/R)$$

3. Inversion formulas. We now look up in our Jackson inversion formulas doc ("formulas") and find that

$$\Phi'(\mathbf{r}') = (a/r')\ \ \Phi(\mathbf{r})$$
$$\sigma'(\mathbf{r}') = (a/r')^3\ \sigma(\mathbf{r})$$

where $\mathbf{r}' = (a^2/r^2)\mathbf{r}$ which implies $r' = a^2/r$ so that $(r'/a) = (a/r)$

So now we know the potential and charge in R' space:

$$\Phi'(\mathbf{r}') = \ (a/r')\ \Phi(\mathbf{r})\ = (a/r')\ [(Q/R)\ \sin^{-1}\left[\ 2R \,/\, \left(\sqrt{(\rho-R)^2 + z^2}\ + \sqrt{(\rho+R)^2 + z^2}\ \right)\right] - (\pi/2)\ (Q/R)$$
$$\sigma'(\mathbf{r}') = (a/r')^3\ \sigma(\mathbf{r})\ = (a/r')^3(Q/2\pi R)\ /\sqrt{R^2-\rho^2} \qquad \rho = \sqrt{(x-c)^2 + y^2} \qquad \text{// both sides}$$

We immediately do our usual combination in the potential and restate these results ($\sin^{-1}+\cos^{-1}\ = \pi/2$)

$$\Phi'(\mathbf{r}') = \ -\ (a/r')\ (Q/R)\ \cos^{-1}\left[\ 2R \,/\, \left(\sqrt{(\rho-R)^2 + z^2}\ + \sqrt{(\rho+R)^2 + z^2}\ \right)\right]$$
$$\sigma'(\mathbf{r}') = \ \ (a/r')^3(1/2\pi)(Q/R)\ /\sqrt{R^2-\rho^2} \qquad \rho = \sqrt{(x-c)^2 + y^2} \qquad \text{// both sides}$$

4. Rescale the Point Charge. As stated, these results apply to the Green's function for the R' space iris where the point charge is located at the inversion origin, and where the point charge size is a certain q' which we will now compute. If we look at the above Φ potential, we see that the constant part was $-(\pi/2)(Q/R)$ and in Φ' this becomes $-(\pi/2)(Q/R)(a/r') = q'/r'$. Therefore $q' = -\ a(\pi/2)(Q/R)$. But we want to rescale our problem so that this point charge is q_1 (just to make up a name). All we need do is multiply everything by

$$\text{factor} = q_1/\ [-\ a(\pi/2)\ (Q/R)] \ = -\ (2/\pi)(1/a)\ q_1(R/Q)$$

Doing this, we find (notice that both have correct units)

$$\Phi'(\mathbf{r}') = \ 2q_1/(\pi r')\ *\ \cos^{-1}\left[\ 2R \,/\, \left(\sqrt{(\rho-R)^2 + z^2}\ + \sqrt{(\rho+R)^2 + z^2}\ \right)\right] \qquad \rho = \sqrt{(x-c)^2 + y^2}$$
$$\sigma'(\mathbf{r}') = \ \ -\ a^2q_1\ /\ (\pi^2r'^3)\ *\ 1\ /\sqrt{R^2-\rho^2} \qquad \text{// both sides}$$

where $\mathbf{r} = (a^2/r'^2)\mathbf{r}'$ which implies $r = a^2/r'$ so that $(r/a) = (a/r')$

5. Convert variables and parameters. Our task now (a fairly large one) is to convert this result from disk variables and parameters ρ, R, c (properties in R space) to the corresponding iris quantities in R' space ρ', R', c' . Since I will want to compare the results I get here to those of the long method, I will use these names

$-c' = S > 0$	distance of point charge from iris origin
$R' = B$	radius of hole in iris

Then we have

$x = (a^2/r'^2)\, x' \qquad y = (a^2/r'^2)\, y' \qquad z = (a^2/r'^2)\, z'$

where x',y',z' are the coordinates of **r'** relative to the red crosshair inversion origin! Meanwhile,

$\rho^2 = (x-c)^2 + y^2$ // the usual cylindrical radial distance relative to disk center

$\mathbf{c} = \beta'\mathbf{c'} \quad \Rightarrow \quad c = \beta' c' = -\beta' S$ // **c** and **c'** are the centers of our spheres

$R = |\beta'|R' = |\beta'|B = -\beta' B$ // sphere radii

$\beta' = a^2/(c'^2 - R'^2) = a^2/(S^2 - B^2) < 0$ // factor that is ratio of centers and radii

Now back to x',y',z' . We want ρ',θ' to be cylindrical coordinates for the iris. Here is what that means:

$\rho'\cos\theta' = x' + S$ // because x' + S is the horizontal component of **r'** in iris cylin's, see picture
$\rho'\sin\theta' = y'$

$\rho'^2 = (x'+S)^2 + y'^2 = x'^2 + S^2 + y'^2 + 2Sx'$ // this will be used in next equation line below

We anticipate that we will have need of this quantity:

$r'^2 = x'^2 + y'^2 + z'^2 = (\rho'^2 - S^2 - 2Sx') + z'^2$

$= (\rho'^2 - S^2 - 2S[\rho'\cos\theta' - S]) + z'^2$

$= \rho'^2 + S^2 - 2S\rho'\cos\theta' + z'^2 = r_1^2$ // distance^2 to iris point **r'** from point charge

so r' and r_1 are the same thing, and I will use r_1 in the end. We will set z' = 0 if we want **r** and **r'** to be in the plane of paper for consideration later of the surface charge.

6. Process the potential. With the previous section as background, we now turn to finding the **iris potential**. As I did in the previous derivation of the iris stuff, I call upon my ellipse theorem,

$$(1/2)\left(\sqrt{(x-a)^2+y^2} + \sqrt{(x+a)^2+y^2}\right)^2 = (x^2 + y^2 + a^2) + \sqrt{(x^2 - y^2 - a^2)^2 + 4x^2y^2}$$

=>

$$\left(\sqrt{(\rho-R)^2+z^2} + \sqrt{(\rho+R)^2+z^2}\right)^2 = 2(\rho^2 + z^2 + R^2) + 2\sqrt{(\rho^2 - z^2 - R^2)^2 + 4\rho^2 z^2}$$

Sticking with our tried and true notation, we define

$$dac2/2 = (\rho^2 + z^2 + R^2) + \sqrt{(\rho^2 - z^2 - R^2)^2 + 4\rho^2 z^2}$$

where dac2 means "denominator of arc cosine, squared" . We are going to install into this expression these objects from above:

$\rho^2 = (x-c)^2 + y^2$ where $x = (a^2/r'^2)\, x'$ and $y = (a^2/r'^2)\, y'$

where $x' = \rho'\cos\theta' - S$ $y' = \rho'\sin\theta'$

$z^2 = (a^2/r'^2)^2\, z'^2$

$R^2 = \beta'^2 B^2$ where $\beta' = a^2/(S^2-B^2) < 0$

As our first step, multiply dac2/2 by $(S^2-B^2)^2 r'^4 \equiv f^2$ so that (argrad = "argument of radical")

$f^2 dac2/2 = (f^2\rho^2 + f^2 z^2 + f^2 R^2) + \sqrt{(f^2\rho^2 - f^2 z^2 - f^2 R^2)^2 + 4\, f^2\rho^2\, f^2 z^2} = t1 + \sqrt{argrad}$

$argrad \equiv (f^2\rho^2 - f^2 z^2 - f^2 R^2)^2 + 4\, f^2\rho^2\, f^2 z^2 = (t_2)^2 + t_3$

We do this to "clear denominators" to make the upcoming algebra easier (on Maple and us).

I throw this into Maple (see "iris greens.mws") and I end up with this result:

argrad =

$$= (B-S)^2 (B+S)^2 a^8 rl^4 (B^2 - 2\, B\, rhop + rhop^2 + zp^2)(B^2 + 2\, B\, rhop + rhop^2 + zp^2)$$

(it took many rounds of corrections before I got this result out!) So we have

$argrad = (B^2-S^2)^2\, a^8 r_1^4\, [(\rho'-B)^2 + z'^2]\, [(\rho'+B)^2 + z'^2]$

which is a familiar looking result. The t1 term is then this:

$$-(S^2 B^2 - 2\, rl^2 B^2 + B^2 rhop^2 + B^2 zp^2 - B^4 - zp^2 S^2 - rhop^2 S^2)\, a^4 rl^2$$

Dividing by $a^4 r_1^2$ we get the factor to be this:

$$= -S^2 B^2 + 2\, rl^2 B^2 - B^2 rhop^2 - B^2 zp^2 + B^4 + zp^2 S^2 + rhop^2 S^2$$

Let's compare this to the "m thing" I got in my double inversion method: (y = yes, term matches)

$$m := S^2 Z^2 - B^2 S^2 + S^2 \rho^2 - B^2 \rho^2 + 2\, rl^2 B^2 + B^4 - B^2 Z^2$$

y y y y y y y

It all matches. Therefore, I know the factor can be written as

$$m = (\rho'^2+z'^2)(S^2-B^2) + B^2(B^2- S^2 + 2r_1^2)$$

So I can summarize my results at this point:

$$f^2 dac2/2 = t1 + \sqrt{argrad} \qquad = a^4 r_1^2 m + a^4 r_1^2 (B^2-S^2) \sqrt{[\,(\rho'-B)^2+z'^2]\,[\,(\rho'+B)^2+z'^2]}$$

$$= a^4 r_1^2 \{\, m + (B^2-S^2) \sqrt{[\,(\rho'-B)^2+z'^2]\,[\,(\rho'+B)^2+z'^2]}\,\}$$

Now since

$$(S^2-B^2)^2 r_1^4 \equiv f^2 \qquad\qquad // \; r' = r_1$$

we end up with

$$dac2 = 2\, a^4 r_1^{-2}\, (S^2-B^2)^{-2}\, \{\, m + (B^2-S^2) \sqrt{[\,(\rho'-B)^2+z'^2]\,[\,(\rho'+B)^2+z'^2]}\,\}$$

so take square root to get (B > S recall)

$$dac = \sqrt{2}\; a^2 r_1^{-1}\, (B^2-S^2)^{-1} \sqrt{m + (B^2-S^2) \sqrt{[\,(\rho'-B)^2+z'^2]\,[\,(\rho'+B)^2+z'^2]}}$$

Recall also that we are looking at

$$\cos^{-1} [\; 2R / (\sqrt{(\rho-R)^2 + z^2} \; + \sqrt{(\rho+R)^2 + z^2} \;) \,]$$

and dac is the denominator. As for the numerator,

$$2R = -2\beta' B = -2a^2 B/(S^2-B^2) = 2a^2 B/(B^2-S^2)$$

$$2R/dac = 2a^2 B/(B^2-S^2) \,/\, \{\sqrt{2}\; a^2 r_1^{-1}\, (B^2-S^2)^{-1} \sqrt{m + (B^2-S^2) \sqrt{[\,(\rho'-B)^2+z'^2]\,[\,(\rho'+B)^2+z'^2]}}\,\}$$

$$= \sqrt{2}\, B\, r_1 \,/\, \{\, \sqrt{m + (B^2-S^2) \sqrt{[\,(\rho'-B)^2+z'^2]\,[\,(\rho'+B)^2+z'^2]}}\,\}$$

which agrees exactly with my double inversion approach. So here then is our final potential

$$\Phi'(\mathbf{r}') = 2q_1/(\pi r') * \cos^{-1} [\; 2R / dac \,]$$

$$= 2q_1/(\pi r_1) \cos^{-1} [\; \sqrt{2}\, B\, r_1 \,/\, \{\, \sqrt{m + (B^2-S^2) \sqrt{[\,(\rho'-B)^2+z'^2]\,[\,(\rho'+B)^2+z'^2]}}\,\} \,]$$

where

$$m = (\rho'^2+z'^2)(S^2-B^2) + B^2(B^2- S^2 + 2r_1^2)$$

$$r_1^2 = \rho'^2 + S^2 - 2S\rho'\cos\theta' + z'^2 \qquad\qquad \theta = 0 \text{ in direction of the point charge}$$

all in full agreement with the double inversion approach.

7. Process the charge density. We now turn to the **iris charge density**. From above we had

$$\sigma'(\mathbf{r}') = -a^2 q_1 / (\pi^2 r'^3) * 1/\sqrt{R^2-\rho^2} \qquad \text{// both sides}$$

If we set z' = 0 as apropos for the disk surface, we find from Maple that

$$R^2-\rho^2 = \frac{(B-rhop)(B+rhop)a^4}{r1^2 (S-B)(S+B)} = (B^2-\rho'^2)\ a^4 / [\ r_1^2(S^2-B^2)]$$

$$= (\rho'^2-B^2)\ a^4 / [\ r_1^2(B^2-S^2)]$$

$$1/\sqrt{R^2-\rho^2} = \sqrt{B^2-S^2}\ r_1\ a^{-2} / \sqrt{\rho'^2-B^2}$$

Therefore we get

$$\sigma'(\mathbf{r}') = -a^2 q_1 / (\pi^2 r'^3) * 1/\sqrt{R^2-\rho^2}$$

$$= -a^2 q_1 / (\pi^2 r'^3) * \sqrt{B^2-S^2}\ r_1\ a^{-2} / \sqrt{\rho'^2-B^2}$$

$$= -q_1 / (\pi^2) * \sqrt{B^2-S^2}\ r_1^{-2} / \sqrt{\rho'^2-B^2}$$

$$= -q_1 / (\pi^2 r_1^2) * \sqrt{B^2-S^2}\ / \sqrt{\rho'^2-B^2} \qquad \text{// both sides}$$

This agrees with my "doing disk by inversion" summary, and agrees with "3. Doing the Second.." page 28.

8. Summary. So here is a **summary of all results**:

$$\Phi'(\mathbf{r}') = 2q_1/(\pi r_1) \cos^{-1} [\ \sqrt{2}\ B\ r_1 / \{ \sqrt{m + (B^2-S^2) \sqrt{[\ (\rho'-B)^2+z'^2]\ [\ (\rho'+B)^2+z'^2]}}\ \}]$$

$$m = (\rho'^2+z'^2)(S^2-B^2) + B^2(B^2- S^2 + 2r_1^2)$$

$$r_1^2 = \rho'^2 + S^2 - 2S\rho'\cos\theta' + z'^2 \qquad \theta = 0 \text{ in direction of the point charge}$$

$$\sigma'(\mathbf{r}') = -q_1 / (\pi^2 r_1^2) * \sqrt{B^2-S^2}\ / \sqrt{\rho'^2-B^2} \qquad \text{// sum over both sides}$$

$$r_1^2 = \rho'^2 + S^2 - 2S\rho'\cos\theta' \qquad \theta = 0 \text{ in direction of the point charge}$$

This summary appears in more generic parameters in "doing the disk by inversion.doc", where I make a side by side comparison between this iris Green's function result (point charge q_1), and the disk Green's function result (also point charge q_1).

9. Explanation of why we get $R^2-\rho^2$ being a multiple of ρ'^2-B^2. (added 12.7.10, fails)

This has always mystified me a bit. In the σ math above we found by inserting quantities that

$$(R^2-\rho^2) = (B^2-\rho'^2)\ k \qquad\qquad k = a^4 / [\ r_1^2(S^2-B^2)]$$

where k stands for the factor sitting there, which is not a "constant" of course since it depends on things like ρ' and θ'

In the "Hard Way" method, we had a similar "amazing result" which was this:

$$(b^2 - s^2) = (B^2- \rho'^2)\ k' \qquad\qquad k' = (d^2+ S^2)^2 / [\ r_1^2(S^2- B^2)]$$

or

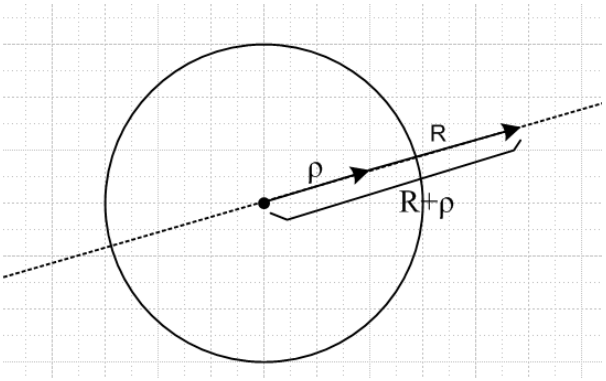
$$(R^2-\rho^2) = (B^2-\rho'^2)\ k'$$

where we note that s = ρ on the charged disk, and b = R = radius of the charged disk. First of all, this does suggest that the a^2 of the easy way inversion is equal to the d^2+S^2 of the hard way inversion, a bit of an obscure fact.

So here is an explanation of why we get the general form shown. Suppose we could show that the following equations were true

$$(R-\rho) = (B-\rho')k_1 \qquad \text{// don't think these are true, see below}$$
$$(R+\rho) = (B+\rho')k_2$$

Then our result above would follow simply by multiplying these equations together, and $k = k_1k_2$. Now, in the first equation above, R-ρ is a distance in charged-disk space between two well defined points.



My explanation has just crashed! I thought I could show that the two endpoints of the R+ρ vector "corresponded to" the two ends of the vector B+ρ' in the image space, but I don't think this is true. Had it been true, we could have claimed that our result follows from the "distance rule" D'/D = stuff which is

part of the inversion formulas doc. The dotted line above, since it does not pass through the inversion origin, maps into some curve in iris space (part of a circle). We know the ρ tip ends up at ρ', but where the disk center and the tip of R+ρ end up is not very obvious, and it is unlikely they end up at the corresponding points for thinking about B+ρ'.

So I remain mystified as to why $(R^2-\rho^2) = (B^2-\rho'^2)$ k is true. All I know is that it falls out of the messy inversion algebra, as Maple showed! There is something "deeper" going on with "inversion" that I don't understand. Perhaps there are other "invariants", perhaps it is studied in some more general transformation theory (fractional? affine? other?). Probably there is some group theory. What happens to a cylindrical coordinate system under an inversion? Does each cylinder become a bagel shaped horn thing?