Parameter facts for Pic 1 and Pic 2 PhL 6.15.10

I am writing this after writing "How to Actively rotate the bowl.doc", where I decided to use θ,φ in Pic 1, but to use θ",φ" in Pic 2. This forces me to think about every angle encountered.

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Overview (1/2 page, written 12.5.10)

In **Section 1** I collect and derive a set of "facts" labeled (a) though (g) which relate only to the Pic 1 picture shown blow, which relates to the first inversion problem.

In **Section 2** I collect and derive a set of "facts" labeled (a) though (g) which relate only to the Pic 2 picture shown blow, which relates to the second inversion problem.

In **Section 3** I *try* to find the connection between quantity (b^2-s^2) from the charged-disk space picture and the quantity (ρ'^2-B^2) from the final iris picture. I try to find this relation using the simple facts collected in this doc, but I am not successful, nor did I try too hard. Later I derive this relationship in "doing the second inversion", and that relation is this:

$$(b^2 - s^2) = (\rho'^2 - B^2)\ (d^2 + S^2)^2 / [\, r_1^2 (B^2 - S^2)] \qquad \text{where} \quad r_1^2 = S^2 + \rho'^2 + 2S\rho' \cos\varphi"$$

This was so messy to derive that I had to have a Maple assist! There is doubtless some reason it comes out so relatively simple, but I don't know what it is. It is something Kelvin would know. It really involves the 3D geometry of things, and my Pic 1 and Pic 2 pictures are just 2D slices.

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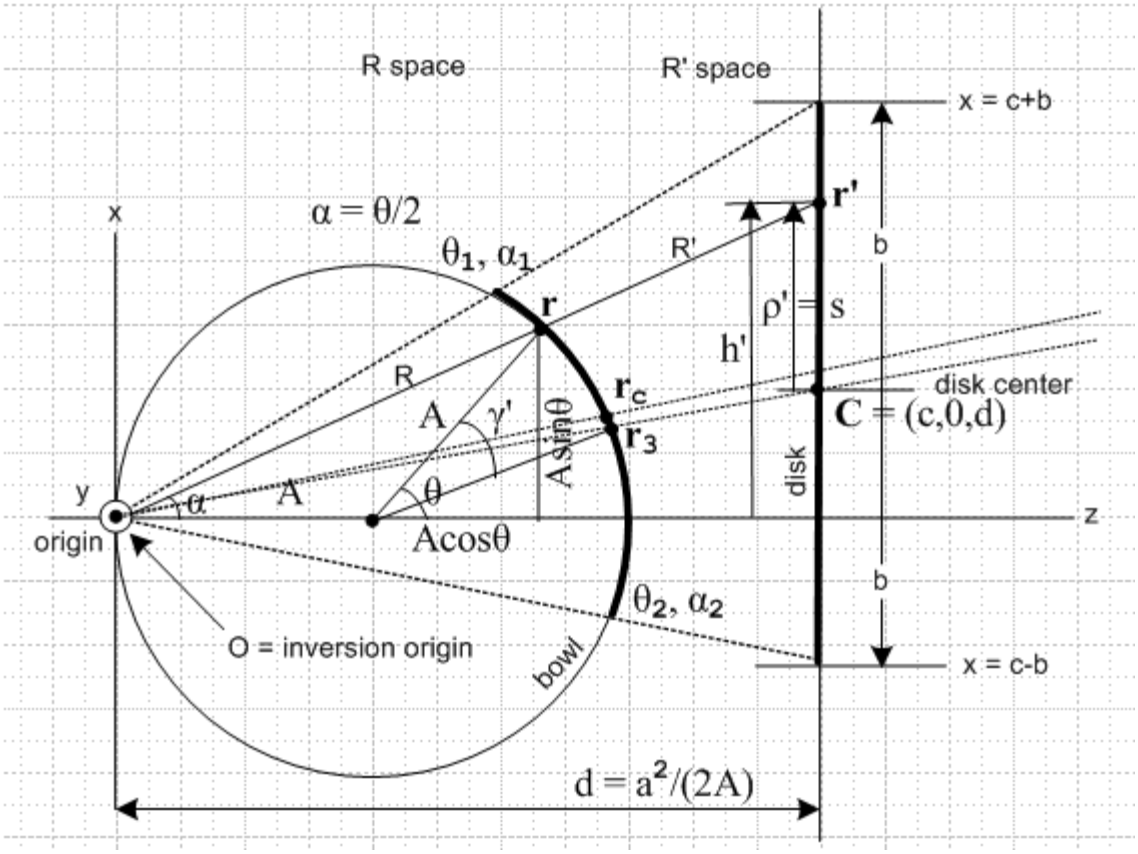
1. Picture 1 : showing the first inversion.



(a) From inversion we know that

$\mathbf{r}' = (a^2/R^2)\,\mathbf{r}$ and thus $R' = (a^2/R)$ (general φ)

(a1) Concerning R we know the following for in plane **r** :

$R^2 = (A\sin\theta)^2 + (A+A\cos\theta)^2 = A^2 + A^2 + 2AA\cos\theta = 2A^2(1+\cos\theta) = 4A^2\cos^2(\theta/2)$

$\Rightarrow \quad R = 2A\cos(\theta/2)$ // φ = 0 only

(a2) Concerning the Green's Point charge at the origin and Q on the disk. For the disk on the right in Pic 1 we add the following constant potential

$-V = -Q/C \qquad C = (2/\pi)b$

We then find that by adjusting Q such that

$Q = -C/a = -(2/\pi)(b/a) = -(2b/\pi a)$

we cause the Green's point charge at the inversion in the bowl R space picture to be +1.

(b) staring at the picture we can see right triangles for these three *positive* polar angles,

$\tan(\theta/2) = (s+c)/d$ // valid when $\varphi = 0$ so **r** is in the plane of paper
$\tan(\theta_1/2) = (b+c)/d$
$\tan(\theta_2/2) = (b-c)/d$

I am willing to restrict my interest (for now) to the situation $c > 0$ and $b > c$, as the picture show. Thus, the RHS of the last two expressions are positive, which corresponds to having positive θ_1 and θ_2. These must be positive if I am calling them polar angles.

We can solve the lower two equations for b and c:

$b = (d/2)\ [\tan(\theta_1/2) + \tan(\theta_2/2)\ ]$
$c = (d/2)\ [\tan(\theta_1/2) - \tan(\theta_2/2)\ ]$

(c) We can define a positive "polar bowl angle" in an obvious manner,

$\theta_b = (\theta_1+\theta_2)/2$ // θ_b lies in the range $(0,\pi)$

(d) Suppose we rotate the circle CW by a small angle ψ. The upper limit moves to $\theta_1 - \psi$ and the lower moves to $\theta_2 + \psi$. We can make these two angles equal by setting $\theta_1 - \psi = \theta_2 + \psi \Rightarrow 2\psi = (\theta_1 - \theta_2)$. We shall give this angle ψ the name θ_c, so we have

$\theta_c = (\theta_1-\theta_2)/2$

I did things differently in Attempt #1, mainly I now think of $\theta_2 > 0$ as well as $\theta_1 > 0$. We inverse solve the last two equations to get

$\theta_1 = (\theta_b + \theta_c)$
$\theta_2 = (\theta_b - \theta_c)$

In the picture I have drawn $\mathbf{r}_c$ appearing at angle θ_c and I make note that the inversion line through $\mathbf{r}_3$ does NOT pass through the center of the disk. For this reason, I don't have any simple $\tan(\theta_c/2)$ type expression for Pic 1.

(e) Now here is more on b and c. First, I copy from Attempt #1, checking every step,

Lemma: (stated after Therefore about 6" below)

$$\tan(A+B) \pm \tan(A-B) = (\tan A + \tan B)/(1-\tan A \tan B) \pm (\tan A - \tan B)/(1+\tan A \tan B)$$

$$= [\ (\tan A + \tan B)(1+\tan A \tan B) \pm (\tan A - \tan B)(1-\tan A \tan B)\ ] / (1 - \tan^2 A \tan^2 B)$$

$$= [\ (\alpha+\beta)(1+\alpha\beta) \pm (\alpha-\beta)(1-\alpha\beta)\ ] / (1 - \alpha^2\beta^2)$$

Let's do the two signs separately. First the + sign

$$N+ = (\alpha+\beta)(1+\alpha\beta) + (\alpha-\beta)(1-\alpha\beta)$$

$$= \alpha + \beta + \alpha^2\beta + \alpha\beta^2 \quad + \quad \alpha - \beta - \alpha^2\beta + \alpha\beta^2$$

$$= \alpha + \alpha\beta^2 \quad + \quad \alpha + \alpha\beta^2 \quad = 2\alpha + 2\alpha\beta^2 = 2\alpha(1+\beta^2)$$

$$N- = (\alpha+\beta)(1+\alpha\beta) - (\alpha-\beta)(1-\alpha\beta)$$

$$= \alpha + \beta + \alpha^2\beta + \alpha\beta^2 \quad - \quad \alpha + \beta + \alpha^2\beta - \alpha\beta^2$$

$$= \beta + \alpha^2\beta \quad + \beta + \alpha^2\beta \quad = 2\beta + 2\beta\alpha^2 = 2\beta(1+\alpha^2)$$

Therefore:

$$\tan(A+B) + \tan(A-B) = 2\tan A\,(1+\tan^2 B)/(1-\tan^2 A\tan^2 B)$$
$$\tan(A+B) - \tan(A-B) = 2\tan B\,(1+\tan^2 A)/(1-\tan^2 A\tan^2 B)$$

Now set $A = (\theta_c/2)$ and $B = (\theta_b/2)$ and we then find that

$$\tan((\theta_c/2)+(\theta_b/2)) + \tan((\theta_c/2)-(\theta_b/2)) = 2\tan(\theta_c/2)\,(1+\tan^2(\theta_b/2))/(1-\tan^2(\theta_c/2)\tan^2(\theta_b/2))$$
$$\tan((\theta_c/2)+(\theta_b/2)) - \tan((\theta_c/2)-(\theta_b/2)) = 2\tan(\theta_b/2)\,(1+\tan^2(\theta_c/2))/(1-\tan^2(\theta_c/2)\tan^2(\theta_b/2))$$

Now replace

$$(\theta_c/2) + (\theta_b/2) = (\theta_1/2)$$
$$(\theta_c/2) - (\theta_b/2) = -(\theta_2/2)$$

And the above then says:

$$\tan(\theta_1/2) - \tan(\theta_2/2) = 2\tan(\theta_c/2)\,(1+\tan^2(\theta_b/2))/(1-\tan^2(\theta_c/2)\tan^2(\theta_b/2))$$
$$\tan((\theta_1/2)) + \tan(\theta_2/2) = 2\tan(\theta_b/2)\,(1+\tan^2(\theta_c/2))/(1-\tan^2(\theta_c/2)\tan^2(\theta_b/2))$$

Then from above we have

$$b = (d/2)\,[\tan(\theta_1/2) + \tan(\theta_2/2)] = d\tan(\theta_b/2)\,(1+\tan^2(\theta_c/2))/(1-\tan^2(\theta_c/2)\tan^2(\theta_b/2))$$
$$c = (d/2)\,[\tan(\theta_1/2) - \tan(\theta_2/2)] = d\tan(\theta_c/2)\,(1+\tan^2(\theta_b/2))/(1-\tan^2(\theta_c/2)\tan^2(\theta_b/2))$$

which lets us relate disk parameters b and c to bowl parameters θ_b and θ_c. To invert these you would have to do this:

$$\theta_b = (\theta_1+\theta_2)/2 = (1/2)\,[2\tan^{-1}((b+c)/d) + 2\tan^{-1}((b-c)/d)] = \tan^{-1}((b+c)/d) + \tan^{-1}((b-c)/d)$$
$$\theta_b = (\theta_1-\theta_2)/2 = (1/2)\,[2\tan^{-1}((b+c)/d) - 2\tan^{-1}((b-c)/d)] = \tan^{-1}((b+c)/d) - \tan^{-1}((b-c)/d)$$

(f) For **r** in the plane (only) we have a similar set of equations involving θ, based on tan(θ/2) = (s+c)/d from above. [never used these result]

$\tan(\theta/2) = (s+c)/d$ $\cos(\theta/2) = d/\sqrt{(s+c)^2 + d^2}$ $\sin(\theta/2) = (s+c)/\sqrt{(s+c)^2 + d^2}$
$\sin\theta = 2d(s+c)/(d^2+(s+c)^2)$
$\cos\theta = (d^2 - (s+c)^2)/(d^2+(s+c)^2)$ // all for **r** in plane only

(g) Now we want to solve for Smythe's $b^2 - s^2$. We know from the **in-plane r** case that
[never used results of this section]

$\tan(\theta/2) = (s+c)/d$
$\tan(\theta_1/2) = (b+c)/d$

Rewrite as

$s = d\tan(\theta/2) - c$

Then we have this fairly simple result

$s^2 = (d\tan(\theta/2) - c)^2$

$b^2 - s^2 = b^2 - (d\tan(\theta/2) - c)^2$ $\varphi = 0$

which gives our Smythian quantity $b^2 - s^2$ entirely in terms of bowl parameters, if we recall that

$b = d\tan(\theta_b/2)\,(1+\tan^2(\theta_c/2))/(1 - \tan^2(\theta_c/2)\tan^2(\theta_b/2))$
$c = d\tan(\theta_c/2)\,(1+\tan^2(\theta_b/2))/(1 - \tan^2(\theta_c/2)\tan^2(\theta_b/2))$

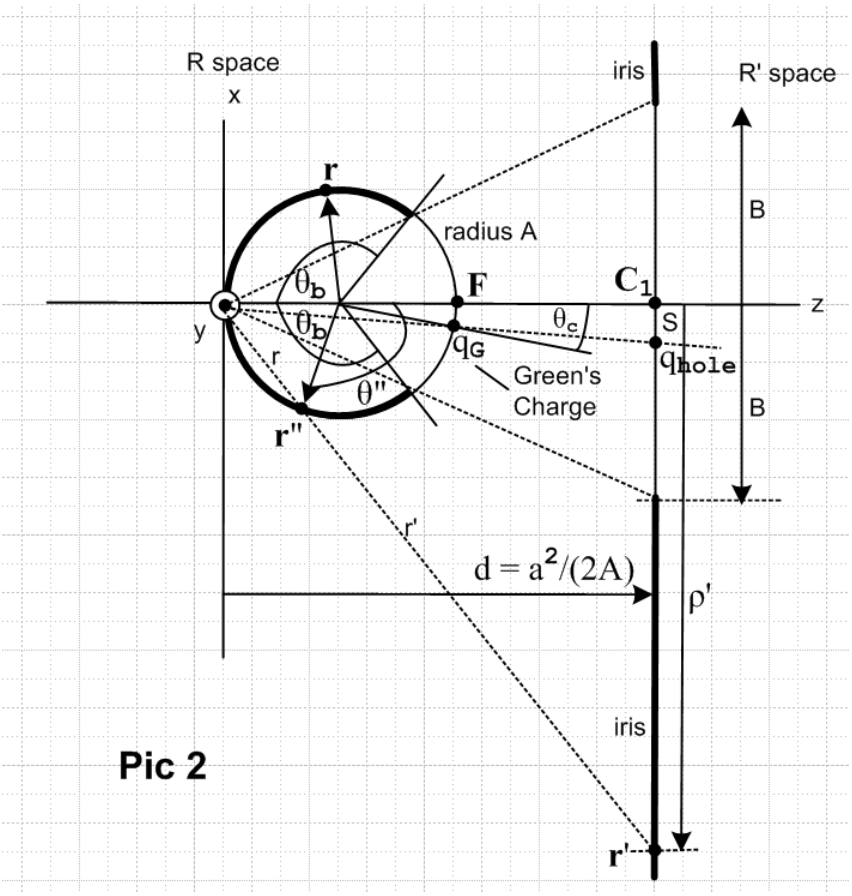
Then we have this

$b^2 - s^2 = F(\theta, \theta_c, \theta_b, d)$

so we have our result entirely in terms of bowl parameters. As we rotate the bowl to get to Pic 2, we will have to deal with θ, but the other three parameters won't change.

Comment: In order to get our $b^2 - s^2$ expression here, we had no need to consider **r** out of the plane.

2. Picture 2 : showing the second inversion.



We arbitrarily use the same d as in the previous inversion to reduce symbol clutter. Some of our "facts" will be true for general θ",φ" while others will only be valid for φ" = π (in the plane).

(a) Start with these facts obtained from the picture, where **r** is in plane

$\tan[(\pi-\theta_b)/2] = B/d = \cot(\theta_b/2)$
$\tan(\theta''/2) = \rho'/d$

and this last item I can expand as

$\tan(\theta''/2) = (\rho'/d)$ $\cos(\theta''/2) = d/\sqrt{\rho'^2 + d^2}$ $\sin(\theta''/2) = \rho'/\sqrt{\rho'^2 + d^2}$
$\sin\theta'' = 2d\,\rho'/(d^2+ \rho'^2)$
$\cos\theta'' = (d^2- \rho'^2)/(d^2+ \rho'^2)$

Repeating the original two items,

$\rho' = d \tan(\theta''/2)$
$B = d \cot(\theta_b/2)$

Then

$$\rho'^2 - B^2 = d^2 \left[\cot^2(\theta_b/2) - \tan^2(\theta''/2) \right]$$

So right off the bat we have our desired quantity entirely in bowl parameters!

(b) We have some nice θ_c trig formulas now in Pic 2 (nothing simple in Pic 1)

$$\tan(\theta_c/2) = S/d$$

which leads to this group of trig formulas:

$$\tan(\theta_c/2) = S/d \qquad \cos(\theta_c/2) = d/\sqrt{S^2 + d^2} \qquad \sin(\theta_c/2) = S/\sqrt{S^2 + d^2}$$
$$\sin\theta_c = 2dS/(d^2+S^2)$$
$$\cos\theta_c = (d^2 - S^2)/(d^2+S^2)$$

(c) Other facts:

$(R'') = (a^2/R'')$ // since these are inversion points (general φ)

$(R'')^2 = \rho'^2 + d^2$ // right triangle (general φ)

$\theta'' = \pi - (\theta - \theta_c)$ // (derived in next section) ($\varphi = 0$ only)

(d) looking at Pic 1 (a1), the following must also be true for Pic 2

$$R'' = 2A \cos(\theta''/2)$$

(e) Concerning the point charges: The point charge on the bowl's sphere is a positive unit charge, since that is how it came to us from the first inversion process where we scaled Q on the disk as discussed above. The bowl's unit point charge is located $\theta, \varphi = \theta_c, \pi$ as shown in the picture.

The in-iris-hole charge is located a distance S directly below the iris center, as shown above. The size of the hole-in-iris charge can be found from basic inversion theory,

$$q'_i = q_i(a/r_i) = (r_i'/a) \ ,$$

and in our case the unprimed charge $q_i = 1$ is located at θ_c , $q_i' = q_{hole}$, $r_i' = \sqrt{S^2 + d^2}$ which is the inversion distance from the inversion origin to the hole charge, as the picture shows. Thus

$$q_{hole} = (\sqrt{S^2 + d^2} / a)$$

(f) If we draw Smythe's r_1 vector on the iris, we get this equation

$$r_1^2 = S^2 + \rho'^2 + 2S\rho' \cos\varphi''$$

(g) We can use these facts

$\tan(\theta_b/2) = (d/B)$
$\tan(\theta_c/2) = (S/d)$

to related c and d to iris parameters. From our Pic 1 section above we had

$b = d \tan(\theta_b/2) (1+ \tan^2(\theta_c/2))/ (1 - \tan^2(\theta_c/2) \tan^2(\theta_b/2))$
$c = d \tan(\theta_c/2) (1+ \tan^2(\theta_b/2))/ (1 - \tan^2(\theta_c/2) \tan^2(\theta_b/2))$

so we install to get

$b = d (d/B) (1+ (S/d)^2)/ (1 - (S/d)^2 (d/B)^2)$
$c = d (S/d) (1+ (d/B)^2)/ (1 -(S/d)^2 (d/B)^2)$

$b = d^2 (1/B) (1+ (S/d)^2)/ (1 - (S/B)^2)$
$c = S (1+ (d/B)^2)/ (1 - (S/B)^2)$

$b = d^2 B (1+ (S/d)^2)/ (B^2- S^2)$
$c = S (B^2+ d^2)/ (B^2- S^2)$

Final results: (these agree with earlier work in Attempt #2A)

$b = B (d^2+ S^2) / (B^2- S^2)$
$c = S (B^2+ d^2) / (B^2- S^2)$

3. Going for the connection between $(b^2- s^2)$ and $(\rho'^2 - B^2)$: in-plane only

[results of this section never used, a good section to just ignore!]

From our document on "How to rotate the bowl", we know that for "corresponding points"

$\cos\theta'' = -\sin\theta_c \sin\theta\cos\varphi - \cos\theta_c\cos\theta$

Now in our special case we have $\varphi = 0$, and we (luckily) studied this case in detail in section " Question: How are (θ,φ) and (θ'',φ'') related to each other?" subsection "Special case $\varphi = 0$". Our conclusion there was

$$\varphi = 0 \Rightarrow \qquad \theta'' = \begin{cases} \pi - (\theta-\theta_c) & (\theta-\theta_c)>0 \\ \pi + (\theta-\theta_c) & (\theta-\theta_c)<0 \end{cases}$$

$$\varphi'' = \begin{cases} \pi & (\theta-\theta_c)>0 \\ 0 & (\theta-\theta_c)<0 \end{cases}$$

This discussion concerned corresponding points $\mathbf{r}'' = R_y(\pi-\theta_c)\mathbf{r}$. In our case, our Pic 1 shows that we have $(\theta-\theta_c) > 0$. Therefore, the point in Pic 2 which corresponds to the point in Pic 1 is this

$\theta'' = \pi - (\theta - \theta_c)$
$\varphi'' = \pi$

Luckily, only θ'' is of concern to us since only it shows up in our Smythian quantity above. So we can compute

$$\tan(\theta''/2) = \tan[(\pi - (\theta - \theta_c))/2] = \cot[(\theta - \theta_c)/2]$$

Therefore we can say

$$\rho'^2 - B^2 = d^2 [\ \cot^2(\theta_b/2) - \tan^2(\theta''/2)\ ]$$

$$= d^2 (\ \cot^2(\theta_b/2) - \cot^2[(\theta - \theta_c)/2]\ )$$

where we now have the Pic 1 angle involved. So we can now consider our two expressions:

$$(b^2 - s^2) = b^2 - (d \tan(\theta/2) - c)^2 \qquad \varphi = 0$$

$$(\rho'^2 - B^2) = d^2 (\ \cot^2(\theta_b/2) - \cot^2[(\theta - \theta_c)/2]\ ) \qquad \varphi = \pi$$

The point is that both lines have the Pic 1 angle θ now. This is the closest I have ever been to connecting these two quantities! We also know that

$$b = d \tan(\theta_b/2) (1 + \tan^2(\theta_c/2))/ (1 - \tan^2(\theta_c/2) \tan^2(\theta_b/2))$$
$$c = d \tan(\theta_c/2) (1 + \tan^2(\theta_b/2))/ (1 - \tan^2(\theta_c/2) \tan^2(\theta_b/2))$$

Now I have both expressions in terms of "bowl parameters" and that was the goal. I regard the amount of bowl rotation $\pi - \theta_c$ as one of our "bowl parameters".

So let's let this result just sit for a while. One could in theory eliminate θ between the two expressions, but let's not attempt that right now.