Doing the Second Inversion PhL 6.15.10

See top level doc for an **overview** of what is going on in the 33 pages below! It is just amazing how complicated it all is, but it got me to the right Problem 38 σ answer, along with a potential Φ to boot. I later had good evidence that the Φ answer is correct as well, when I discovered the simpler disk-to-disk single inversion way to solve this problem.

Overview: This is just a very long, pure nuts and bolts, grind it out calculation. I give a detailed overview in the top level review doc. That overview *by itself* is 11 pages long, more of a "meta" doc. If you want a quick overview, just read every line of the contents below.

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1. Preliminaries. Here is where we stand at this point:

(1) I have done the first inversion in the old documents, to get σ and Φ for the bowl from the disk.

(2) In doc "How to rotate the bowl", I have derived expressions for σ_{post} and Φ_{post} for the post-rotated bowl.

(3) In doc " Parameter facts for Pic 1 and Pic 2.doc" I have, from scratch, written down the various geometric and inversion facts for our two "pictures".

2. Do the second inversion, formally.

We shall now do "the second inversion" right here. First, here is how we left things after the first inversion and after the subsequent bowl rotation (from "how to rotate"):

$\sigma_{post}(\mathbf{r}'') = \sigma(\mathbf{r}) = -(a^2/\pi^2R^3)/\sqrt{b^2-s^2}$ $\qquad d = a^2/2A$

$R = 2A[\sin\theta_c \sin\theta''\cos\varphi'' - \cos\theta_c \cos\theta'' + 1]$ $\qquad s = \sqrt{(x'-c)^2 + y'^2}$

$x' = (a^2A/R^2)\ [-\cos\theta_c \sin\theta''\cos\varphi'' - \sin\theta_c \cos\theta'']$

$y' = (a^2A/R^2)\ [\sin\theta''\sin\varphi''$

$z' = (a^2A/R^2)\ [\sin\theta_c \sin\theta''\cos\varphi'' - \cos\theta_c \cos\theta'' + 1]$ $\qquad // = d$

Let's forget the potential for right now and concentrate on σ. Here is the basic second inversion claim:

$\sigma'(\mathbf{r}') = (R''/a)^3\ \sigma_{post}(\mathbf{r}'')$ $\qquad$ // see Jackson META page 8

where we are of course in Pic 2 now. Of course the point **r''** on the sphere corresponds to point **r'** on the iris. From Pic 2 fact (d) we have

$R'' = 2A\cos(\theta''/2)$

So here is the charge density on the iris:

$\sigma_{iris}(\mathbf{r}') = (R''/a)^3\ \sigma_{post}(\mathbf{r}'') = (2A/a)^3 \cos^3(\theta''/2)\ \sigma_{post}(\mathbf{r})$

$= -(2A/a)^3 \cos^3(\theta''/2)\ (a^2/\pi^2R^3)/\sqrt{b^2-s^2}$

$= -(8A^3/\pi^2aR^3) \cos^3(\theta''/2)/\sqrt{b^2-s^2}$ $\qquad dim = 1/L^2$

where, of course, s and R are given by the complicated expressions above.

3. Evaluation of $\sigma_{iris}(\mathbf{r}')$

We have

$\sigma_{iris}(\mathbf{r}') = -(8A^3/\pi^2aR^3) \cos^3(\theta''/2)/\sqrt{b^2-s^2}$

with

$R = 2A[\sin\theta_c \sin\theta''\cos\varphi'' - \cos\theta_c \cos\theta'' + 1]$
$s = \sqrt{(x'-c)^2 + y'^2}$
$x' = (a^2A/R^2)\ [-\cos\theta_c \sin\theta''\cos\varphi'' - \sin\theta_c \cos\theta'']$
$y' = (a^2A/R^2)\ [\sin\theta''\sin\varphi'']$

We want to get rid of angle functions in favor of iris parameters! To this end, I call upon these facts:

$\tan(\theta_c/2) = S/d$ $\cos(\theta_c/2) = d/\sqrt{S^2 + d^2}$ $\sin(\theta_c/2) = S/\sqrt{S^2 + d^2}$
$\sin\theta_c = 2dS/(d^2+S^2)$
$\cos\theta_c = (d^2- S^2)/(d^2+S^2)$

$\tan(\theta''/2) = (\rho'/d)$ $\cos(\theta''/2) = d/\sqrt{\rho'^2 + d^2}$ $\sin(\theta''/2) = \rho'/\sqrt{\rho'^2 + d^2}$
$\sin\theta'' = 2d\ \rho'/(d^2+ \rho'^2)$
$\cos\theta'' = (d^2- \rho'^2)/(d^2+ \rho'^2)$

$B = d \cot(\theta_b/2)$

$r_1^2 = S^2 + \rho'^2 + 2S\rho' \cos\varphi''$

So let's now try blindly "working on" some of our angle functions.

$(\sin\theta''\sin\varphi'') = [2d\ \rho'/(d^2+ \rho'^2)] \sin\varphi''$

Next,

$(\cos\theta_c\sin\theta''\cos\varphi'' + \sin\theta_c\cos\theta'')$

$= (d^2- S^2)/(d^2+S^2)* 2d\ \rho'/(d^2+ \rho'^2) * \cos\varphi'' + 2dS/(d^2+S^2)* (d^2- \rho'^2)/(d^2+ \rho'^2)$

$= [(d^2- S^2)\ 2d\ \rho' \cos\varphi'' + 2dS\ (d^2- \rho'^2)] / [(d^2+S^2)\ (d^2+ \rho'^2)]$

$= 2d\ [(d^2- S^2)\ \rho' \cos\varphi'' + S\ (d^2- \rho'^2)] / [(d^2+S^2)\ (d^2+ \rho'^2)]$

Next,

$[\ 1 - \cos\theta_c\cos\theta'' + \sin\theta_c\sin\theta''\cos\varphi''\ ]$

$= 1 - (d^2- S^2)/(d^2+S^2)*(d^2- \rho'^2)/(d^2+ \rho'^2) + 2dS/(d^2+S^2)\ 2d\ \rho'/(d^2+ \rho'^2) \cos\varphi''$
$= 1 - [(d^2- S^2)\ (d^2- \rho'^2) - 2dS2d\ \rho' \cos\varphi'']\ /\ [(d^2+S^2)\ (d^2+ \rho'^2)]$

$= \{\ (d^2+S^2)\ (d^2+ \rho'^2) - (d^2- S^2)\ (d^2- \rho'^2) + 2dS2d\ \rho' \cos\varphi''\}/\ [(d^2+S^2)\ (d^2+ \rho'^2)]$

Now as I have done several times before, we go off and find that

$(d^2+S^2)\,(d^2+\rho'^2) - (d^2-S^2)\,(d^2-\rho'^2)$

$= (d^4 + d^2S^2 + d^2\rho'^2 + S^2\rho'^2) - (d^4 - d^2S^2 - d^2\rho'^2 + S^2\rho'^2)$

$= (d^2S^2 + d^2\rho'^2) - (-d^2S^2 - d^2\rho'^2) = 2(d^2S^2 + d^2\rho'^2) = 2d^2(S^2+\rho'^2)$

So we then have

$[] = \{\ 2d^2(S^2+\rho'^2) + 2dS2d\ \rho'\cos\varphi"\}/\ [(d^2+S^2)\,(d^2+\rho'^2)]$

$= \{\ 2d^2(S^2+\rho'^2) + 4d^2S\ \rho'\cos\varphi"\}/\ [(d^2+S^2)\,(d^2+\rho'^2)]$

$= 2d^2\ \{(S^2+\rho'^2) + 2S\ \rho'\cos\varphi"\}/\ [(d^2+S^2)\,(d^2+\rho'^2)]$

$= 2d^2\ \{S^2+\rho'^2 + 2S\ \rho'\cos\varphi"\}/\ [(d^2+S^2)\,(d^2+\rho'^2)]$

But *happily* from above we have

$r_1^2 = S^2 + \rho'^2 + 2S\rho'\cos\varphi"$

So we then end up with

$[\ 1 - \cos\theta_c\cos\theta" + \sin\theta_c\sin\theta"\cos\varphi"\ ] = 2d^2\ r_1^2\ /\ [(d^2+S^2)\,(d^2+\rho'^2)]$

which I think is a key result. This also tells us that

$R^2 = 2A^2[\ 1 - \cos\theta_c\cos\theta" + \sin\theta_c\sin\theta"\cos\varphi"\ ]$

$= 4A^2\ d^2\ r_1^2\ /\ [(d^2+S^2)\,(d^2+\rho'^2)]$

$= (2Adr_1)^2\ /\ [(d^2+S^2)\,(d^2+\rho'^2)]$

$R = 2Adr_1/\sqrt{(d^2+S^2)\,(d^2+\rho'^2)}$ // agrees with previous work in Attempt #2A

Here then is a summary of our angle functions (with a common denom) and R:

$(\cos\theta_c\ \sin\theta"\cos\varphi" + \sin\theta_c\ \cos\theta") = 2d\ [(d^2-S^2)\ \rho'\cos\varphi" + S\ (d^2-\rho'^2)]\ /\ [(d^2+S^2)\,(d^2+\rho'^2)]$

$(\sin\theta"\sin\varphi) = [2d\ \rho'/(d^2+\rho'^2)]\ \sin\varphi" = [2d\ \rho'\ (d^2+S^2)\ \sin\varphi"\ ]\ /\ [(d^2+S^2)\,(d^2+\rho'^2)]$

$(1 - \cos\theta_c\ \cos\theta" + \sin\theta_c\ \sin\theta"\cos\varphi"\ ) = 2d^2\ r_1^2\ /\ [(d^2+S^2)\,(d^2+\rho'^2)]$

$R^2 = (2Adr_1)^2 / [(d^2+S^2)(d^2+\rho'^2)]$

We continue now, recalling from above,

$s = \sqrt{(x'-c)^2 + y'^2}$
$x' = -(Aa^2/R^2)(\cos\theta_c \sin\theta''\cos\varphi'' + \sin\theta_c \cos\theta'')$
$y' = (Aa^2/R^2)(\sin\theta''\sin\varphi'')$

Notice that

$(Aa^2/R^2) = (Aa^2/(2Adr_1)^2)[(d^2+S^2)(d^2+\rho'^2)] = (Aa^2/4A^2d^2r_1^2)[(d^2+S^2)(d^2+\rho'^2)]$

$= (2dA/4Ad^2r_1^2)[(d^2+S^2)(d^2+\rho'^2)] = (1/2dr_1^2)[(d^2+S^2)(d^2+\rho'^2)]$

$= (1/2d)(1/r_1^2)[(d^2+S^2)(d^2+\rho'^2)]$

Thus we have

$x' = -(Aa^2/R^2)(\cos\theta_c \sin\theta''\cos\varphi'' + \sin\theta_c \cos\theta'')$

$= -(1/2d)(1/r_1^2)[(d^2+S^2)(d^2+\rho'^2)]) * 2d [(d^2- S^2)\rho' \cos\varphi'' + S(d^2- \rho'^2)] / [(d^2+S^2)(d^2+\rho'^2)]$

$= -(1/2d)(1/r_1^2)\ 2d\ [(d^2- S^2)\rho' \cos\varphi'' + S(d^2- \rho'^2)]$

$= -(1/r_1^2)\ [(d^2- S^2)\rho' \cos\varphi'' + S(d^2- \rho'^2)]$ // not too bad

Next we do

$y' = (Aa^2/R^2)(\sin\theta''\sin\varphi'')$

$= (1/2d)(1/r_1^2)[(d^2+S^2)(d^2+\rho'^2)] * [2d\rho'(d^2+S^2)\sin\varphi''] / [(d^2+S^2)(d^2+\rho'^2)]$

$= (1/2d)(1/r_1^2)[2d\rho'(d^2+S^2)\sin\varphi'']$

$= (1/r_1^2)\rho'(d^2+S^2)\sin\varphi''$

Now I need something nice for c. But we have something, which is this (param facts Pic 2)

$b = B(d^2+ S^2) / (B^2- S^2)$
$c = S(B^2+ d^2) / (B^2- S^2)$

and above

$$x' = -(1/r_1^2)\ [\rho'(d^2 - S^2)\cos\varphi'' + S(d^2 - \rho'^2)]$$
$$y' = (1/r_1^2)\ [\rho'(d^2+S^2)\sin\varphi'']$$

So now we can insert things to obtain:

$$s = \sqrt{(x'-c)^2 + y'^2} \qquad s^2 = (x'-c)^2 + y'^2$$

Let's remember now what we are trying to do ! We want this:

$$\sigma_{\mathbf{iris}}(\mathbf{r}') = -\ (8A^3/\pi^2 a R^3)\cos^3(\theta''/2)/\sqrt{b^2-s^2} = -(2A/R)^3\,(1/\pi^2 a)\cos^3(\theta''/2)/\sqrt{b^2-s^2}$$

and we have something nice for R, and we have been "working on" the s^2 nightmare. We can simplify the above by inserting R and $\cos(\theta''/2)$ as follows

$$\cos(\theta''/2) = d/\sqrt{\rho'^2 + d^2}$$

$$R = 2Adr_1/\sqrt{(d^2+S^2)(d^2+\rho'^2)} \qquad \Rightarrow \qquad (2A/R) = \sqrt{(d^2+S^2)(d^2+\rho'^2)}/(dr_1)$$

So we have

$$(2A/R)^3 = (d^2+S^2)^{3/2}(d^2+\rho'^2)^{3/2}/d^3 r_1^3$$

Then we get

$$\sigma_{\mathbf{iris}}(\mathbf{r}') = -\ (d^2+S^2)^{3/2}(d^2+\rho'^2)^{3/2}(1/\pi^2 a\, d^3 r_1^3)\cos^3(\theta''/2)/\sqrt{b^2-s^2}$$

$$= -\ (d^2+S^2)^{3/2}(d^2+\rho'^2)^{3/2}(1/\pi^2 a\, d^3 r_1^3)\, d^3(\rho'^2+d^2)^{-3/2}/\sqrt{b^2-s^2}$$

$$= -\ (d^2+S^2)^{3/2}\ (1/\pi^2 a\, r_1^3)/\sqrt{b^2-s^2}$$

We are now down to this endgame situation:

$$\sigma_{\mathbf{iris}}(\mathbf{r}') = -\ (d^2+S^2)^{3/2}\ (1/\pi^2 a\, r_1^3)/\sqrt{b^2-s^2} \qquad // \ q_{\mathbf{hole}} = \sqrt{S^2+d^2}/a$$

$$s^2 = (x'-c)^2 + y'^2$$

$$x' = -(1/r_1^2)\ [\rho'(d^2 - S^2)\cos\varphi'' + S(d^2 - \rho'^2)]$$
$$y' = (1/r_1^2)\ [\rho'(d^2+S^2)\sin\varphi'']$$

$$b = B\ (d^2+S^2)/(B^2-S^2)$$
$$c = S\ (B^2+d^2)/(B^2-S^2)$$

Admittedly, this is for the situation in which

$q_{hole} = (\sqrt{S^2 + d^2} / a)$

If we wanted the result for some arbitrary charge q_1 in the hole, we would divide by q_{hole} to get our unit charge result, then multiply by q_1 . Then we get

$$\sigma_{iris}(\mathbf{r}') = - q_1 (a/\sqrt{S^2 + d^2}) (d^2+S^2)^{3/2} (1/ \pi^2 a r_1^3) /\sqrt{b^2-s^2}$$

$$= - (q_1/ \pi^2 r_1^3) (d^2+S^2) / \sqrt{b^2-s^2} \qquad \text{// dim = } Q/L^2 \text{ correct}$$

Before going on, we can take a look at Attempt #2's interpretation of Smythe's claim for this result

$$\sigma_{Smythe} = - (q_1/\pi^2 r_1^2)\sqrt{B^2-S^2} / \sqrt{\rho'^2 - B^2}$$

where I have doubled Smythe's result to get sum of charges on top and bottom of iris since this is the way I started things on the disk with Jackson. These results are enticingly similar. Smythe's result has the φ" dependence all in the r_1^2 factor.

To show our two results agree, I would have to show that

$$(d^2+S^2) / \sqrt{b^2-s^2} = r_1 \sqrt{B^2-S^2} / \sqrt{\rho'^2 - B^2}$$

$$(d^2+S^2)^2 / (b^2-s^2) = r_1^2 (B^2- S^2)/ (\rho'^2- B^2)$$

$$(b^2-s^2) = (d^2+S^2)^2 (\rho'^2- B^2) / [r_1^2 (B^2- S^2)]$$

with the cosφ" appearing only in that r_1 factor.

4. Does the z' = d condition have anything to say?

Although we did not need to use it, we have this condition floating around in our σ analysis, you see it at the start of this doc,

$$z' = d = (a^2/R^2)(Z\text{^}+A) \qquad = (Aa^2/R^2) (\sin\theta_c \sin\theta"\cos\varphi" - \cos\theta_c \cos\theta" + 1)$$

which I write as

$$(Aa^2/R^2) (\sin\theta_c \sin\theta"\cos\varphi" - \cos\theta_c \cos\theta" + 1) = d \qquad \text{LHS = RHS}$$

Let's just evaluate LHS and see what happens. We will use our result from above. I just treat this like another angular factor:

$$(\sin\theta_c \sin\theta"\cos\varphi" - \cos\theta_c \cos\theta" + 1) =$$

$$=2dS/(d^2+S^2)* 2d\ \rho'/(d^2+\rho'^2) * \cos\varphi'' - (d^2-S^2)/(d^2+S^2)* (d^2-\rho'^2)/(d^2+\rho'^2) + 1$$

$$= 1 + [\ 2dS\ 2d\ \rho' * \cos\varphi'' - (d^2-S^2)\ (d^2-\rho'^2)] / [(d^2+S^2)\ (d^2+\rho'^2)]$$

$$= \{(d^2+S^2)\ (d^2+\rho'^2) + 2dS\ 2d\ \rho' * \cos\varphi''\ - (d^2-S^2)\ (d^2-\rho'^2)\ \} / [(d^2+S^2)\ (d^2+\rho'^2)]$$

The two factors without the angle are the same as I dealt with above, where I found,

$$(d^2+S^2)\ (d^2+\rho'^2) - (d^2-S^2)\ (d^2-\rho'^2) = 2d^2(S^2+\rho'^2)$$

So we then have

$$(\sin\theta_c \sin\theta''\cos\varphi'' - \cos\theta_c \cos\theta'' + 1) = \{2d^2(S^2+\rho'^2) + 2dS\ 2d\ \rho' * \cos\varphi''\ \} / [(d^2+S^2)\ (d^2+\rho'^2)]$$

$$= \{2d^2(S^2+\rho'^2) + 4d^2S\ \rho' * \cos\varphi''\ \} / [(d^2+S^2)\ (d^2+\rho'^2)]$$

$$= 2d^2\{S^2+\rho'^2 + 2S\rho'\ \cos\varphi''\ \} / [(d^2+S^2)\ (d^2+\rho'^2)]$$

$$= 2d^2r_1^2 / [(d^2+S^2)\ (d^2+\rho'^2)]$$

So we are now left with this LHS (we expect this to come out being d)

$$\text{LHS} = (Aa^2/R^2)\ (\sin\theta_c \sin\theta''\cos\varphi'' - \cos\theta_c \cos\theta'' + 1)$$

$$R^2 = 4A^2d^2r_1^2 / [(d^2+S^2)\ (d^2+\rho'^2)]$$

$$\Rightarrow \quad \text{LHS} = (Aa^2/4A^2d^2r_1^2)\ [(d^2+S^2)\ (d^2+\rho'^2)]\ * 2d^2r_1^2 / [(d^2+S^2)\ (d^2+\rho'^2)]$$

$$= (Aa^2/4A^2d^2r_1^2) * 2d^2r_1^2$$

$$= (a^2/2A) = d$$

So this was just an algebra check and it came out right.

5. Continue processing the $b^2 - s^2$ nightmare.

Above I arrived at this situation

$$s^2 = (x'-c)^2 + y'^2 \qquad = (c-x')^2 + y'^2$$

$$x' = -(1/r_1^2)\ [\rho'\ (d^2-S^2)\ \cos\varphi'' + S\ (d^2-\rho'^2)]$$
$$y' = (1/r_1^2)\ [\rho'\ (d^2+S^2)\ \sin\varphi''\ ]$$

$b = B\ (d^2 + S^2) / (B^2 - S^2)$
$c = S\ (B^2 + d^2) / (B^2 - S^2)$

If the two trig terms had the same factor, at least sinφ" would go away and I would at least end up with s^2 being a function only of φ" which I physically know must be true (and of course it is true in Smythe's answer). So I guess I will try to hunt down a possible sign error here! // I cannot find a sign error, so just proceed anyway:

$$s^2 = \{c - x'\}^2 + y'^2$$

$$= \{ S\ (B^2 + d^2) / (B^2 - S^2) + (1/r_1^2)\ [\rho' (d^2 - S^2) \cos\varphi'' + S (d^2 - \rho'^2)] \}^2 + (1/r_1^4)\ \rho'^2 (d^2 + S^2)^2 \sin^2\varphi''$$

$$= (1/r_1^4) [\ \{ r_1^2 S\ (B^2 + d^2) / (B^2 - S^2) + [\rho' (d^2 - S^2) \cos\varphi'' + S (d^2 - \rho'^2)] \}^2 + \rho'^2 (d^2 + S^2)^2 \sin^2\varphi'']$$

$$= (1/[r_1^4 (B^2 - S^2)^2])\ *$$

$$[\ \{ r_1^2 S\ (B^2 + d^2) + (B^2 - S^2) [\rho' (d^2 - S^2) \cos\varphi'' + S (d^2 - \rho'^2)] \}^2 + (B^2 - S^2)^2\ \rho'^2 (d^2 + S^2)^2 \sin^2\varphi'']$$

$$= (1/[r_1^4 (B^2 - S^2)^2]) * (f_1^2 + f_2)$$

This evaluation is a job for Maple. We also have

$$r_1^2 = S^2 + \rho'^2 + 2S\rho' \cos\varphi'' \qquad \cos\varphi'' = (r_1^2 - S^2 - \rho'^2)/2S\rho'$$

$$b^2 - s^2 = b^2 - (1/[r_1^4 (B^2 - S^2)^2] * f$$

$$= (b^2 [r_1^4 (B^2 - S^2)^2] - f) /[r_1^4 (B^2 - S^2)^2]$$

But

$$b^2 (B^2 - S^2)^2 = B^2\ (d^2 + S^2)^2 / (B^2 - S^2)^2\ * (B^2 - S^2)^2 = B^2\ (d^2 + S^2)^2$$

so we have

$$b^2 - s^2 = (B^2 (d^2 + S^2)^2 r_1^4 - f) / /[r_1^4 (B^2 - S^2)^2] = \text{num/den}$$

I entered this in Maple and got these results: ("inversion Smythe algebra.mws"

```
restart;
f1 := r1^2*S*(B^2+d^2) + (B^2-S^2)*(rho*(d^2-S^2)*cosphi + S*(d^2-rho^2));
```

$$f1 := r1^2\,S\,(B^2+d^2)+(B^2-S^2)\,(\rho\,(d^2-S^2)\,cosphi+S\,(d^2-\rho^2))$$

```
f2 := (B^2-S^2)^2 * rho^2 * (d^2 + S^2)^2 * (1 - (cosphi)^2);
```

$$f2 := (B^2-S^2)^2\,\rho^2\,(d^2+S^2)^2\,(1-cosphi^2)$$

```
f := f1^2 + f2;
```

$$f := (r1^2\,S\,(B^2+d^2)+(B^2-S^2)\,(\rho\,(d^2-S^2)\,cosphi+S\,(d^2-\rho^2)))^2+(B^2-S^2)^2\,\rho^2\,(d^2+S^2)^2\,(1-cosphi^2)$$

```
cosphi := (r1^2 - S^2 - rho^2)/(2*S*rho);
```

$$cosphi := \frac{1}{2}\,\frac{r1^2-S^2-\rho^2}{S\,\rho}$$

```
num := B^2*(d^2+S^2)^2 * r1^4 - f;
```

$$m := B^2\,(d^2+S^2)^2\,r1^4-\left(r1^2\,S\,(B^2+d^2)+(B^2-S^2)\left(\frac{1}{2}\,\frac{(d^2-S^2)\,(r1^2-S^2-\rho^2)}{S}+S\,(d^2-\rho^2)\right)\right)^2$$

$$-(B^2-S^2)^2\,\rho^2\,(d^2+S^2)^2\left(1-\frac{1}{4}\,\frac{(r1^2-S^2-\rho^2)^2}{S^2\,\rho^2}\right)$$

```
g := factor(%);
```

$$g := r1^2\,(B-\rho)\,(B+\rho)\,(d^2+S^2)^2\,(S-B)\,(S+B)$$

```
final := g / (r1^4 * (B^2-S^2)^2);
```

$$final := \frac{(B-\rho)\,(B+\rho)\,(d^2+S^2)^2\,(S-B)\,(S+B)}{r1^2\,(B^2-S^2)^2}$$

```
simplify(%);
```

$$\frac{(B-\rho)\,(B+\rho)\,(d^2+S^2)^2}{(-B^2+S^2)\,r1^2}$$

.

This, my result is this:

$b^2 - s^2 = (B^2 - \rho'^2)\,(d^2 + S^2)^2 / [\,r_1^2 (S^2 - B^2)]$

$= [\,(d^2 + S^2)^2 / r_1^2\,]\,(\rho'^2 - B^2) / (B^2 - S^2)$

which is looking very good!!!

6. Install $b^2 - s^2$ result and compare with Smythe

My result from above was this

$\sigma_{iris}(\mathbf{r}') = -\,q_1\,(a/\sqrt{S^2+d^2})\,(d^2+S^2)^{3/2}\,(1/\pi^2 a\,r_1^3)/\sqrt{b^2-s^2}$

$= -\,(q_1/\pi^2\,r_1^3)\,(d^2+S^2)/\sqrt{b^2-s^2}$ // dim = Q/L^2 correct

and I now install my new finding

$$b^2 - s^2 = [\,(d^2 + S^2)^2/\, r_1^2\,]\,(\rho'^2 - B^2)\,/\,(B^2 - S^2)$$

$$\sqrt{b^2-s^2} = [\,(d^2 + S^2)/\, r_1\,]\,\sqrt{\rho'^2 - B^2}\,/\,\sqrt{B^2-S^2}$$

$$1/\sqrt{b^2-s^2} = r_1\,\sqrt{B^2-S^2}\,/\,[(d^2 + S^2)\,\sqrt{\rho'^2 - B^2}\,]$$

which then gives my final result

$$\sigma_{\mathbf{iris}}(\mathbf{r}') = -\,(q_1/\,\pi^2\, r_1^3)\;(d^2+S^2)\,/\,\sqrt{b^2-s^2}$$

$$= -\,(q_1/\,\pi^2\, r_1^3)\;(d^2+S^2)\;r_1\,\sqrt{B^2-S^2}\,/\,[(d^2 + S^2)\,\sqrt{\rho'^2 - B^2}\,]$$

$$= -\,(q_1/\,\pi^2\, r_1^2)\;\sqrt{B^2-S^2}\,/\,\sqrt{\rho'^2 - B^2}$$

The Smythe result quoting from above is this:

$$\sigma_{\mathbf{Smythe}} = -\,(\,q_1/\pi^2 r_1^2)\sqrt{B^2-S^2}\,/\,\sqrt{\rho'^2 - B^2}$$

FINALLY I have derived Smythe's result. Take note: 7:30 PM June 14, 2010. At some point I will report just how LONG I have been working on getting his result! His result is correct for the charge on one of the surfaces as in his book, and I have doubled his result to match my Jackson.

For the first time ever I believe Smythe's result and I have validated the inversion/rotation/inversion method of deriving it from the σ on a charged isolated disk. *Now that I know* it is right, I can try to understand how to do it without all the messy algebra that only Maple could figure out!

7. Do the second inversion of the potential and process it.

Now let's try to write the corresponding iris potential.

(1) Starting Point. The potential after rotation and before second inversion is given by this (from how to rotate bowl)

$$\Phi_{\mathbf{post}}(\mathbf{r}'') = \Phi(\mathbf{r}) = g(\mathbf{r}|\xi) = -\,(2/\pi Q)\,\sin^{-1}\,[\;2b\,/\,(\,\sqrt{(s-b)^2 + (w-d)^2}\;+\sqrt{(s+b)^2 + (w-d)^2}\;)\,] + 1/Q$$

$$\sigma_{\mathbf{post}}(\mathbf{r}'') = \sigma(\mathbf{r}) = -\,(a^2/\pi^2 Q^3)\,/\sqrt{b^2-s^2}$$

$$Q = \sqrt{x''^2 + y''^2 + (z''-A)^2 + 2A[\sin\theta_c x'' - \cos\theta_c(z''-A)] + A^2}$$

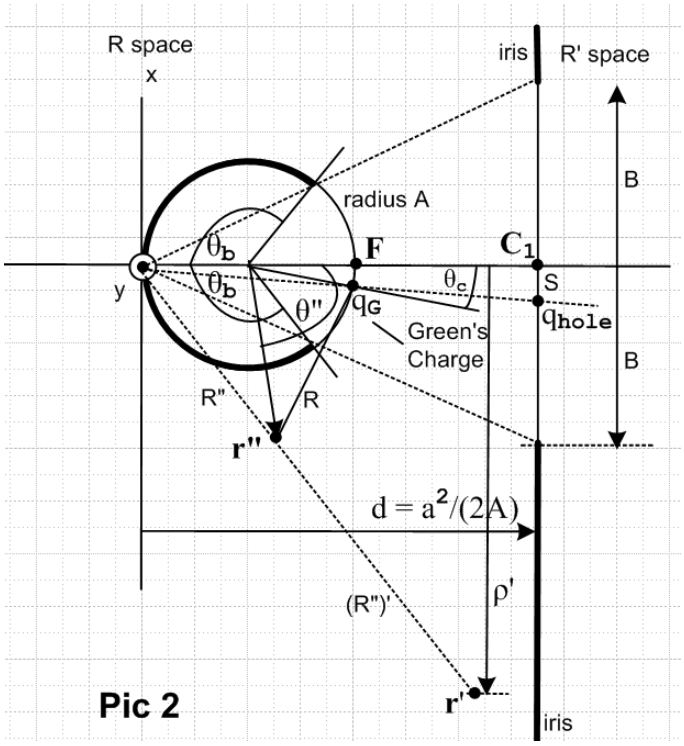
$$s = \sqrt{(u-c)^2 + v^2}$$

$$u = (a^2/Q^2)[\,-\cos\theta_c x'' - \sin\theta_c(z''-A)]$$

$v = (a^2/Q^2)y''$

$w = (a^2/Q^2)(\sin\theta_c x'' - \cos\theta_c(z''-A) + A)$

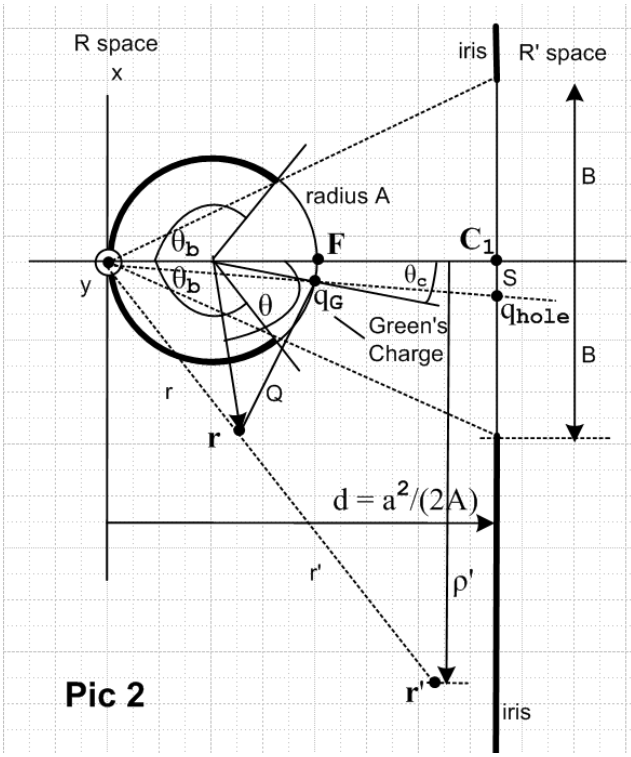
where **r**" = (x",y",z") are Cartesians based at the inversion origin. We are now evaluating at a general point **r**", so it is no longer on the sphere, and the corresponding point **r**' is not on the iris. Here is a picture:



Pic 2

(2) Major Change In Notation.

I now want to "do" the second inversion for the potential using the canonical **r** and **r**' vector notation. So I will now replace **r**" by **r**, adjusting all our expressions and creating a new picture.



Pic 2

$$\Phi_{post}(\mathbf{r}) = -(2/\pi Q)\sin^{-1}\left[\ 2b / \left(\sqrt{(s-b)^2 + (w-d)^2}\ + \sqrt{(s+b)^2 + (w-d)^2}\ \right)\right] + 1/Q$$

$$\sigma_{post}(\mathbf{r}) = -\ (a^2/\pi^2 Q^3)/\sqrt{b^2 - s^2}$$

$$Q = \sqrt{x^2 + y^2 + (z-A)^2 + 2A[\sin\theta_c x - \cos\theta_c (z-A)] + A^2}$$

$$s = \sqrt{(u-c)^2 + v^2}$$

$$u = (a^2/Q^2)[\ -\cos\theta_c x - \sin\theta_c (z-A)]$$

$$v = (a^2/Q^2) y$$

$$w = (a^2/Q^2)(\ \sin\theta_c x - \cos\theta_c (z-A) + A)$$

where **r** = (x,y,z) are inversion origin Cartesians. So we now carry on with the second inversion!

(3) Jackson's potential inversion rule is this (I select one of the rules) (look at Pic 2)

$$\varphi'(\mathbf{r'}) = (a/r')\ \varphi(\mathbf{r})$$

which we apply in our situation to get

$$\Phi_{iris}(\mathbf{r'}) = (a/r')\ \Phi_{post}(\mathbf{r})$$

so we get

$$\Phi_{iris}(\mathbf{r'}) = (a/r')\ \{ -(2/\pi Q)\sin^{-1}\left[\ 2b / \left(\sqrt{(s-b)^2 + (w-d)^2}\ + \sqrt{(s+b)^2 + (w-d)^2}\ \right)\right] + 1/Q\ \}$$

$$Q = \sqrt{x^2 + y^2 + (z-A)^2 + 2A[\sin\theta_c x - \cos\theta_c (z-A)] + A^2}$$

$$Q^2 = (x^2 + y^2 + (z-A)^2 + 2A[\sin\theta_c x - \cos\theta_c (z-A)] + A^2$$

$$s = \sqrt{(u-c)^2 + v^2}$$
$$u = (a^2/Q^2)[\ -\cos\theta_c x - \sin\theta_c(z-A)]$$
$$v = (a^2/Q^2)y$$
$$w = (a^2/Q^2)(\ \sin\theta_c x - \cos\theta_c(z-A) + A)$$

(4) Express in iris coordinates. But naturally we want to see our result in terms of the coordinates of point **r'**, not **r**. We know that

$$\mathbf{r} = (a/r')^2\mathbf{r}'$$

$$x = (a/r')^2 x'$$
$$y = (a/r')^2 y'$$
$$z = (a/r')^2 z'$$

so we can install these things to get

$$\Phi_{iris}(\mathbf{r}') = (a/r')\ \{-\ (2/\pi Q)\sin^{-1}[\ 2b / (\ \sqrt{(s-b)^2 + (w-d)^2}\ + \sqrt{(s+b)^2 + (w-d)^2}\ )\ ] + 1/Q\ \}$$

$$Q^2 = ((a/r')^4\ x'^2 + (a/r')^4\ y'^2 + ((a/r')^2 z' - A)^2 + 2A[\sin\theta_c(a/r')^2 x' - \cos\theta_c((a/r')^2 z' - A)] + A^2$$

$$s = \sqrt{(u-c)^2 + v^2}$$
$$u = (a^2/Q^2)[\ -\cos\theta_c(a/r')^2 x' - \sin\theta_c((a/r')^2 z' - A)]$$
$$v = (a^2/Q^2)\ (a/r')^2 y'$$
$$w = (a^2/Q^2)(\ \sin\theta_c(a/r')^2 x' - \cos\theta_c[(a/r')^2 z' - A] + A)$$

(5) Convert to iris center coordinates. But we would rather have our Cartesian coordinates centered on the iris center, so define

$$x' = X_i \qquad r'^2 = X_i^2 + Y_i^2 + (Z_i+d)^2 = \rho'^2 + (Z_i+d)^2$$
$$y' = Y_i$$
$$z' = (Z_i+d) \qquad \text{// If } Z_i = 0 \text{ for on iris plane, then } z' = d$$

and we write it again Sam,

$$\Phi_{iris}(\mathbf{r}') = (a/r')\ \{\ -(2/\pi Q)\sin^{-1}[\ 2b / (\ \sqrt{(s-b)^2 + (w-d)^2}\ + \sqrt{(s+b)^2 + (w-d)^2}\ )\ ] + 1/Q\ \}$$

$$Q^2 = ((a/r')^4\ X_i^{\ 2} + (a/r')^4\ Y_i^2 + ((a/r')^2(Z_i+d) - A)^2 + 2A[\sin\theta_c(a/r')^2\ X_i - \cos\theta_c((a/r')^2(Z_i+d) - A)] + A^2$$

$$s = \sqrt{(u-c)^2 + v^2}$$
$$u = (a^2/Q^2)[\ -\cos\theta_c(a/r')^2\ X_i - \sin\theta_c((a/r')^2(Z_i+d) - A)]$$
$$v = (a^2/Q^2)\ (a/r')^2\ Y_i$$
$$w = (a^2/Q^2)(\ \sin\theta_c(a/r')^2\ X_i - \cos\theta_c[(a/r')^2(Z_i+d) - A] + A)$$

Although it is pretty messy, this gives our iris potential in terms of iris coordinates!

(6) Convert to cylindrical iris coordinates with the strange history names $\mathbf{r}' = (\rho', Z_i, \varphi)$. This used to be φ'', notice that both $\mathbf{r}$ and $\mathbf{r}'$ have the same φ. then we have'

$$X_i = \rho'\cos\varphi$$
$$Y_i = \rho'\sin\varphi$$
$$Z_i = Z_i$$

So write it again Sam #2:

$$\Phi_{iris}(\mathbf{r}') = (a/r')\{ -(2/\pi Q)\sin^{-1}[\ 2b / (\sqrt{(s-b)^2 + (w-d)^2} + \sqrt{(s+b)^2 + (w-d)^2}\ )\ ] + 1/Q \}$$

$$Q^2 = ((a/r')^4\ \rho'^2 + (((a/r')^2(Z_i+d) - A)^2 + 2A[\sin\theta_c (a/r')^2\ \rho'\cos\varphi - \cos\theta_c((a/r')^2(Z_i+d) - A)] + A^2$$

$$s = \sqrt{(u-c)^2 + v^2} \qquad r'^2 = \rho'^2 + (Z_i+d)^2$$
$$u = (a^2/Q^2)[\ -\cos\theta_c (a/r')^2\ \rho'\cos\varphi - \sin\theta_c (\ (a/r')^2(Z_i+d) - A)]$$
$$v = (a^2/Q^2)\ (a/r')^2\ \rho'\sin\varphi$$
$$w = (a^2/Q^2)\{\ \sin\theta_c (a/r')^2\ \rho'\cos\varphi - \cos\theta_c[\ (a/r')^2(Z_i+d) - A] + A\}$$

(7) It will probably help to <u>move some factors around</u>:

$$Q^2 = (a/r')^4\ (\quad \rho'^2 + (\ Z_i+d-A(r'/a)^2\ )^2 + 2A\ (r'/a)^2\ \{\ \sin\theta_c\rho'\cos\varphi - \cos\theta_c\ [\ Z_i+d\ -A(r'/a)^2]\ \} + (r'/a)^4A^2\ \ )$$
$$= (a/r')^4\ \ T^2 \qquad \text{// define } T^2$$

$$u = (a/r')^2(a^2/Q^2)[\ -\cos\theta_c\ \rho'\cos\varphi - \sin\theta_c (\ (Z_i+d) - A(r'/a)^2)]$$
$$v = (a/r')^2\ (a^2/Q^2)\ \rho'\sin\varphi$$
$$w = (a/r')^2\ (a^2/Q^2)(\ \sin\theta_c\ \rho'\cos\varphi - \cos\theta_c[\ (Z_i+d) - A(r'/a)^2] + A(r'/a)^2)$$

We notice now that

$$(a/r')^2(a^2/Q^2) = (r'^2/T^2) \qquad Q = (a/r')^2\ \ T$$

so things get a little better, where I finally factor the Q out on the Φ line,

$$\Phi_{iris}(\mathbf{r}') = (a/r')\ (1/Q)\{ -(2/\pi)\sin^{-1}[\ 2b / (\sqrt{(s-b)^2 + (w-d)^2} + \sqrt{(s+b)^2 + (w-d)^2}\ )\ ] + 1 \}$$
$$= (r'/a)\ (1/T)\{ -(2/\pi)\sin^{-1}[\ 2b / (\sqrt{(s-b)^2 + (w-d)^2} + \sqrt{(s+b)^2 + (w-d)^2}\ )\ ] + 1 \}$$

$$T^2 = (\quad \rho'^2 + (\ Z_i+d-A(r'/a)^2\ )^2 + 2A(r'/a)^2\ \{\ \sin\theta_c\rho'\cos\varphi - \cos\theta_c\ [\ Z_i+d\ -A(r'/a)^2]\ \} + (r'/a)^4A^2\ \ )$$
$$u = (r'^2/T^2)\ [\ -\cos\theta_c\ \rho'\cos\varphi - \sin\theta_c\{\ (Z_i+d) - A(r'/a)^2\}]$$
$$v = (r'^2/T^2)\ \rho'\sin\varphi \qquad Q = (a/r')^2\ \ T$$
$$w = (r'^2/T^2))(\ \sin\theta_c\ \rho'\cos\varphi - \cos\theta_c[\ (Z_i+d) - A(r'/a)^2] + A(r'/a)^2)$$

(8) Correct for charge size. Now recall this from our σ work above,

"Admittedly, this is for the situation in which

$q_{hole} = (\sqrt{S^2 + d^2} / a)$

If we wanted the result for some arbitrary charge q_1 in the hole, we would divide by q_{hole} to get our unit charge result, then multiply by q_1 ."

Doing this process, we get

$\Phi_{iris}(\mathbf{r'})$

$= q_1 (a/\sqrt{S^2 + d^2}\ )(r'/a)\ (1/T)\{ - (2/\pi) \sin^{-1} [\ 2b / (\ \sqrt{(s-b)^2 + (w-d)^2}\ + \sqrt{(s+b)^2 + (w-d)^2}\ )\ ]+1\ \}$

$= q_1 (r'/\sqrt{S^2 + d^2}\ )\ (1/T)\{ - (2/\pi) \sin^{-1} [\ 2b / (\ \sqrt{(s-b)^2 + (w-d)^2}\ + \sqrt{(s+b)^2 + (w-d)^2}\ )\ ]+1\ \}$

(9) Combine the two terms inside {...}: write

$\{\} = (1 - (2/\pi) \sin^{-1}X\ = (2/\pi)\ [\ \pi/2 - \sin^{-1}X\ ]\ = (2/\pi) \cos^{-1}X$

Proof: let $\sin^{-1}X = \theta$, some angle. then

$[\ \pi/2 - \sin^{-1}X\ ] = [\ \pi/2 - \theta]$

$\cos[\ \pi/2 - \theta]\ = \sin\theta \quad \Rightarrow \quad [\ \pi/2 - \theta]\ = \cos^{-1}[\ \sin\theta]\ = \cos^{-1}[\ X]$

$\Rightarrow\ [\ \pi/2 - \sin^{-1}X\ ] = \cos^{-1}X$

Then we can write

$\Phi_{iris}(\mathbf{r'}) = q_1\ (r'/\sqrt{S^2 + d^2}\ )\ (1/T)(2/\pi)\cos^{-1}[\ 2b / (\ \sqrt{(s-b)^2 + (w-d)^2}\ + \sqrt{(s+b)^2 + (w-d)^2}\ )\ ]$

$T^2 = (\quad \rho'^2 + (\ Z_i + d - A(r'/a)^2\ )^2 + 2A(r'/a)^2\ \{\ \sin\theta_c \rho' \cos\varphi - \cos\theta_c\ [\ Z_i + d - A(r'/a)^2]\ \}\ + (r'/a)^4 A^2\ )$

$u = (r'^2/T^2)\ [\ -\cos\theta_c\ \rho'\cos\varphi - \sin\theta_c\{\ (Z_i + d) - A(r'/a)^2\}]$

$v = (r'^2/T^2)\ \rho'\sin\varphi \qquad\qquad Q = (a/r')^2\ T$

$w = (r'^2/T^2))(\ \sin\theta_c\ \rho'\cos\varphi - \cos\theta_c[\ (Z_i + d) - A(r'/a)^2] + A(r'/a)^2)$

(10) Get rid of A. At this point let's replace

$A(r'/a)^2 = (r'^2/2d)$

and write it again

$\Phi_{iris}(\mathbf{r'}) = (2q_1/\pi T)\ (r'/\sqrt{S^2 + d^2}\ )\ \cos^{-1}[\ 2b / (\ \sqrt{(s-b)^2 + (w-d)^2}\ + \sqrt{(s+b)^2 + (w-d)^2}\ )\ ]$

$$T^2 = \left(\rho'^2 + (Z_i+d-(r'^2/2d))^2 + 2(r'^2/2d) \{ \sin\theta_c \rho'\cos\varphi - \cos\theta_c [Z_i+d -(r'^2/2d)] \} + (r'^2/2d)^2 \right)$$

$s = \sqrt{(u-c)^2 + v^2}$ $\qquad r'^2 = \rho'^2 + (Z_i+d)^2$

$u = (r'^2/T^2) [-\cos\theta_c \rho'\cos\varphi - \sin\theta_c \{ (Z_i+d) - (r'^2/2d)\}]$

$v = (r'^2/T^2) \rho'\sin\varphi$ $\qquad Q = (a/r')^2 \; T$

$w = (r'^2/T^2))(\sin\theta_c \rho'\cos\varphi - \cos\theta_c[(Z_i+d) - (r'^2/2d)] +(r'^2/2d))$

$r'^2 = \rho'^2 + (Z_i+d)^2$

$\sin\theta_c = 2dS/(d^2+S^2)$

$\cos\theta_c = (d^2- S^2)/(d^2+S^2)$

$b = B \; (d^2+ S^2) / (B^2- S^2)$

$c = S \; (B^2+ d^2) / (B^2- S^2)$

and now we are completely rid of both a and A, and everything is in terms of iris-center cylindrical coordinates. It took an amazing amount of work to arrive at this point! Still, the following facts are not clear:

- How do you show potential is independent of d ?
- How do you show potential is symmetric under $Z_i \rightarrow -Z_i$?

I think we are getting closer to some Maple work now. Surely there are many errors in what I have done above, could be in any of those 10 steps!

Here is one hopeful fact: (scratch paper) $\qquad r'^2 = \rho'^2 + (Z_i+d)^2$

$[Z_i+d -(r'^2/2d)] = (d^2- Z_i^2- \rho'^2)/(2d)$ // rechecked on scratch

so here finally is something symmetric under $Z_i \rightarrow -Z_i$. Continue here next time! It is 10 PM 6.19.10.

(11) Process the T factor: Let's now insert the θ_c trig functions and the above "hopeful fact"

$T^2 =$

$$\left(\rho'^2 + (d^2- Z_i^2- \rho'^2)^2/(2d)^2 \right.$$

$$\left. + 2(r'^2/2d) \{ 2dS/(d^2+S^2) * \rho'\cos\varphi - (d^2- S^2)/(d^2+S^2) (d^2- Z_i^2- \rho'^2)/(2d) \} + (r'^2/2d)^2 \right)$$

Mult through by $(2d)^2$ [I now group the last term with the first two]

$$(2d)^2 T^2 = \left((2d)^2\rho'^2 + (d^2- Z_i^2- \rho'^2)^2 + r'^4 \right.$$

$$\left. +2r'^2 \{ (2d)^2S/(d^2+S^2) * \rho'\cos\varphi - (d^2- S^2)/(d^2+S^2) * (d^2- Z_i^2- \rho'^2)\} \right)$$

Now mult through by (d^2+S^2) // $r'^2 = \rho'^2 + (Z_i+d)^2$

$$(2d)^2\,(d^2+S^2)\,T^2 = (\;\; (d^2+S^2)\,[\,(2d)^2\rho'^2 + (d^2 - Z_i^2 - \rho'^2)^2 + r'^4\;\,]$$

$$+ 2r'^2\,\{\;(2d)^2 S\rho'\cos\varphi - (d^2 - S^2)(d^2 - Z_i^2 - \rho'^2)\}\;\;)$$

We throw this into Maple as follows (**"Smythe38phiwork.mws"**)

```
T1a := (d^2+S^2)*( 4*d^2*rho^2 + (d^2-Z^2-rho^2)^2 + r^4);
```

$$T1a := (d^2+S^2)\left(4\,d^2\,\rho^2 + (d^2 - Z^2 - \rho^2)^2 + r^4\right)$$

```
r := (rho^2 + (Z+d)^2)^(1/2);
```

$$r := \sqrt{\rho^2 + (Z+d)^2}$$

```
T1b := 4*d^2 *S*rho*cos(phi) - (d^2 - S^2)*(d^2-Z^2-rho^2);
```

$$T1b := 4\,d^2\,S\,\rho\,\cos(\phi) - (d^2 - S^2)\,(d^2 - Z^2 - \rho^2)$$

```
T1 := T1a + 2*r^2*T1b:simplify(%):factor(%);
```

$$4\,(\rho^2 + Z^2 + 2\,Z\,d + d^2)\,d^2\,(S^2 + Z^2 + 2\,S\,\rho\,\cos(\phi) + \rho^2)$$

where the last line is $(2d)^2\,(d^2+S^2)\,T^2$. Therefore,

$$(2d)^2\,(d^2+S^2)\,T^2 = (2d)^2\,[\,\rho^2 + (Z_i+d)^2\,]\,(\,\rho^2 + S^2 + Z_i^2 + 2S\rho\cos\varphi\,)$$

$$= (2d)^2\,r'^2\,(\,\rho^2 + S^2 + Z_i^2 + 2S\rho\cos\varphi\,)$$

Now the (...) quantity is our Φ analogue of the Smythe r_1 factor, just that now we are off the plane so we have this extra Z^2 term. So let's redefine

$$r_1^2 = (\,\rho^2 + S^2 + Z_i^2 + 2S\rho\cos\varphi\,) \qquad r'^2 = \rho'^2 + (Z_i+d)^2$$

where r' is the distance from inversion origin to point **r'** , as shown in the above figure. Then we have

$$(2d)^2\,(d^2+S^2)\,T^2 = (2d)^2\,r'^2\,r_1^2$$

$$(d^2+S^2)\,T^2 = r'^2\,r_1^2$$

$$\sqrt{S^2+d^2}\;T = r'\,r_1$$

$$1/T = \sqrt{S^2+d^2}\,/\,(r'r_1)$$

and this is the kind of result I like to see!!! Our "factor out front" in Φ is then

$$(2q_1/\pi T)\,(r'/\sqrt{S^2+d^2}\;) \;=\; (2q_1/\pi)\,(r'/\sqrt{S^2+d^2}\;)(\,\sqrt{S^2+d^2}\,/\,(r'r_1)\;) \;=\; (2q_1/\pi r_1)$$

So we now have

$$\Phi_{iris}(\mathbf{r}') = (2q_1/\pi r_1)\cos^{-1}\left[\ 2b \,/\, \left(\sqrt{(s-b)^2 + (w-d)^2}\ + \sqrt{(s+b)^2 + (w-d)^2}\ \right)\right]$$

where $r_1^2 = (\rho'^2 + S^2 + Z_i^2 + 2S\rho\cos\varphi)$

So two comments! (1) the leading factor is symmetric in Z_i (2) the leading factor contains no d's . So things are looking very promising.

(12) Process the s variable. We have $s = \sqrt{(u-c)^2 + v^2}$ where [we include w for later use]

$$u = (r'^2/T^2)\left[-\cos\theta_c\,\rho'\cos\varphi - \sin\theta_c\{(Z_i+d) - (r'^2/2d)\}\right]$$
$$v = (r'^2/T^2)\,\rho'\sin\varphi$$
$$w = (r'^2/T^2)\{\ \sin\theta_c\,\rho'\cos\varphi - \cos\theta_c[(Z_i+d) - (r'^2/2d)] + (r'^2/2d)\ \}$$

$$\sin\theta_c = 2dS/(d^2+S^2) \qquad \cos\theta_c = (d^2 - S^2)/(d^2+S^2) \qquad 1/T = \sqrt{S^2+ d^2}\ /\ (r'r_1)$$

First, notice that

$$(r'^2/T^2) = r'^2\,(S^2 + d^2)/(r'^2 r_1^2) = (S^2 + d^2)r_1^{-2}$$

Now throw in this and the trig functions to get

$$u = (S^2 + d^2)r_1^{-2}\left[-(d^2 - S^2)/(d^2+S^2) * \rho'\cos\varphi - 2dS/(d^2+S^2) * \{(Z_i+d) - (r'^2/2d)\}\right]$$
$$v = (S^2 + d^2)r_1^{-2}\,\rho'\sin\varphi$$
$$w = (S^2 + d^2)r_1^{-2}\ \{\ 2dS/(d^2+S^2)*\rho'\cos\varphi - (d^2 - S^2)/(d^2+S^2)\,[(Z_i+d) - (r'^2/2d)] + (r'^2/2d)\ \}$$

$$u = r_1^{-2}\left[-(d^2 - S^2)\rho'\cos\varphi - S\{2d(Z_i+d) - r'^2\}\ \right]$$
$$v = r_1^{-2}\,\rho'\sin\varphi\,(S^2 + d^2)$$
$$w = r_1^{-2}\ \{\ 2dS\rho'\cos\varphi - (d^2 - S^2)\,[(Z_i+d) - (r'^2/2d)] + (S^2 + d^2)\,(r'^2/2d)\ \}$$

Multiply through by $(2d)r_1^2$

$$(2d)\,r_1^2 u = 2d\left[-(d^2 - S^2)\rho'\cos\varphi - S\{2d(Z_i+d) - r'^2\}\ \right] \equiv u_1$$
$$(2d)\,r_1^2 v = 2d\,\rho'\sin\varphi\,(S^2 + d^2) \equiv v_1$$
$$(2d)\,r_1^2 w = \{\ (2d)^2\,S\rho'\cos\varphi - (d^2 - S^2)\,[2d(Z_i+d) - r'^2] + (S^2 + d^2)\,r'^2\ \} \equiv w_1$$

Note in passing: I should have made use of this fact, but did not do so:

$$\{2d(Z_i+d) - r'^2\} = 2d(Z_i+d) - \rho'^2 - (Z_i+d)^2 = (Z_i+d)(2d - Z_i - d) - \rho'^2 = (Z_i+d)(d-Z_i) - \rho'^2$$

$$= (d + Z_i)(d-Z_i) - \rho'^2 = d^2 - Z_i^2 - \rho'^2$$

Now we are ready to look at s.

$$s = \sqrt{(u-c)^2 + v^2} \qquad s^2 = (u-c)^2 + v^2$$

$$[(2d)\, r_1^2]^2 s^2 = ([(2d)\, r_1^2]u - [(2d)\, r_1^2]c)^2 + ([(2d)\, r_1^2]v)^2$$

$$= (u_1 - [(2d)\, r_1^2]c)^2 + v_1^2$$

Now install

$$c = S\ (B^2 + d^2) / (B^2 - S^2)$$

to get

$$[(2d)\, r_1^2]^2 s^2 = (u_1 - [(2d)\, r_1^2]c)^2 + v_1^2$$

$$= (\quad u_1 - [(2d)\, r_1^2]\ S\ (B^2 + d^2) / (B^2 - S^2)\quad)^2 + v_1^2$$

$$[\ (B^2 - S^2)\ (2d)\ r_1^2\ ]^2 s^2 = \equiv s1s$$

$$= [\quad (B^2 - S^2)u_1 - [(2d)\, r_1^2]\ S\ (B^2 + d^2)\quad]^2 + [\ (B^2 - S^2)\ v_1\quad]^2 \qquad // L^{12}$$

The result for s1s is a big mess, so let's check for errors. My Maple entry is good. Well, I took this mess and I replaced cosφ and sinφ with the r_1 expression, then it is "not so bad"

$$s1s = [\ (B^2 - S^2)\ (2d)\ r_1^2\ ]^2 s^2 =$$

$$4 d^2 (d^2 + S^2)^2 (-B^2 r1^2 Z^2 - S^4 Z^2 - B^2 r1^2 \rho^2 + S^2 r1^2 \rho^2 + S^2 r1^2 Z^2 - B^4 Z^2 - B^2 S^2 r1^2 + B^4 r1^2 + 2 B^2 S^2 Z^2 + B^2 r1^4)$$

$$= (2d)^2\ (d^2 + S^2)^2\ h$$

where I use h as a symbol for this set of 10 terms, and Maple now knows P10. The good news is that the d-dependence is entirely in the first two factors, which is promising. We have

$$[\ (B^2 - S^2)\ (2d)\ r_1^2\ ]^2 s^2 = (2d)^2\ (d^2 + S^2)^2\ h$$

$$[\ (B^2 - S^2)\ r_1^2\ ]^2 s^2 = (d^2 + S^2)^2\ h$$

What does this tell us

(13) Using the ellipse theorem and get final result for Φ. At this point, I find that s is not a perfect square (since the above 10 terms do not factor), and I really did not expect it would come out that way. I now call upon the famous ellipse theorem from ellipses.doc which says

$$(1/2)\ (\ \sqrt{(x-a)^2 + y^2}\quad +\ \sqrt{(x+a)^2 + y^2}\ )^2 = \ (x^2 + y^2 + a^2) + \sqrt{(x^2 - y^2 - a^2)^2 + 4x^2y^2}$$

$$(\ \sqrt{(s-b)^2 + (w-d)^2} + \sqrt{(s+b)^2 + (w-d)^2}\ )^2 = \ 2(s^2 + (w-d)^2 + b^2) + 2\sqrt{(s^2 - (w-d)^2 - b^2)^2 + 4s^2(w-d)^2}$$

Now recall that so far we have

$$\Phi_{\mathbf{iris}}(\mathbf{r}') = (2q_1/\pi r_1) \cos^{-1} [\ 2b / (\sqrt{(s-b)^2 + (w-d)^2}\ + \sqrt{(s+b)^2 + (w-d)^2}\ )]$$

So the denom of the $\cos^{-1}$ arg can be replaced by the ellipse theorem above (with a sqrt), and s only appears squared on the RHS. So let's write this as

$$dac2 = 2(s^2 + (w-d)^2 + b^2) + 2\sqrt{(s^2 - (w-d)^2 - b^2)^2 + 4s^2(w-d)^2}$$

where dac2 means "denominator of arc cosine squared". Our ingredients at this point are these:

$$[\ (B^2 - S^2)\ r_1^2\ ]^2 s^2 = (d^2 + S^2)^2 h$$

$$(2d)\ r_1^2 w = \{\ (2d)^2\ S\rho'\cos\varphi - (d^2 - S^2)\ [\ 2d(Z_i + d) - r'^2] + (S^2 + d^2)\ r'^2\ \} \qquad \equiv w_1$$

$$b = B\ (d^2 + S^2) / (B^2 - S^2)$$

As our first step, multiply dac2 by $(B^2 - S^2)^2 r_1^4 \equiv f^2$:

$$f^2\ dac2/2 = (f^2 s^2 + f^2(w-d)^2 + f^2 b^2) + \sqrt{(f^2 s^2 - f^2(w-d)^2 - f^2 b^2)^2 + 4f^2 s^2 f^2 (w-d)^2}$$

Next, multiply both sides by $(2d)^2\ r_1^4 \equiv g^2$

$$f^2\ g^2\ dac2/2 = (f^2\ g^2 s^2 + f^2(gw-gd)^2 + g^2 f^2 b^2)$$

$$+ \sqrt{(f^2\ g^2 s^2 - f^2(gw-gd)^2 - f^2\ g^2 b^2)^2 + 4f^2\ g^2 s^2 f^2 (gw-gd)^2}$$

Next, multiply both sides by $(B^2 - S^2)^2 \equiv k^2$

$$f^2\ g^2\ k^2\ dac2/2 = (f^2\ g^2\ k^2 s^2 + f^2\ k^2\ (gw-gd)^2 + g^2 f^2\ k^2 b^2)$$

$$+ \sqrt{(f^2\ g^2\ k^2 s^2 - f^2\ k^2\ (gw-gd)^2 - f^2\ g^2\ k^2 b^2)^2 + 4f^2\ g^2\ k^2 s^2 f^2\ k^2\ (gw-gd)^2}$$

Now we make these identifications

$$f^2 s^2 = (B^2 - S^2)^2 r_1^4 \quad s^2 = (d^2 + S^2)^2\ h$$
$$gw = (2d)\ r_1^2 = w_1$$
$$kb = (B^2 - S^2)\ B\ (d^2 + S^2) / (B^2 - S^2) = B(d^2 + S^2)$$

We then have

$$f^2\ g^2\ k^2\ dac2/2 = (g^2\ k^2\ (d^2 + S^2)^2\ h + f^2\ k^2\ (w_1 - gd)^2 + g^2 f^2\ B^2 (d^2 + S^2)^2)$$

$$+\sqrt{(g^2\,k^2(d^2+S^2)^2\,h - f^2\,k^2\,(w_1\text{-}gd)^2 - f^2\,g^2\,B^2(d^2+S^2)^2)^2 + 4g^2\,k^2(d^2+S^2)^2\,h\,f^2\,k^2\,(w_1\text{-}gd)^2}$$

where

$f = (B^2 - S^2)r_1^2$

$g = (2d)\,r_1^2$

$k = (B^2 - S^2)$

w_1 = as already in Maple

h = as already in Maple

My Big Hope is that the argument of the radical will be a perfect square, but not totally necessary. Call it

argrad =

$$[g^2\,k^2(d^2+S^2)^2\,h - f^2\,k^2\,(w_1\text{-}gd)^2 - f^2\,g^2\,B^2(d^2+S^2)^2]^2 + 4g^2\,k^4(d^2+S^2)^2\,h\,f^2\,(w_1 - gd)^2$$

To my great amazement, Maple gives me this relatively simple result

argrad = $16\,rl^{12}\,d^4\,(d^2+S^2)^4\,(B-S)^6\,(B+S)^6\,(-2\,B\,\rho+\rho^2+Z^2+B^2)\,(2\,B\,\rho+\rho^2+Z^2+B^2)$

It is very close to a perfect square! Let's write it out by hand

argrad = $4^2 r_1^{12} d^4\,(d^2+S^2)^4\,(B^2 - S^2)^6\;[\,(\rho'-B)^2+Z_i^2]\,[\,(\rho'+B)^2+Z_i^2]$ // checked

I just did a backscan looking for sign errors in linear ρ' terms. The u,v,w is always -.+.+. I don't see an error! So I guess the above is correct and we don't have a perfect square. Let's just forge ahead then. We had,

$$f^2\,g^2\,k^2\,dac2/2 = (g^2\,k^2\,(d^2+S^2)^2\,h + f^2\,k^2\,(w_1\text{-}gd)^2 + g^2f^2\,B^2(d^2+S^2)^2) + \sqrt{\text{argrad}}$$

For the LHS we know,

$$f^2\,g^2\,k^2\,dac2/2 = (B^2 - S^2)^2 r_1^4\,(2d)^2\,r_1^4\,(B^2 - S^2)^2\,dac2/2 = (B^2 - S^2)^4 r_1^8 (2d)^2\,dac2/2$$

leaving us with

$$(B^2 - S^2)^4 r_1^8 (2d)^2\,dac2/2 = (g^2\,k^2\,(d^2+S^2)^2\,h + f^2\,k^2\,(w_1\text{-}gd)^2 + g^2f^2\,B^2(d^2+S^2)^2) + \sqrt{\text{argrad}}$$

Let's go now and compute the non-radical stuff on the right, which I have not done yet.

$$j \equiv (g^2\,k^2\,(d^2+S^2)^2\,h + f^2\,k^2\,(w_1\text{-}gd)^2 + g^2f^2\,B^2(d^2+S^2)^2)$$

Maple gives us a pretty good result for this:

j = $4\,d^2\,rl^6\,(S-B)^2\,(S+B)^2\,(S^2Z^2 - B^2S^2 + S^2\rho^2 - B^2\rho^2 + 2\,rl^2\,B^2 + B^4 - B^2Z^2)\,(d^2+S^2)^2$

And I will call the 7 term factor by the name m, so

$$m := S^2 Z^2 - B^2 S^2 + S^2 \rho^2 - B^2 \rho^2 + 2 rl^2 B^2 + B^4 - B^2 Z^2$$

which I point out is symmetric in Z and has no d's appearing anywhere! So we have

$$j = (2d)^2 r_1^6 (S^2-B^2)^2 (d^2+S^2)^2 m$$

Installing this "j" thing above we get

$$(B^2- S^2)^4 r_1^8 (2d)^2 \, dac2/2 = j + \sqrt{argrad}$$

$$= (2d)^2 r_1^6 (S^2-B^2)^2 (d^2+S^2)^2 m + \sqrt{argrad}$$

Now back to our result

$$argrad = 4^2 r_1^{12} d^4 (d^2+S^2)^4 (B^2- S^2)^6 \, [(\rho'-B)^2+Z_i^2] \, [(\rho'+B)^2+Z_i^2] \qquad L$$

$$\sqrt{argrad} = 4 r_1^6 d^2 (d^2+S^2)^2 (B^2- S^2)^3 \sqrt{[(\rho'-B)^2+Z_i^2] \, [(\rho'+B)^2+Z_i^2]}$$

Then we have

$$(B^2- S^2)^4 r_1^8 (2d)^2 \, dac2/2 = (2d)^2 r_1^6 (S^2-B^2)^2 (d^2+S^2)^2 m$$

$$+ 4 r_1^6 d^2 (d^2+S^2)^2 (B^2- S^2)^3 \sqrt{[(\rho'-B)^2+Z_i^2] \, [(\rho'+B)^2+Z_i^2]}$$

and we can cancel a few things (each term is L^{20})

$$(B^2- S^2)^2 r_1^2 \, dac2/2 = (d^2+S^2)^2 m$$

$$+ (d^2+S^2)^2 (B^2- S^2) \sqrt{[(\rho'-B)^2+Z_i^2] \, [(\rho'+B)^2+Z_i^2]}$$

Then we have

$$dac2 = 2 (B^2- S^2)^{-2} r_1^{-2} (d^2+S^2)^2 \, [m + (B^2- S^2) \sqrt{[(\rho'-B)^2+Z_i^2] \, [(\rho'+B)^2+Z_i^2]} \,]$$

and we now take the square root

$$dac = \sqrt{2} (B^2- S^2)^{-1} r_1^{-1} (d^2+S^2) \sqrt{m + (B^2- S^2) \sqrt{[(\rho'-B)^2+Z_i^2] \, [(\rho'+B)^2+Z_i^2]}}$$

which is at least manageable. Our $\cos^{-1}$ argument is 2b/dac with

$$b = B (d^2+ S^2) / (B^2- S^2)$$

so

$$\text{arg of } \cos^{-1} = (\sqrt{2}\, Br_1)\, 1/\sqrt{m + (B^2 - S^2)\sqrt{[(\rho'-B)^2+Z_i^2]\,[(\rho'+B)^2+Z_i^2]}}$$

and our potential is now

$$\Phi_{iris}(\mathbf{r}') = (2q_1/\pi r_1)\cos^{-1}\left[\ 2b \,/\, \left(\sqrt{(s-b)^2 + (w-d)^2} + \sqrt{(s+b)^2 + (w-d)^2}\ \right)\right]$$

$$= (2q_1/\pi r_1)\cos^{-1}[\ \text{arg of } \cos^{-1}\ ]$$

$$= (2q_1/\pi r_1)\cos^{-1}\left[(\sqrt{2}\ Br_1/\sqrt{m + (B^2 - S^2)\sqrt{[(\rho'-B)^2+Z_i^2]\,[(\rho'+B)^2+Z_i^2]}}\ \right]$$

with

$$m := S^2 Z^2 - B^2 S^2 + S^2 \rho^2 - B^2 \rho^2 + 2\, r1^2\, B^2 + B^4 - B^2 Z^2$$

$$r_1^2 = (\rho'^2 + S^2 + Z_i^2 + 2S\rho'\cos\varphi)$$

The good news is : (1) it is brief enough that I can actually write it down. (2) it is a function of Z^2 and this has the looked-for symmetry for $Z \rightarrow -Z$; (3) there are no appearances of symbol "d" in the result. (4) arc cos argument is dimensionless. All these facts are good evidence for a lack of major errors in the algebra.

I am able to write m this way

$$m = (\rho'^2+Z_i^2)(S^2-B^2) + B^2(B^2 - S^2 + 2r_1^2)$$ // rechecked, this is correct.

so here is my final result

$$\Phi_{iris}(\mathbf{r}') = (2q_1/\pi r_1)\cos^{-1}[\ (Br_1\sqrt{2}\ ) *$$

$$1/\sqrt{(\rho'^2+Z_i^2)(S^2-B^2) + B^2(B^2 - S^2 + 2r_1^2) + (B^2 - S^2)\sqrt{[(\rho'-B)^2+Z_i^2]\,[(\rho'+B)^2+Z_i^2]}}\ ]$$

where $r_1^2 = (\rho'^2 + S^2 + Z_i^2 + 2S\rho'\cos\varphi)$ => $2r_1 dr_1 = 2Z_i dZ_i$

(14) Show that Φ = 0 on the iris metal. Obviously we need to run some checks on this result! The first check is to see if we get 0 on the iris! So set $Z_i = 0$ and see what happens:

$$\Phi_{iris}(\mathbf{r}'; Z_i = 0) = (2q_1/\pi r_1)\cos^{-1}[\ (Br_1\sqrt{2}\ ) *$$

$$1/\sqrt{(\rho'^2)(S^2-B^2) + B^2(B^2 - S^2 + 2r_1^2) + (B^2 - S^2)\sqrt{[(\rho'-B)^2]\,[(\rho'+B)^2]}}\ ]$$

where $r_1^2 = (\rho'^2 + S^2 + 2S\rho'\cos\varphi)$

Let's examine the argument of the radical here:

$$\text{argrad} = (\rho'^2)(S^2-B^2) + B^2(B^2 - S^2 + 2r_1^2) + (B^2 - S^2)\sqrt{[(\rho'-B)^2]\,[(\rho'+B)^2]} = m + n$$

Clearly r_1 is finite, so in order to get $\Phi = 0$ we are going to need arg of $\cos^{-1} = 1$ so $\cos^{-1} = 0$. So we are going to need this:

$$(Br_1\sqrt{2}\,)/\sqrt{\text{argrad}} = 1$$

$$B^2 r_1^2\, 2 = \text{argrad}$$

So we are going to want to see this

$$(\rho'^2)(S^2-B^2) + B^2(B^2- S^2 + 2r_1^2) + (B^2- S^2)\sqrt{[\,(\rho'-B)^2]\,[\,(\rho'+B)^2]} = B^2 r_1^2 2$$

We might imagine that

$$\sqrt{[\,(\rho'-B)^2]\,[\,(\rho'+B)^2]} = \pm\,(\rho'-B)(\rho'+B) = \pm\,(\rho'^2-B^2)$$

so we might then have

$$(\rho'^2)(S^2-B^2) + B^2(B^2- S^2 + 2r_1^2) \pm (B^2- S^2)\,(\rho'^2-B^2) = B^2 r_1^2 2$$

$$(\rho'^2)(S^2-B^2) + B^2(B^2- S^2 + 2r_1^2) \pm \rho'^2(B^2-S^2) \mp B^2(B^2-S^2) = B^2 r_1^2 2$$

Now if we go with the upper sign, we get some cancellation

$$(\rho'^2)(S^2-B^2) + B^2(B^2- S^2 + 2r_1^2) + \rho'^2(B^2-S^2) - B^2(B^2-S^2) = B^2 r_1^2 2$$

$$B^2(\,2r_1^2) = B^2 r_1^2 2$$

and there you have it!!! This says that $(Br_1\sqrt{2}\,)/\sqrt{\text{argrad}} = 1$ and then $\cos^{-1}(1) = 0$ and $\Phi = 0$. I just knew it was going to come out this way, just as it does for the disk.

So go back to the sign issue again. We have

$\sqrt{(\rho'-B)^2} = +(\rho'-B) > 0$ when $\rho' > B$ which means we are "on the iris"

If we are in the hole, then we have

$\sqrt{(\rho'-B)^2} = +(B-\rho') > 0 \qquad = -(\rho'-B)$

and this is the lower sign situation. So in the hole we do NOT get $\Phi = 0$.

What DO we get in the hole? Well, from above we had this situation

$$\text{argrad} = (\rho'^2)(S^2-B^2) + B^2(B^2- S^2 + 2r_1^2) \pm \rho'^2(B^2-S^2) \mp B^2(B^2-S^2)$$

and if we are in the hole, we want to take the lower signs, so we have

$$\text{argrad} = (\rho'^2)(S^2-B^2) + B^2(B^2- S^2 + 2r_1^2) - \rho'^2(B^2-S^2)+B^2(B^2-S^2)$$

$$= - 2\rho'^2(B^2-S^2) + B^2 [\, 2(B^2-S^2) + 2r_1^2]$$

$$= - 2\rho'^2(B^2-S^2) + B^2 [\, 2(B^2-S^2) + 2(\rho'^2 + S^2 + 2S\rho'\cos\varphi)]$$

$$= - 2\rho'^2(-S^2) + B^2 [\, 2(B^2-S^2) + 2(S^2 + 2S\rho'\cos\varphi)]$$

$$= - 2\rho'^2(-S^2) + B^2 [\, 2(B^2) + 2(2S\rho'\cos\varphi)]$$

$$= 2\rho'^2S^2 + 2B^2 [\, B^2 + 2S\rho'\cos\varphi]$$

So we could insert this into our $\Phi_{\mathbf{iris}}$ and get our result for potential in the hole

$$\Phi_{\mathbf{iris}}(\mathbf{r}'; Z_i = 0) = (2q_1/\pi r_1) \cos^{-1}[\, (Br_1\sqrt{2}\,) / \sqrt{2\rho'^2S^2 + 2B^2 [\, B^2 + 2S\rho'\cos\varphi]}\,]$$

As a special case, we could ask about the potential in the hole at $\rho' = S$, which is the circle which contains our point charge. We get

$$\text{argrad} = 2\rho'^2S^2 + 2B^2 [\, B^2 + 2S\rho'\cos\varphi] \qquad r_1^2 = (S^2 + S^2 + 2S^2\cos\varphi)$$
$$= 2S^4 + 2B^2 [\, B^2 + 2S^2\cos\varphi]$$

Now as a special case of this special case, let's look at $\varphi = \pi$, which is the location of the point charge. We then have

$$\text{argrad} = 2S^4 + 2B^2 [\, B^2 - 2S^2] = 2 [\, S^4 + B^4 - 2B^2S^2] = 2(B^2-S^2)^2 \qquad r_1 \rightarrow 0$$

We then are faced with

$$\cos^{-1}((Br_1\sqrt{2}\,) / [\sqrt{2}\,(B^2-S^2)\,]) = \cos^{-1}[Br_1/(B^2-S^2)] = \cos^{-1}(0) = \pi/2$$

The potential at this point is then

$$\Phi = (2q_1/\pi r_1)\, \pi/2 = q/r_1$$

as expected, and it of course blows up, despite that fact that the iris is an infinite object.

(15) Can we perhaps use the ellipse theorem a second time ?

$$(1/2)\left(\sqrt{(x-a)^2+y^2}\ +\ \sqrt{(x+a)^2+y^2}\ \right)^2 = \ (x^2+y^2+a^2)+\sqrt{(x^2-y^2-a^2)^2+4x^2y^2}$$

which says

$$(1/\sqrt{2})\left(\sqrt{(x-a)^2+y^2}\ +\ \sqrt{(x+a)^2+y^2}\ \right) = \sqrt{(x^2+y^2+a^2)+\sqrt{(x^2-y^2-a^2)^2+4x^2y^2}}$$

We have this "quantity of interest"

$$\sqrt{(\rho'^2+Z_i^2)(S^2-B^2)+B^2(B^2-S^2+2r_1^2)+\ (B^2-S^2)\sqrt{[\ (\rho'-B)^2+Z_i^2]\ [\ (\rho'+B)^2+Z_i^2]}}$$

Can we identify this somehow with the RHS of the previous line, for some choice of the three arguments a, x and y ? Proportional would be OK. Consider

$$[\ (\rho'-B)^2+Z_i^2]\ [\ (\rho'+B)^2+Z_i^2] = (\rho'-B)^2(\rho'+B)^2+Z_i^2(\rho'+B)^2+Z_i^2(\rho'-B)^2+Z_i^4$$

$$= (\rho'^2-B^2)^2+2Z_i^2(\rho'^2+B^2)+Z_i^4$$

$$= \rho'^4+B^4-2\rho'^2B^2+2Z_i^2(\rho'^2+B^2)+Z_i^4$$

$$= \rho'^4+B^4\ +Z_i^4-2\rho'^2B^2+2Z_i^2(\rho'^2+B^2)$$

$$(x^2-y^2-a^2)^2+4x^2y^2 \qquad = x^4+y^4+a^4+2x^2y^2-2a^2x^2-2a^2y^2\ = f(x,y)$$

It sure seems close, but the signs are wrong. Secondly, there is no φ dependence here, but there is inside the r_1 factor in the rest of the radical which has to be $(x^2+y^2+a^2)$. So I am afraid I cannot make this "fit".

Try one more time:

$$(x^2+y^2+a^2) \ = \ [(\rho'^2+Z_i^2)(S^2-B^2)+B^2(B^2-S^2+2r_1^2)]/\ (B^2-S^2)$$

$$= \ [-(\rho'^2+Z_i^2)+B^2+2r_1^2(B^2/\ (B^2-S^2)\ ]$$

$$(x^2+y^2+a^2)^2 = \ [-(\rho'^2+Z_i^2)+B^2+2r_1^2(B^2/\ (B^2-S^2)\ ]^2$$

$$(x^2-y^2-a^2)^2+4x^2y^2 \ = \ [(\rho'^2-B^2)^2+2Z_i^2(\rho'^2+B^2)+Z_i^4]$$

Subtract to get

$$2x^2(y^2+a^2)+4x^2y^2 = [-(\rho'^2+Z_i^2)+B^2+2r_1^2(B^2/\ (B^2-S^2)\ ]^2 - \ [(\rho'^2-B^2)^2+2Z_i^2(\rho'^2+B^2)+Z_i^4]$$

Even if it works, it is a complete mess, so forget it!

(16) Compute the charge density?

$$\Phi_{iris}(\mathbf{r}') = (2q_1/\pi r_1)\cos^{-1}[\,(Br_1\sqrt{2}\,)\,*$$

$$1/\sqrt{(\rho'^2+Z_i^2)(S^2-B^2) + B^2(B^2-S^2+2r_1^2) + (B^2-S^2)\sqrt{[\,(\rho'-B)^2+Z_i^2]\,[\,(\rho'+B)^2+Z_i^2]}\,}\,]$$

where $r_1^2 = (\,\rho'^2 + S^2 + Z_i^2 + 2S\rho\cos\varphi\,)$ $\Rightarrow$ $2r_1dr_1 = 2Z_idZ_i$

We *expect to find* that [$E = 4\pi\sigma$ in Jackson units, p 12] [put in Smythe for **one side**]

$$\sigma = -(\,q_{hole}/2\pi^2r_1^2)\sqrt{B^2-S^2}\,/\,\sqrt{\rho'^2-B^2}$$

$$\partial_{Zi}\,\Phi_{iris}(\mathbf{r}')|^{Zi=0} = -4\pi\sigma_{Smythe} = -4\pi\,[\,-(\,q_1/2\pi^2r_1^2)\sqrt{B^2-S^2}\,/\,\sqrt{\rho'^2-B^2}\,]$$

$$= (\,2q_1/\pi r_1^2)\sqrt{B^2-S^2}\,/\,\sqrt{\rho'^2-B^2}\,]$$

One little thing to note is this: the "outside factor" has Z_i dependence through r_1. But, the contribution from this term to the derivative will be 0 because $\cos^{-1} = 0$ on the iris, as we already showed. So only dependence inside the $\cos^{-1}$ will contribute. We might abbreviate

$$u = (Br_1\sqrt{2}\,)/\sqrt{m+n}$$

$$\Phi_{iris}(\mathbf{r}') = (2q_1/\pi r_1)\cos^{-1}(u)$$

$$\partial_{Zi}\,\Phi_{iris}(\mathbf{r}')|^{Zi=0} = (2q_1/\pi r_1)\,[\,-1/\sqrt{1-u^2}\,\,]\,\partial_{Zi}u$$

where

$$u^2 = 2B^2r_1^2/(m+n)$$

We know that at $Z_i = 0$ we will have $m+n = (Br_1\sqrt{2}\,)^2$ as we already showed. So we have

$$u^2 = 2B^2r_1^2/\,2B^2r_1^2 = 1$$

so $\sqrt{1-u^2} = 0$ in this case, and we must get cancellation from something in $\partial_{Zi}u$. It turns out that we will want to keep the first order in Z_i term in both $\partial_{Zi}u$ up top, and in $\sqrt{1-u^2}$ down below. Here then is the complete calculation which I hate to admit took me more than a whole day to get correct.

(a) Review of input information and strategy:

$$u = (Br_1\sqrt{2}\,)/\sqrt{m+n} \qquad 1 = (Br_{1o}\sqrt{2}\,)/\sqrt{m_0+n_0}$$

$$u^2 = 2B^2r_1^2/(m+n) \qquad 1 = 2B^2r_{1o}^2/(m_0+n_0)$$

$r_1^2 = (\rho'^2 + S^2 + Z_i^2 + 2S\rho\cos\varphi)$ $\qquad r_{1o}^2 = (\rho'^2 + S^2 + 2S\rho\cos\varphi)$

$m = (\rho'^2+Z_i^2)(S^2-B^2) + B^2(B^2- S^2 + 2r_1^2)$ $\qquad m_0 = - (B^2- S^2)(\rho'^2-B^2) + 2B^2r_{1o}^2$

$n = (B^2- S^2) \sqrt{[(\rho'-B)^2+Z_i^2] [(\rho'+B)^2+Z_i^2]}$ $\qquad n_0 = + (B^2- S^2)(\rho'^2-B^2)$ $\qquad \rho' > B$

$\Phi_{iris}(\mathbf{r}') = (2q_1/\pi r_1) \cos^{-1}(u)$ $\qquad$ Note that $m_0 + n_0 = 2B^2r_{1o}^2$

$\partial_{Zi} \Phi_{iris}(\mathbf{r}')|^{Zi=0} \approx (2q_1/\pi r_1) [-1/\sqrt{1-u^2}] \partial_{Zi}u$

$1-u^2 = [(m+n) - 2B^2r_1^2]/(m+n)$

$\partial_{Zi}r_1 = (Z_i/r_1)$ $\qquad \partial_{Zi}r_1^2 = 2r_1\partial_{Zi}r_1 = 2r_1 (Z_i/r_1) = 2Z_i$

$- 2u\partial_{Zi}u = 2B^2 [r_1^2\partial_{Zi}(m+n) - (m+n) 2Z_i] / (m+n)^2$

Our interest will be in the ratio $\partial_{Zi}u/ \sqrt{1-u^2}$ which is a 0/0 situation for $Z_i = 0$, so we shall keep the lowest terms for both in Z_i as we do our calculation.

(b) Compute facts about n.

As a start, let's expand n for small Z_i:

$(a^2 + b^2)^{1/2} = a + (1/2)a^{-1}b^2 + ...$ $\qquad$ assume $\rho' > B$

$\sqrt{[(\rho'-B)^2+Z_i^2]} \approx (\rho'-B) + (1/2) Z_i^2/ (\rho'-B) = (\rho'-B)[1+ (1/2)Z_i^2/ (\rho'-B)^2]$

$\sqrt{[(\rho'+B)^2+Z_i^2]} \approx (\rho'+B) + (1/2) Z_i^2/ (\rho'+B) = (\rho'+B)[1+ (1/2)Z_i^2/ (\rho'+B)^2]$

$\sqrt{[(\rho'-B)^2+Z_i^2] [(\rho'+B)^2+Z_i^2]} \approx (\rho'^2- B^2) [1 + (1/2)Z_i^2 \{ 1/(\rho'-B)^2 + 1/(\rho'+B)^2 \}]$

$= (\rho'^2- B^2) [1 + Z_i^2 (\rho'^2+B^2)/ (\rho'^2- B^2)^2]$

So we then have, for small Z_i

$n = (B^2- S^2) \sqrt{[(\rho'-B)^2+Z_i^2] [(\rho'+B)^2+Z_i^2]} \approx (B^2- S^2) (\rho'^2- B^2) [1 + Z_i^2 (\rho'^2+B^2)/ (\rho'^2- B^2)^2]$

$\partial_{Zi}n = (B^2- S^2) (\rho'^2- B^2) 2Z_i(\rho'^2+B^2)/ (\rho'^2- B^2)^2 = (B^2- S^2) 2Z_i(\rho'^2+B^2) / (\rho'^2- B^2)$

$n = n_0 + (B^2- S^2) Z_i^2 (\rho'^2+B^2)/ (\rho'^2- B^2)$

(c) Compute facts about m.

Next we have

$$m = (\rho'^2+Z_i^2)(S^2-B^2) + B^2(B^2- S^2 + 2r_1^2)$$

$$\partial_{z_i} m = 2Z_i(S^2-B^2) + 2B^2\,\partial_{z_i} r_1^2 \;=\; 2Z_i(S^2-B^2) + 4B^2 Z_i = 2Z_i\,[\,(S^2-B^2) + 2B^2] \;= 2Z_i(S^2+B^2)$$

$$m = m_0 + Z_i^2(S^2-B^2) \;+ 2B^2Z_i^2 \quad = m_0 + Z_i^2(S^2+B^2)$$

(d) Compute facts about m+n

$$m+n = \{m_0 + Z_i^2(S^2+B^2)\} \;+ \{n_0 + (B^2- S^2)\, Z_i^2(\rho'^2+B^2) / (\rho'^2- B^2)\}$$

$$= m_0 + n_0 + Z_i^2\,\{\,(S^2+B^2) + \;(B^2- S^2)\,(\rho'^2+B^2) / (\rho'^2- B^2)\,\}$$

$$= m_0 + n_0 + Z_i^2\,\{\,(S^2+B^2)\,(\rho'^2- B^2) + \;(B^2- S^2)\,(\rho'^2+B^2)\,\}/(\rho'^2- B^2)$$

$$= m_0 + n_0 + Z_i^2\,\{\,2B^2(\rho'^2-S^2)\,\}/(\rho'^2- B^2)$$

$$= m_0 + n_0 + 2\,B^2Z_i^2\,(\rho'^2-S^2)/(\rho'^2- B^2)$$

$$= \;2B^2r_{1o}^2 + 2\,B^2Z_i^2\,(\rho'^2-S^2)/(\rho'^2- B^2)$$

Then we know that

$$\partial_{z_i}(m+n) = \;4\,B^2Z_i\,(\rho'^2-S^2)/(\rho'^2- B^2)$$

(e) Compute $\partial_{z_i}\mathbf{u}$.

We start with this from above,

$$u^2 = 2B^2r_1^2/(m+n)$$

$$2u\partial_{z_i}u = 2B^2\,[\,(m+n)\,\partial_{z_i}(r_1^2) - r_1^2\,\partial_{z_i}(m+n)\,]\,/(m+n)^2$$

$$\partial_{z_i}u = B^2\,[\,(m+n)\,\partial_{z_i}(r_1^2) - r_1^2\,\partial_{z_i}(m+n)\,]\,/(m+n)^2$$

where we just set u = 1 on the LHS. We also replace m+n in the denom with $2B^2r_{1o}^2$ because its correction would be at least order Z^3 which we are going to ignore.

$$\partial_{z_i}u = B^2\,[\,(m+n)\,\partial_{z_i}(r_1^2) - r_1^2\,\partial_{z_i}(m+n)\,]\,/(2B^2r_{1o}^2)^2$$

Now, our two derivatives are linear in Z, so we shall only keep linear terms in this numerator difference and ignore higher order terms. Therefore, we can write the above as

$$\partial_{Zi}u = B^2 [(2B^2 r_{1o}^2)\, \partial_{Zi}(r_1^2) - r_{1o}^2\, \partial_{Zi}(m+n)\,] /(2B^2 r_{1o}^2)^2$$

$$= [(2B^2)\, \partial_{Zi}(r_1^2) - \partial_{Zi}(m+n)\,] /(2Br_{1o})^2$$

We now install our derivatives computed above, namely

$$\partial_{Zi}(m+n) = 4 B^2 Z_i (\rho'^2 - S^2)/(\rho'^2 - B^2)$$

$$\partial_{Zi} r_1^2 = 2Z_i$$

to get

$$\partial_{Zi}u = [(2B^2)\, \partial_{Zi}(r_1^2) - \partial_{Zi}(m+n)\,] /(2Br_{1o})^2$$

$$= [(2B^2)\, 2Z_i - 4 B^2 Z_i (\rho'^2 - S^2)/(\rho'^2 - B^2)\,] /(2Br_{1o})^2$$

$$= 4Z_i B^2 [\, 1 - (\rho'^2 - S^2)/(\rho'^2 - B^2)\,] /(2Br_{1o})^2$$

$$= Z_i [\, 1 - (\rho'^2 - S^2)/(\rho'^2 - B^2)\,] /(r_{1o})^2$$

$$= Z_i [(\rho'^2 - B^2) - (\rho'^2 - S^2)\,] /[r_{1o}^2 (\rho'^2 - B^2)]$$

$$= Z_i [S^2 - B^2\,] /[r_{1o}^2 (\rho'^2 - B^2)]$$

$$= - Z_i (B^2 - S^2) /[r_{1o}^2 (\rho'^2 - B^2)]$$

(f) <u>Compute</u> $\sqrt{1-u^2}$.

Start with this

$$1-u^2 = 1 - 2B^2 r_1^2/(m+n) = [\,(m+n) - 2B^2 r_1^2]/\,(m+n)$$

We anticipate that the difference here will have Z_i^2 terms and higher, so we can replace the denominator with $2B^2 r_{1o}^2$ since the correction would be 4th order. So

$$1-u^2 = [\,(m+n) - 2B^2 r_1^2]/\,(2B^2 r_{1o}^2)$$

Into this we install two facts already found,

$$m+n = 2B^2 r_{1o}^2 + 2 B^2 Z_i^2 (\rho'^2 - S^2)/(\rho'^2 - B^2)$$

$$r_1^2 = r_{1o}^2 + Z_i^2$$

The leading terms cancel and we are left with

$$1-u^2 = [\ 2\,B^2Z_i^2\,(\rho'^2-S^2)/(\rho'^2-B^2) - \ 2B^2Z_i^2]/\,(2B^2r_{1o}^2)$$

$$= \ Z_i^2\,B^2\,[\ (\rho'^2-S^2)/(\rho'^2-B^2) - 1]/\,(B^2r_{1o}^2)$$

$$= \ Z_i^2\ [\ (\rho'^2-S^2)/(\rho'^2-B^2) - 1]/\,(r_{1o}^2)$$

$$= \ Z_i^2\ [\ (\rho'^2-S^2) - (\rho'^2-B^2)]/\,[r_{1o}^2(\rho'^2-B^2)]$$

$$= \ Z_i^2\ [\ B^2-S^2]/\,[r_{1o}^2(\rho'^2-B^2)]$$

Therefore we have found that

$$\sqrt{1-u^2} \ = Z_i\,\sqrt{B^2-S^2}\,/\,[r_{1o}\sqrt{\rho'^2-B^2}\,]$$

$$1/\sqrt{1-u^2} \ \ = r_{1o}\,\sqrt{\rho'^2-B^2}\,/\,[Z_i\,\sqrt{B^2-S^2}\,]$$

(g) <u>Compute</u> $\partial_{Zi}\,\mathbf{\Phi}_{iris}(\mathbf{r'})|^{Zi=0}$

From our starting info we have,

$$\partial_{zi}\,\Phi_{iris}(\mathbf{r'})|^{zi=0} \approx \ (2q_1/\pi r_{1o})\,[\ -1/\sqrt{1-u^2}\,]\,\partial_{zi}u$$

and we now install our two pieces from the last two sections above,

$$\partial_{zi}u = -\,Z_i\,(B^2-S^2)\,/[r_{1o}^2(\rho'^2-B^2)]$$

$$1/\sqrt{1-u^2} \ \ = r_{1o}\,\sqrt{\rho'^2-B^2}\,/\,[Z_i\,\sqrt{B^2-S^2}\,]$$

to get

$$\partial_{zi}\,\Phi_{iris}(\mathbf{r'})|^{zi=0} \approx \ (2q_1/\pi r_{1o})\,[\ -1/\sqrt{1-u^2}\,]\,\partial_{zi}u$$

$$= (2q_1/\pi r_{1o})\,[\ 1/\sqrt{1-u^2}\,]\,(-\partial_{zi}u)$$

$$= (2q_1/\pi r_{1o})\,[r_{1o}\sqrt{\rho'^2-B^2}\,/\,[Z_i\,\sqrt{B^2-S^2}\,]]\,Z_i\,(B^2-S^2)\,/[r_{1o}^2(\rho'^2-B^2)]$$

$$= \ (2q_1/\pi r_{1o}^2)\,(\sqrt{B^2-S^2}\,/\,\sqrt{\rho'^2-B^2}\,)$$

Our expected answer was this: (from section 16 above.)

$$\partial_{zi}\,\Phi_{iris}(\mathbf{r'})|^{zi=0} \ \ = (\ 2q_1/\pi r_{1o}^2)\sqrt{B^2-S^2}\,/\,\sqrt{\rho'^2\,-B^2}$$

and we have (finally after 2 days of algebra screwing up) the expected and known-correct result. So this is a powerful test of our potential!