Smythe Problem 40 PhL 6.22.10

Problem 40: For a spherical bowl at potential V_o, find the potential at any point on the cap.

40. Invert the preceding problem, apply Green's reciprocation theorem, and show that the potential at a point P on a sphere, part of whose surface is occupied by a thin conducting bowl at potential V_0, is $(2V_0/\pi)\sin^{-1}(\cos\alpha/\cos\theta)$ where θ is the angle, measured at the bowl end of the rotational symmetry axis, between this axis and P and α that between this axis and the edge of the bowl.

As shown below, the answer has this form: $V(\theta_c) = (q_1/Q_2)\,V_0 = (-Q_{induced_on_bowl} / Q_{in_ring})V_0$.

I made three mistakes while doing this problem:

- misunderstood the Green's Reciprocation Theorem so could not find the simple ratio idea
- misunderstood the angles in the picture Smythe did not draw, see picture below
- made a one-letter algebra error converting the integral from θ to x, this cost about 15 hours.

Once these mistakes were corrected, the problem was in fact quite easy to do. In debugging the algebra error, I used Maple to do numerical integrations as checks so I could narrow down the location of the bug.

This problem tells you, given an isolated charged bowl at potential V_0, the potential at any point on the complementary sphere surface (ie, on the cap).

OVERVIEW (5 PAGES, WRITTEN 12.3.10) 2
PART 1: STATEMENT OF THE PROBLEM 6
Trying to Construct the Desired Situation by adding inversion "pairs" 7
PART 2: WHAT DOES THE RECIP THEOREM HAVE TO DO WITH PROBLEM 40? 8
Section added 12.3.10. What does "Green's Reciprocation Theorem" say? 8
Note 1 on Smythe's derivation of Green's Reciprocation Theorem. 10
Note 2 on Green's Reciprocation Theorem. 10
What does Green's Reciprocation Theorem have to say about our situation? 10
Aside on a famous special case of the above rule. 10
Here is an interesting and practical application of this recip rule 11
Aside on a useless application of the recip rule. 11
Trying to apply the Famous Simple Case: 12
Interesting Special Case Idea. [wrong!] 12
One Attempted Application of the Interesting Special Case. [wrong!] 12
Asymmetric Detail Reminder: 13
Conclusion 13

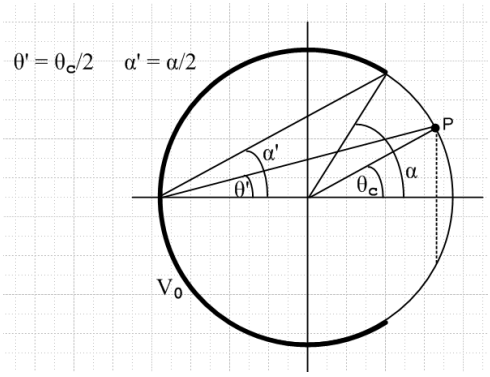
PART 3: RETURNING TO THIS PROBLEM JUNE 29, 2010, 7 DAYS LATER. 13
Reconsideration of situations S and S" 14
Ready now to actually solve problem 40 15
1. Finding the transformed R space charge density 15
2. Writing the integral of σ(θ) which should give q_1 18
3. Transforming the integral to a standard form 19
4. Looking up the integral, success! 20
PART 4: JUNK LEFT OVER FROM MAKING AN ERROR CONVERTING FROM Θ TO X. 21
Appendix A: Working backwards from Smythe's Result 24
Appendix B: Another Hint from a Specialty Paper on Elliptic Integrals. 25
Appendix C: Combining the complete integrals. 27
Appendix D: try some numbers. 27
Appendix E: try some numbers at an earlier point 29
Appendix F: try numerical integration of the integral after transformed from θ to x 30
Debug Problem 40 Problems 31

Overview (5 pages, written 12.3.10)

I did a massive amount of work trying to solve this problem (28 pages!), and finally succeeded. Simple superpositions did not work. I had to go off and learn Green's Reciprocation Theorem and it turns out that the problem can be solved by applying that theorem to two situations for two conductors. One situation is that of Problem 40, and the other that of Problem 39. The solution involves a certain integral in whose transformation I made a simple algebraic error which led to lots of wasted effort (all of Part 4 retained below for historical amusement). In the end, all came out well.

Part 1.

Here I first just state the problem. Consider this picture of the charged spherical bowl of radius A at potential V_0, where P is just a point in space on the bowl's cap:



The claim is that the potential at point P is given by this very simple expression,

$$V_P = V_o\,(2/\pi)\,\sin^{-1}\,[\,\cos\alpha' / \cos\theta'\,]$$

so this is what we want to show. Sounds like no big deal.

Since Smythe says to make use of the results of Problem 39, I next attempted to **superpose different pairs** of situations (one of which was the Problem 39 situation) such that the result would be the above picture. But I could not find a pair that worked, so I gave up on this approach.

Part 2.

Since Smythe also says to make use of Green's Reciprocation Theorem, I thought it would be good to understand this theorem which I had never heard of in my life. I fumbled around a lot with this, but now the theorem is clearly stated and trivially derived in the first subsection of Part 2 below, along with Note 1 which explains a critical detail. This theorem says that if you have a set of N conductors, if you concoct two situations S (charges Q_i and potentials V_i) and S" (charges Q_i" and potentials V_i"), then the following will be true, where the Σ_i is the sum over the N metal conductors,

$$\Sigma_i\, V_i\, Q_i" \;=\; \Sigma_i\, V_i" Q_i \qquad\qquad (*)\;\; //\;\; \text{This is Smythe p 33 (1)}$$

In particular, if you have only two conductors, this says

$$Q_1V_1" + Q_2V_2" = Q_1"V_1 + Q_2"V_2 \qquad 1 = \text{metal bowl} \qquad 2 = \text{thin metal ring thru P}$$

I apply this theorem to a very simple pair of situations:

S has $Q_1 = Q$ and $Q_2 = 0$
S" has $Q_1" = 0$ and $Q_2" = Q$.

This is not the pair of situations which will solve problem 40, but it is a "famous pair". We then have

$$Q_1V_1" + Q_2\;\; V_2" \;\; = Q_1"V_1 + Q_2"V_2$$
$$Q\;V_1" + \;\; 0 \;\;\; V_2" \;\;\; = 0\;\; V_1 + QV_2$$

$$\Rightarrow\;\; V_1" = V_2$$

I then apply this to the case of a full metal sphere surrounded by a thin metal ring, see below. This is the classic "reciprocity theorem" for two conductors, and I had at least heard of this notion, dimly. All by itself, it is a very powerful theorem I think.

In this section I took some wrong turns, and I have left the wrong conclusions in place, but have marked them wrong in red.

Part 3.

I proceed to apply this same 2-conductor Green's Reciprocation Theorem to our R-space situation (we ignore R' space for the time being). Here then are our two proposed "concocted" situations S and S":

Situation S" is the situation of our Problem 40: we just have a charged bowl Q,V_0 and nothing else:

S": bowl = 1 has $Q_1" = Q$ and potential $V_1" = V_0$
ring = 2 has $Q_2" = 0$ and potential $V_2" = V(\theta_c)$

where $V(\theta_c)$ is the potential at the ring, the thing we are trying to solve for! We don't know Q, but we do know V_0.

Situation S is the R-space situation of Problem 39: grounded bowl with induced charge $-q_1$, ring with line charge density λ.

S: bowl = 1 has $Q_1 = -q_1$ and potential $V_1 = 0$
ring = 2 has $Q_2 = 2\pi a\lambda$ and potential $V_2 = V_2$

We must think of the metal ring as having some tiny diameter so that V_2 will be large but not infinite. We now apply our reciprocation theorem:

$$Q_1V_1" + Q_2 \quad V_2" \quad = Q_1"V_1 + Q_2"V_2$$

$$-q_1V_0 + 2\pi a\lambda\, V(\theta_c) = \quad Q \quad 0 + \quad 0 \quad V_2$$

We now see that V_2 is multiplied by 0, so as long a V_2 is not infinite, we get 0. So we never had to actually deal with V_2. Also, Q is multiplied by 0, so we don't have to know the charge on the bowl (which, if we did, would require us to know the capacitance of the bowl) . We here is what we know:

$$-q_1V_0 + 2\pi a\lambda\, V(\theta_c) = 0$$

$$\Rightarrow \; V(\theta_c) \; = V_0(-q_1) / (2\pi a\lambda) \; = \; - V_0\,(q_1/Q_2) \qquad\qquad Q_2 = 2\pi a\lambda$$

But in Problem 39, we have *access to* both quantities λ (charge density on the R-space ring) and $-q_1$ (charge induced on the bowl). If we then compute λ and $-q_1$, we have our answer! I carry out these computations in Subsections 1,2,3,4 outlined below. We find of course that q_1 is proportional to Q_2, so our job is really to compute the ratio (q_1/Q_2), then we have our answer.

In **Subsection 1**, I start with the known R'-space charge on the iris (each side)

$$\sigma'(\rho') = \; - \; (q/2\pi^2)\,(\sqrt{B^2-S^2}\,/\,\sqrt{\rho'^2 - B^2}\,)\,/\,(\rho'^2-S^2)$$

Using the inversion rule

$$\sigma(\mathbf{r}) \; = (a/r)^3\,\sigma'(\mathbf{r'}) \; = (r'/a)^3\,\sigma'(\mathbf{r'}) \qquad\qquad rr' = a^2 \qquad\qquad a/r \; = r'/a$$

I then compute the charge density on the bowl in R space (same on each side). After some amount of fiddling, I show that the bowl's charge on each side is this:

$$\sigma(\theta) = - \quad Q_2 \, (1/[8\pi^2 A^2]) \sec^3(\theta/2) \, (\sqrt{1+ \tan^2(\theta_c/2)} \,)$$

$$(\sqrt{\tan^2(\alpha/2)-\tan^2(\theta_c/2)} \,/\, \sqrt{\tan^2(\theta/2) - \tan^2(\alpha/2)} \,) \,/\, (\tan^2(\theta/2)-\tan^2(\theta_c/2))$$

where A is the bowl radius, α the polar bowl angle, θ_c a polar angle location on the bowl, and $Q_2 = 2\pi a\lambda$ as appears above, the total charge on the R space ring.

In **Subsection 2** I integrate the above to obtain $-q_1$, the total charge on the bowl induced by the ring. I find that

$$q_1/Q_2 = \quad + (1/2\pi) \sqrt{1+ \tan^2(\theta_c/2)} \, \sqrt{\tan^2(\alpha/2)-\tan^2(\theta_c/2)}$$
$$\int_{\alpha}^{\pi} \sin\theta \, d\theta \, \sec^3(\theta/2) / \, [\sqrt{\tan^2(\theta/2) - \tan^2(\alpha/2)} \, (\tan^2(\theta/2)-\tan^2(\theta_c/2))\,]$$

Recall that our *goal* is this, where the rightmost expression is Smythe's claimed result,

$$V(\theta_c) = \quad - V_0 \, (q_1/Q_2) \quad = \; V_0 \, (2/\pi) \sin^{-1} [\, \cos(\alpha/2) \,/\, \cos(\theta_c/2) \,]$$

$$\Rightarrow \; q_1/Q_2 = \; - \; V_0 \, (2/\pi) \sin^{-1} [\, \cos(\alpha/2) \,/\, \cos(\theta_c/2) \,]$$

At some point, in Appendix E I went off and did a numeric integration of my q_1/Q_2 expression above for certain values of α and θ_c , and I found that this agreed with an evaluation of Smythe's claimed result, so I then knew I was on the right track. [This helped me located my algebra mistake in Subsection 3.]

In **Subsection 3** I transform the integral you see above. I started off making these definitions:

$$A = \tan(\alpha/2) \qquad C = \tan(\theta_c/2) \quad A > C \qquad x = \tan(\theta/2)$$

where I accidentally overloaded symbol A since it is the bowl radius, but managed OK doing that. My integral of interest was

$$I = \int_{\alpha}^{\pi} \sin\theta \, d\theta \qquad \sec^2(\theta/2) \quad \sec(\theta/2) / \, [\sqrt{\tan^2(\theta/2) - A^2} \, (\tan^2(\theta/2) - C^2) \,]$$

$$= \; 4 \int_{A}^{\infty} dx \, x / \, [\sqrt{x^2+1} \, \sqrt{x^2 - A^2} \;\; (x^2 - C^2) \,] \;\; = 4 \, J$$

(red x was erroneously omitted in my first efforts) so now I had to go off and compute

$$J = J(A,C) = \int_{A}^{\infty} dx \, x / \, [(x^2 - C^2) \sqrt{x^2+1} \, \sqrt{x^2 - A^2} \,] \;\; = (1/2) \;\; \int_{A^2}^{\infty} \; dy \, / \, [(y - C^2) \sqrt{y+1} \, \sqrt{y - A^2} \,]$$

My result would agree with Smythe's result if I could show that

$$J(A,C) = [\sqrt{1+C^2}\,\sqrt{A^2-C^2}\,]^{-1}\ \sin^{-1}[\sqrt{C^2+1}\ /\sqrt{A^2+1}\,]$$

In **Subsection 4** I got the Wolfram integrator to do the y-form J integral, and it gave exactly the desired result. I also found something in GR7, but Wolfram was easier. This integral is an example of what GR would call an integral of an algebraic expression.

Part 4.

This part is only interesting in a historical sense. When I transformed the above integral in Subsection 3, I made an algebraic error omitting the factor of x from the transformed integral (shown in red above). This error caused the integral to be a messy elliptic function deal which was completely wrong and led me on a wild goose chase. I did not know where I was making the error until I found it, so I was debugging by doing numerical integrations to see at what point I was picking up the error. I narrowed it down to the transforming of the integral from θ to x. Even though I had checked this transformation about 4 times, I did not detect the error until I knew for sure it was in this transformation, then I found it. So it goes.

Part 1: Statement of the Problem

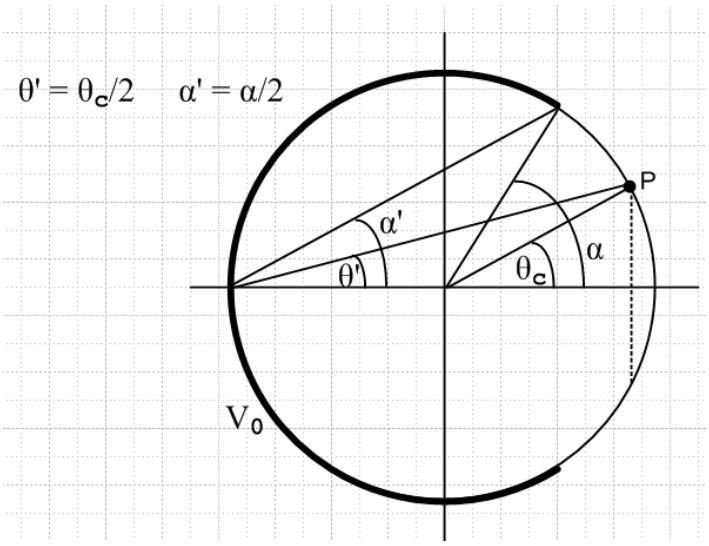
We have just done problem 39 so we know about an iris with a ring of charge in the hole -- this is what Smythe below will call "the preceding problem". We know -- using inversion -- how to connect this to a spherical bowl with a ring of charge lying symmetrically on the complementary part of the sphere. In the connection, both bowl and iris are at ground. In theory, we know all the charge densities in both R and R' space. I even know the potentials everywhere in both spaces.

Now comes problem 40:

> **40.** Invert the preceding problem, apply Green's reciprocation theorem, and show that the potential at a point P on a sphere, part of whose surface is occupied by a thin conducting bowl at potential V_0, is $(2V_0/\pi)\sin^{-1}(\cos\alpha/\cos\theta)$ where θ is the angle, measured at the bowl end of the rotational symmetry axis, between this axis and P and α that between this axis and the edge of the bowl.

Words often do not a picture make, but I think I have the correct interpretation of the angles. Let's *rename* Smythe's α to be α' and θ to be θ' and then let's restate his Problem 40 with a picture:

Restated Problem 40: Consider this picture of the charged spherical bowl: (there is no point charge at point P and there is no ring of charge on which P is a point)



The claim is that the potential at point P is given by

$$V_P = V_o\,(2/\pi)\,\sin^{-1}\,[\,\cos\alpha' / \cos\theta'\,]$$

In terms of my favored sphere-centered angles, we would write this as follows:

$$V(\theta_c) = V_o\,(2/\pi)\,\sin^{-1}\,[\,\cos(\alpha/2) /\cos(\theta_c/2)\,]$$

so this is what I want to show.

Notice that this is not quite the "charged bowl problem" where we ask "what is the potential at some point for a bowl charged with Q". We don't -- after all -- know the capacitance of this bowl. But it is asking about an isolated bowl being held at V_0. So far in my efforts, I have never seen a way to make a connection between: (1) The Green's Function potential for an object, and (2) the potential of that object charged and in isolation. They are just different problems.

Trying to Construct the Desired Situation by adding inversion "pairs"

If we start with our Problem 39 iris/bowl inversion analysis (**pair1**), how can we get the bowl up to potential V_0 from potential 0? I think we are always allowed to superpose two inversion pairs to get a sum situation which is then a valid inversion pair. The first pair is our starting point. The second pair (**pair2**) could be a constant V_0 potential in R space, and the matching point charge in R' space. In the sum pair, call it **pair3**, we then have in R space V_0 on the bowl (and $V(r = \infty) = V_0$ there) and we have known σ on the bowl and λ on the ring. Over in R' space we have a new point charge at the origin, and we have σ' on the iris and λ' on the ring, as in the starting pair. One difference is that the iris is now no longer at a constant potential, so we regard the σ on the iris as "sticky charge". [This is really an application of the Red Flag superposition theorem, see elsewhere.]

Comment: From my full Problem 38 work, I know the exact potential everywhere in both R space and R' space for both pair1 and pair3. In R space we have this pair3 situation: bowl at V_0, $V(\infty) = V_0$, a known

negative σ on the bowl which integrates to some $Q_{\mathtt{bowl}}$, a known positive λ on the ring which integrates to some $Q_{\mathtt{ring}}$. I could compute both these numbers and don't think they have the same magnitude, although I know that in R' space they *do* have the same magnitude (I explicitly did this calculation). That is to say, I know that $Q'_{\mathtt{iris}} = -Q'_{\mathtt{ring}}$.

Plan A: Consider **pair4** being a ring of charge in R space with total charge $-Q_{\mathtt{ring}}$ with its matching ring in R' space with total charge $-Q'_{\mathtt{ring}}$. I think I know "everything" about this pair4 (potential in R space, and potential in R' space, using Jackson formula say). What happens if I define **pair5** = pair3+pair4 ? In R space for pair5 we have a bowl with known sticky charge σ, but with a non-constant potential, and we have no ring at all, so this sticky-charge bowl is in isolation. But this is not the bowl that problem 40 is asking about, so I don't think this plan A is fruitful.

Part 2: What does the Recip Theorem have to do with Problem 40?

Section added 12.3.10. What does "Green's Reciprocation Theorem" say?

I must say I had never heard of this thing before. The term reciprocation does not appear in green Jackson, but does appear in blue Jackson, though only in the problems and only in a simplified form. The theorem concerns two "situations". In each situation, you have the same set of N "charge locations".

In situation S, you place point charges q_1, q_2...q_N at these locations. A set of potentials are then carefully defined as follows. If you were to remove q_1 (or set q_1=0) and measure the potential at location 1 due to all the *other* charges, that potential is called V_1. This is a potential you would measure only if you set $q_1 = 0$. If you don't set $q_1 = 0$, then of course the measured potential would be ∞ at location 1. In similar fashion we define V_2 to be the potential of all other charges if q_2 were removed, and so on.

Situation S' we have the exact same N charge locations, but we have different charges which we call q_1', q_2' and so on. And in S' we define potential V_1' to be that which would be measured were q_1' removed. So all the definitions in S' are the same, we just have different values for the charges and the potentials defined in this special way.

Smythe discusses this on page 34 (he uses primes instead of double primes). He tells us to consider a certain matrix and form sums of the rows and sums of the columns. The theorem states that the sum of the sums of the rows equals the sums of the sums of the columns. Here is the whole ball of wax:

$$
\begin{array}{ccccccccc}
0 & + & \frac{q_2' q_1}{4\pi\epsilon r_{21}} & + & \frac{q_3' q_1}{4\pi\epsilon r_{31}} & + \cdots + & \frac{q_n' q_1}{4\pi\epsilon r_{n1}} & = & q_1 V_1' \\
\frac{q_1' q_2}{4\pi\epsilon r_{12}} & + & 0 & + & \frac{q_3' q_2}{4\pi\epsilon r_{32}} & + \cdots + & \frac{q_n' q_2}{4\pi\epsilon r_{n2}} & = & q_2 V_2' \\
\frac{q_1' q_3}{4\pi\epsilon r_{13}} & + & \frac{q_2' q_3}{4\pi\epsilon r_{23}} & + & 0 & + \cdots + & \frac{q_n' q_3}{4\pi\epsilon r_{n3}} & = & q_3 V_3' \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{q_1' q_n}{4\pi\epsilon r_{1n}} & + & \frac{q_2' q_n}{4\pi\epsilon r_{2n}} & + & \frac{q_3' q_n}{4\pi\epsilon r_{3n}} & + \cdots + & 0 & = & q_n V_n' \\
\underline{q_1' V_1} & & \underline{q_2' V_2} & & \underline{q_3' V_3} & \cdots & \underline{q_n' V_n} & &
\end{array}
$$

The order of summation being immaterial, the sum may be obtained by adding the terms of the last row or last column. Equating these gives

$$\sum_{s=1}^{n} q_s' V_s = \sum_{s=1}^{n} q_s V_s' \qquad (2)$$

where equation (2) is the Green's Reciprocation Theorem. Look at the first column. If you add up all entries, you get q_1' times a sum of terms which you realize is V_1 as we defined it above. The entries are not potential energies in either situation S or S'. I don't really know how to "interpret" the theorem. It is simply a comparison of two different situations S and S' each of which involves the same locations of a set of charges. As you see, the proof of the theorem is a complete no-brainer.

You can then consider applying this theorem to a set of surface charges lying on metal surfaces. In two situations S and S' we have the same set of metal surfaces, and the two situations differ only in how charges exist on these surfaces. Using Σ to mean integration, we can define $Q_1 = \Sigma_i q_{1i}$ to be the total charge on surface s_1, where $q_{1i} = \sigma_1(\mathbf{r}_i)dA_i$ is a point charge on surface s_1. If we were to remove just the one charge q_{1i} and measure the potential at that point on surface s_1, that would be V_{1i} . The above theorem could then be stated this way

$$\Sigma_I \Sigma_i \; q_{Ii}' \, V_{Ii} \;=\; \Sigma_I \Sigma_i \; q_{Ii} \, V_{Ii}'$$

where we have simply ordered each sum first over surfaces I, and then for each surface as a sum over the i charges on the surface. Now comes the big trick: as explained in Note 1 below, removing the one charge q_{1i} to measure V_{1i} gives the same V_{1i} you would get by *not* removing that charge! This V_{1i} would then be the actual potential you measure at this point i on surface 1. But these charges i are on a piece of metal s_1 and therefore V_{1i} is the same at all points i, and we can then just call it V_1. So $V_{Ii} = V_I$ and same for primed. Our theorem above then says

$$\Sigma_I \, V_I \, \Sigma_i \; q_{Ii}' \;=\; \Sigma_I \, V_I' \, \Sigma_i \; q_{Ii}$$

$$\Sigma_I \, V_I \, Q_I' \;=\; \Sigma_I \, V_I' Q_I \qquad (*) \;\; // \;\; \text{This is Smythe p 33 (1)}$$

where Q_I is the total charge on surface s_I in situation S, and so on. So now our theorem addresses a set of metal surfaces in two situations. The V_I are the actual potentials of the metal surfaces in situation S, and the Q_I are the actual total charges on these surfaces. The surfaces must be conductors to make this

theorem work as stated in (*) above. So this is a theorem we can apply to real world problems, since we no longer have that caveat of "all other charges" which prevented V_i from being the measured potential.

In the discussion below, I replace situation S' by situation S" and all primes are double primes. This is to avoid confusion with single prime referring to R' space.

Note 1 on Smythe's derivation of Green's Reciprocation Theorem.

In the derivation, a small detail is swept under the Smythe rug. When we start off, V_s is the potential at the location of charge q_s due to all the OTHER charges in configuration S (ie, the unprimed config) But when we later start thinking of $q_s = \sigma dA$ on a metal surface, we really should think of V_s as the potential at this point due to all *other* charges in the configuration, that is, excluding this little σdA. But we can then make use of Stakgold's Chapter 6 discussion of layers and what happens to a potential as you approach the metal. Stakgold showed that as you approach the surface, the contribution from σdA right under your landing spot is in fact 0 in the limit of small dA. Therefore, the potential due to all the OTHER charges and due to ALL the charges is the same in this case. That is why when we group together all the charges on a piece of metal, we can just take V to be the common potential of that piece of metal. Thus, in Smythe's first scenario with a bunch of point charges, we had to "be careful" about the meaning of V_i because it had to exclude the contribution of charge q_i. But in the final statement of the recip rule for metal objects, we no longer have to "be careful" and V_i is then just the potential that a piece of metal has in some configuration. I misunderstood this on my first reading.

Note 2 on Green's Reciprocation Theorem.

We might think of the great sphere as an extra piece of metal in this theorem. But if all the other pieces of metal are localized, then we know that in *any* scenario, $V_i = 0$ for this great sphere, so on both sides of our equation we could include it, but then each included term is 0. Thus, we can ignore the great sphere in the statement of the recip theorem.

What does Green's Reciprocation Theorem have to say about our situation?

Think R space only. We have a bowl and a ring only. Think of the ring as a thin conductor which holds the λ charge in question. So we have a problem with *two conductors*. The fully general recip rule for this situation in R space is this

$$Q_1V_1" + Q_2V_2" = Q_1"V_1 + Q_2"V_2$$

This "rule" applies to two "situations" that you are allowed to invent. In situation S, the two conductors have charges Q_i which you may specify, and in situation S" they have charges Q_i" which you may specify.

Note: I use double primes here so I can reserve single primes to refer to the inversion picture of a given situation, so that R space and R' space form an inversion pair. Smythe uses single primes in his discussion of the recip rule.

Aside on a famous special case of the above rule. As Smythe points out, you could make these choices for your two situations:

S	$Q_1 = Q$	$Q_2 = 0$
S"	$Q_1" = 0$	$Q_2" = Q$

Then the recip rule reads

$$Q\,V_1" = Q\,V_2 \quad \Rightarrow \quad V_1" = V_2$$

and this is discussed on Smythe page b35 (d54), where b = book, d = djvu. Here is what this special case says in an English paragraph.

" Suppose you have two conductors 1 and 2.

In situation S you charge #1 to Q, the #2 is uncharged and on #2 you measure some potential and you call that potential V_2.

In situation S" you charge #2 to Q, the #1 is uncharged and on #1 you measure some potential and you call that potential $V_1"$.

These two potentials will be the same. "

Here the word "reciprocity" makes a lot of sense (Smythe and others say reciprocation).

Here is an interesting and practical application of this recip rule. Suppose the two conductors are a *full sphere* of radius a, and a ring lying on a larger sphere of radius b. So sphere = #1 and ring = #2.

In situation S, we charge the sphere and the ring is uncharged. This "uncharged ring" of metal being "symmetrically located" relative to the sphere can be deleted from the picture with no change to the situation S. The reason is that we know that the potential everywhere on the ring will be the same whether or not the metal is present. Adding the metal does not cause charges on the ring to move to a new pattern, for example. Therefore, we know that the potential for situation S is $V_2 = Q/b$, very simple.

Suppose we are asked to solve the problem of situation S": our ring #2 is charged with Q and the sphere #1 is uncharged. This causes some tricky induced charge distribution on sphere #1. [I could calculate the potential I think for this situation using a little image charge ring inside the sphere, and then I could evaluate that potential on the sphere surface, but that is not the point here.] . Although the induced charge on the sphere is some fairly complicated distribution, and the potential in space is also complicated, from the recip rule we know right off the bat that the potential on the sphere is $V_1" = Q/b$. This is I think an impressive result!

Once again, here would be the assigned problem: An uncharged metal sphere is surrounded by an infinitesimally thin metal ring of radius b carying total charge Q. The center of the ring and sphere are the same point. What is the potential on the sphere? Answer: $V_{sphere} = Q/b$. The result is independent of the sphere's radius.

Aside on a useless application of the recip rule. Suppose you take situation S" to have both $Q_i" = 0$. Then your recip rule says that $Q_1V_1" + Q_2V_2" = 0$. Since all $Q_i" = 0$, we know that all $V_i" = 0$. This rule then says that 0 + 0 = 0, which is surely "useless".

Question: Smythe gives a hint that we are supposed to somehow use "the recip rule" in solving Problem 40. My question is this: is it in the form of the "famous simple case" above, or is it some other form where we make up our S and S' situations? Right now I have no idea. [some other form!]

Trying to apply the Famous Simple Case:

Rule says $V_1" = V_2$.
Situation S: bowl #1 charged to Q, ring uncharged. Since I don't know the potential of an isolated charged bowl anywhere (including on the ring location) , I don't know V_2. Nor do I know σ_1 on the bowl.
Situation S": ring #2 charged to Q, bowl uncharged. There will be some $\sigma_1"$ induced on the bowl by the ring charge, which I don't know. There will be some potential everywhere which I don't know. So I certainly don't know $V_1"$.

In other words, I only know how to use the "famous simple case" when the bowl is a full sphere. When it is a partial sphere, I am dead in the water in both sides of the above recip relation.

Interesting Special Case Idea. [wrong!] Let's go back to our "most general statement" : $Q_1V_1" + Q_2V_2" = Q_1"V_1 + Q_2"V_2$. If in situation S" we set one of the charges to 0, say $Q_1" = 0$, then $V_2" = 0$ [wrong! This was my misunderstanding of the meaning of $V_2"$ as excluding charge $Q_2"$ which is wrong] and the rule says $Q_1V_1" = Q_2"V_2$. Notice that we have not set $Q_2 = 0$ here, it could be anything! So this is then an application of the rule which is a little more general that our "famous special case" . So call this our "interesting special case". It is this:

Interesting Special Case: (we can restate this in other ways doing 1 ↔ 2 and/or prime ↔ noprime)
In situation S we have some arbitrary Q_1 and Q_2.
In situation S" we have $Q_1" = 0$ and arbitrary $Q_2"$.
Then we know that $Q_1V_1" = Q_2"V_2$.

One Attempted Application of the Interesting Special Case. [wrong!] In trying to apply our "interesting special case", one possible situation S would be the pair1 situation from above. We know σ on the bowl, and λ on the ring, we could integrate both to get our Q_1 and Q_2, and neither is 0. In this case, we would know that V_1 = potential on an uncharged bowl due to Q_2 on the ring [wrong] . And we would know that V_2 = potential on an uncharged ring due to Q_1 on the bowl. [Note that I don't know how to compute *either* of these potentials V_1 or V_2. When the bowl was a sphere, I knew how to compute $V_2 = Q_1/b$ as discussed above.] [Note also that finding V_2 is very close to what we are trying to solve for in Problem 40.]

What then might we take as our candidate situation S" ? We set bowl's $Q_1" = 0$ so we have an uncharged bowl sitting there (it might have some $\sigma_1" \neq 0$ of course). The ring has some $Q_2"$. Then $V_1"$ is the potential on the bowl. This of course is our "famous special case" situation for two conductors, and I have already noted above that I don't know how to compute $V_1"$. If we go the other way, then I don't know how to compute $V_2"$ by the same argument.

Somehow both our situations S and S" involve the exact same unknown two potentials, whatever names you might want to give them. So this "application of the interesting special case" also does not

bear any fruit. This interesting special case brings nothing new to the table that the famous special case does not already bring to the table. Neither case helps us one iota.

Asymmetric Detail Reminder: [correct] One detail: if you have a bowl with Q_1, it makes a potential on the location of the metal ring whether or not there is an actual metal ring there! In other words, suppose there were no ring. The bowl makes some potential which is the same at all points where the ring would go. When we materialize the ring, nothing happens, all stays the same.

The opposite is not true. Suppose we have a metal ring of charge Q_2 . It makes a potential in space. If we add the uncharged metal bowl, surely that potential will be changed, and an induced charge distribution will form on the bowl (adding up to 0).

Conclusion: [I was confused.] In Problem 40, Smythe tells us to "invert the preceding problem and apply Green's reciprocation theorem". The "preceding problem" was Problem 39 which talks about a ring of charge inside a metal iris and we compute σ_{iris} in detail. We know that the inversion of that problem is a ring of charge plus a bowl. So certainly we know what geometric "inversion" he is generally talking about. But I don't see any way that the Green's recip theorem gives us any strategy to solve the problem. It just relates two potentials neither of which I know.

It is true that one of these two potentials is roughly the thing we are trying to solve for in this problem -- namely, the potential on the uncharged ring due to a charged bowl, which is the same as saying the potential of a charged bowl. If this is the form of the recip rule that we are really supposed to use, then Smythe is *implying* that there must be some way to calculate the other potential -- namely, the constant potential on an uncharged bowl due to a charged ring. So I could attempt to solve that problem as an integral equation for σ on the bowl, but such a solution would then have nothing at all to do with "inversion" or with the results of Problem 39, and then Smythe's hint concerning the previous problem is a complete red herring.

In any event, I am unable to do this Problem 40, but have done my due diligence in trying it.

Part 3: Returning to this problem June 29, 2010, 7 days later.

This is a new start on the problem, after fixing up my wrong interpretation of the Recip Rule.

1. First of all I see that, in the above, I *misinterpreted* the meaning of V_i in the recip equation. This is not the potential due to all OTHER pieces of metal $\neq$ i, it is the potential due to ALL pieces of metal (including that due to total or induced charge on metal piece i). In other words, V_i is just the potential of metal piece "i" in a given situation S. The source of my confusion is explained in Note 1 which I have added above. Two sections involving a "special interesting case" above are based on my misinterpretation, so are "wrong" but I leave them in place, marked as such.

2. I studied the iris with point Green's charge in hole and confirmed that at the Green's charge, the potential diverges as q/r_1, just as you would expect, and this is not affected by the fact that the iris (which holds the induced charge) happens to have an infinite extent.

3. If you have an infinitesimal ring of line charge λ, the potential at the ring diverges logarithmically, as a simple integration shows. This is the same thing that happens in the simpler infinite line charge case. But you are allowed to think of the ring as having some small diameter much smaller than the bowl sphere.

and in this case the potential is large but finite on the ring. As in (2) above, the presence of an infinite iris does not change the fact that the total potential diverges logarithmically on an infinitesimal ring of line charge in the hole.

4. It is useful to rethink the basic 2-object case as outlined above, where in each of the two situations S and S", one object has charge Q and the other has no charge. Call these metal objects 1 and 2. If we charge up object to $Q_1 > 0$, object 2 remains with $Q_2 = 0$, and you can think of the great sphere holding $-Q_1$. There will be some induced charge $\sigma_2 \neq 0$ on object 2 which integrates to 0. Also, if you think of starting with charged object 1 and you "bring in" object 2, you can see that object 2 will end up with a *positive* potential.

5. As a special case of the above, you can think of object 2 as a small BB. There will be an induced charge on the BB that is mainly a dipole pointed away from object 1. Although $Q_2 = 0$, this dipole does in fact contribute to the total potential in the situation. However, as the BB is made smaller and smaller, this dipole moment shrinks and eventually "goes away" and we are left with a "point BB" which simply acts as a non-interfering *probe* of the local potential, itself contributing nothing to the potential.

6. In the case of our bowl and ring, the same idea as in item 5 applies to the ring. For a finite diameter ring, there is an induced dipole all around the ring, but this p goes to 0 as the ring's wire diameter shrinks and eventually you can regard this ring as just a probe. It only works as a probe because the potential of the bowl is symmetrical around the ring. If the ring were tipped (so not symmetric relative to the other object), then you cannot regard it as a probe, because it is probing multiple points in space which don't have the same potential. Perhaps it would probe (measure) the average potential on the ring.

Reconsideration of situations S and S"

I reserve situation S' to be the V=0 iris plus ring charge in the hole. In R' space, so like S'.

I want situation S to be the inversion of S', which is the V=0 bowl + ring charge. In R space, so like S.

Situation S" will be a *third* situation in the next paragraph.

Let **situation S"** be the following. We have a charged bowl $Q_1" = Q$, and an uncharged ring $Q_2" = 0$. The potential of the bowl is some prescribed $V_1" = V_0$, and the potential of the infinitesimal-wire ring is $V_2" = V(\theta_c)$ which is just a probe of the potential of the bowl. $V_2"$ is the "thing we want to know". The great sphere has $Q_3" = -Q$ and exists at $V_3" = 0$ so, as noted above, we don't need to include it in our recip relation.

Let **situation S** be the R-space result of our inversion problem. The ring holds some charge $Q_2 > 0$ due to the line charge λ on it. The bowl has some negative induced charge $Q_1 = -q_1$ where $q_1 > 0$. We know that the great sphere will then have charge $Q_3 = -(Q_2 - q_1)$ and $V_3 = 0$. Importantly, we know $V_1 = 0$ because the bowl is grounded in our inverted bowl + ring situation S. The ring potential V_2 is infinite if we make the wire have zero diameter, true, but let's assume some tiny finite diameter. Then the potential at the wire surface of the ring is some large but finite value we will continue to just call $V_2 > 0$. Since we

"know all about" situation S, we presumably know the value of q_1. We get this by integrating the charge density on the bowl, which we supposedly know. We also know Q_2 .

Now, what does the recip rule have to say about our two situations S and S" ? We start with:

$Q_1"V_1 + Q_2" V_2 = Q_1V_1" + Q_2V_2"$ bowl = 1 ring = 2

Filling in our stuff from the previous two paragraphs, this says:

$Q * 0 + 0 * V_2 = -q_1 * V_0 + Q_2 * V(\theta)$

Now we see the advantage of having a finite wire diameter: The second term on the LHS vanishes even though V_2 might be very large. Thus we really have

$-q_1 * V_0 + Q_2 * V(\theta_c) = 0$

which tells us something about the thing we want to know,

$V(\theta_c) = (q_1/Q_2)\, V_0 = (- Q_{induced_on_bowl} / Q_{in_ring})\, V_0$

so all we have to do is obtain these two charges in our inverted problem. The light shineth! This is the correct form of the "interesting special case" I was fumbling incorrectly with in my first attempt.

So what is really going on here? Our physical world is two pieces of metal called 1 and 2. We have two situations S, S" with $Q_1"V_1 + Q_2" V_2 = Q_1V_1" + Q_2V_2"$. We would like to find a connection between $V_1"$ and $V_2"$, the metal potentials in situation S", so it would be nice if we could concoct a situation S such that $Q_1"V_1 + Q_2" V_2 = 0$. Our method is to arrange S such that $V_1 = 0$ (because S is a Green's function type situation for this object) which kills one of the two terms. Then it happens that in S" we have $Q_2" = 0$ already, so that kills the second term. Then $Q_1V_1" + Q_2V_2"$. If we can determine the ratio of the object charges in concocted situation S, then we can find the ratio of the two $V_i"$ in situation S".

Ready now to actually solve problem 40

1. Finding the transformed R space charge density

In problem 39 where we are discussing the iris with ring charge in its hole, if that ring-in-hole charge is q, then the charge density on the iris is this: (the iris is in R' space, so I prime the density and variables)

$$\sigma'(\rho') = - (q/2\pi^2) (\sqrt{B^2-S^2} / \sqrt{\rho'^2 - B^2}) / (\rho'^2-S^2)$$

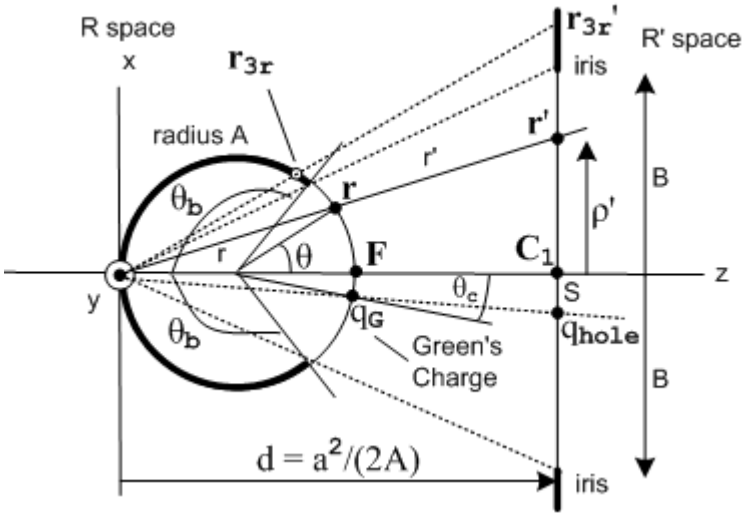
We found that the total charge on the iris was -q. The above is the charge *on either side* of the iris. Notice that the expression above has both a "square root pole" and a "full pole" in the denominator relative to variable ρ'^2.

Our job is to invert this to find the charge density on the bowl, which we will then integrate to get the answer to problem 40! The σ on either side of the bowl will be the same, just as on the iris, so we will eventually double things to get the total charge on the bowl.

From our inversion META notes we steal this claim [I have checked this many times.]

$$\sigma(\mathbf{r}) = (a/r)^3\,\sigma'(\mathbf{r}') = (r'/a)^3\,\sigma'(\mathbf{r}') \qquad rr' = a^2 \qquad a/r = r'/a$$

where **r** and **r'** are inversion coordinates. We roll out our problem 38 Pic 2, where now we just imagine that we have a charge ring on the iris passing through the q_{hole} position, and a ring charge on the bowl passing through the q_G position. We also need to imagine **r** and **r'** such that **r** is on the bowl and **r'** up on the iris. The picture was not quite drawn for this purpose, but it will do! Also, imagine $\alpha = \pi - \theta_b$ is the angle like θ going up to the edge of the bowl.



So our answer so far is then this, for the charge on one bowl side,

$$\sigma(\mathbf{r}) = (r'/a)^3\,\sigma'(\mathbf{r}') = -\,(r'/a)^3\,(q/2\pi^2)\,(\sqrt{B^2-S^2}\,/\,\sqrt{\rho'^2 - B^2}\,)\,/\,(\rho'^2-S^2)$$

As usual, we want to express things in bowl coordinates, not iris coordinates. We can see that

$$\rho'/d = \tan(\theta/2) \qquad d/r' = \cos(\theta/2) \qquad B/d = \tan(\alpha/2) \qquad S/d = \tan(\theta_c/2) > 0$$

Meanwhile, we know that (think ring = sum of 1,000 point charges)

$$q_i = q_i'(a/r_i') \quad \Rightarrow \quad q_G = q_{hole}\,(a/\sqrt{d^2+S^2}\,) \quad \Rightarrow \quad Q_2 = q\,(a/\sqrt{d^2+S^2}\,)$$

where q is the total charge on the iris ring, while Q_2 is the total charge on the R-space ring. We can then replace (always trying to get rid of R' space parameters in favor of R space parameters)

$$q = Q_2\,(\sqrt{d^2+S^2}\,/a)$$

and we then have for our one-side bowl charge

$$\sigma(\mathbf{r}) = (r'/a)^3\,\sigma'(\mathbf{r}') = -\ (r'/a)^3\,(q/2\pi^2)\,(\sqrt{B^2-S^2}\,/\,\sqrt{\rho'^2 - B^2}\,)\,/\,(\rho'^2-S^2)$$

$$= -\ (r'/a)^3\,(1/2\pi^2)\,Q_2\,(\sqrt{d^2+ S^2}\,/a)\,(\sqrt{B^2-S^2}\,/\,\sqrt{\rho'^2 - B^2}\,)\,/\,(\rho'^2-S^2)$$

$$= -\ (d\sec(\theta/2)/a)^3\,(1/2\pi^2)\,Q_2\,(\sqrt{d^2+ S^2}\,/a)\,(\sqrt{B^2-S^2}\,/\,\sqrt{d^2\tan^2(\theta/2) - B^2}\,)\,/\,(d^2\tan^2(\theta/2)-S^2)$$

$$= -\ (d^3\sec^3(\theta/2)/a^3)\,(1/2\pi^2)\,Q_2\,(\sqrt{d^2+ S^2}\,/a)\,(\sqrt{B^2-S^2}\,/\,\sqrt{d^2\tan^2(\theta/2) - B^2}\,)\,/\,(d^2\tan^2(\theta/2)-S^2)$$

$$= -\ (d^3\sec^3(\theta/2)/a^4)\,(1/2\pi^2)\,Q_2\,(\sqrt{d^2+ S^2}\,)\,(\sqrt{B^2-S^2}\,/\,\sqrt{d^2\tan^2(\theta/2) - B^2}\,)\,/\,(d^2\tan^2(\theta/2)-S^2)$$

which at least has the right dimensions of Q/L^2. We are not surprised that σ_{bowl} here depends on S, which controls the position of the R-space ring relative to the bowl. We now use $a^2 = 2Ad$ to get

$$= -\ (d^3\sec^3(\theta/2)/(4A^2d^2))\,(1/2\pi^2)\,Q_2\,(\sqrt{d^2+ S^2}\,)\,(\sqrt{B^2-S^2}\,/\,\sqrt{d^2\tan^2(\theta/2) - B^2}\,)\,/\,(d^2\tan^2(\theta/2)-S^2)$$

$$= -\ (d\sec^3(\theta/2)/(4A^2))\,(1/2\pi^2)\,Q_2\,(\sqrt{d^2+ S^2}\,)\,(\sqrt{B^2-S^2}\,/\,\sqrt{d^2\tan^2(\theta/2) - B^2}\,)\,/\,(d^2\tan^2(\theta/2)-S^2)$$

$$= -\ Q_2\,(d/[8\pi^2A^2])\,\sec^3(\theta/2)\,(\sqrt{d^2+ S^2}\,)\,(\sqrt{B^2-S^2}\,/\,\sqrt{d^2\tan^2(\theta/2) - B^2}\,)\,/\,(d^2\tan^2(\theta/2)-S^2)$$

and now we replace according to the above claims.,

$$\rho' = d\ \tan(\theta/2) \qquad B = d\,\tan(\alpha/2) \qquad S = d\,\tan(\theta_c/2)$$

$$= -\ Q_2\,(d/[8\pi^2A^2])\,\sec^3(\theta/2)\,(\sqrt{d^2+ d^2\tan^2(\theta_c/2)}\,)$$

$$(\sqrt{d^2\,\tan^2(\alpha/2)-d^2\,\tan^2(\theta_c/2)}\,/\,\sqrt{d^2\tan^2(\theta/2) - d^2\,\tan^2(\alpha/2)}\,)\,/\,(d^2\tan^2(\theta/2)-d^2\,\tan^2(\theta_c/2))$$

If we now pull out all the factors of d , they nicely all go away, and we get

$$\sigma(\theta) = -\ Q_2\,(1/[8\pi^2A^2])\,\sec^3(\theta/2)\,(\sqrt{1+ \tan^2(\theta_c/2)}\,)$$

$$(\sqrt{\tan^2(\alpha/2)-\tan^2(\theta_c/2)}\,/\,\sqrt{\tan^2(\theta/2) - \tan^2(\alpha/2)}\,)\,/\,(\tan^2(\theta/2)-\tan^2(\theta_c/2))$$

where now our S dependence is replaced by θ_c dependence. This then is our major result showing the charge density on the bowl. It is pretty complicated, but the fact that d went away makes us think we did it correctly. We have replaced ρ' basically with tan(θ/2), and we still see in the denominator both the "square root pole" and the "full pole" that we started with. In addition, we have picked up a factor $\sec^3(\theta/2)$ which arises from $(r'/a)^3$, which is to say, from the way the charge density transforms as you go from R' space to R space. Apart from this, the charge densities are "the same" in both spaces. So our task

now is to integrate this charge density over the bowl to get the q_1 which appears in our S situation description given above.

Comment: $\sigma(\theta)$ shown above is the σ on each surface of the grounded bowl induced by a charge ring at θ_c which ring has total charge Q_2.

2. Writing the integral of $\sigma(\theta)$ which should give q_1

To do this, we consider a ring of charge on the bowl (bowl is radius A) which has charge $dq = \sigma(\theta)\, 2\pi A^2 \sin\theta d\theta$. Why? The radius of this ring is $A\sin\theta$, so one dimension of a dA charge patch is $(A\sin\theta)d\varphi$. The other orthogonal dimension is $Ad\theta$ (both these "dimensions" are pieces of circles"). Then $dA = A^2\sin\theta d\theta d\varphi$ and we integrate around φ to get $dA = 2\pi A^2 \sin\theta d\theta$. So the $\sin\theta$ cannot be avoided.

We then want to integrate this ring from $\theta = \pi - \theta_b \equiv \alpha$ up to π to cover the entire bowl. Then we double everything which I will now just do right here. We then get

$$q_1/Q_2 = +(1/[4\pi^2A^2])\sqrt{1+\tan^2(\theta_c/2)}\sqrt{\tan^2(\alpha/2)-\tan^2(\theta_c/2)} \;*\; 2\pi A^2 \;*$$
$$\int_{\alpha}^{\pi} \sin\theta\, d\theta\, \sec^3(\theta/2)/\left[\sqrt{\tan^2(\theta/2)-\tan^2(\alpha/2)}\,(\tan^2(\theta/2)-\tan^2(\theta_c/2))\right]$$

$$= +(1/2\pi)\sqrt{1+\tan^2(\theta_c/2)}\sqrt{\tan^2(\alpha/2)-\tan^2(\theta_c/2)}$$
$$\int_{\alpha}^{\pi} \sin\theta\, d\theta\, \sec^3(\theta/2)/\left[\sqrt{\tan^2(\theta/2)-\tan^2(\alpha/2)}\,(\tan^2(\theta/2)-\tan^2(\theta_c/2))\right]$$

which does not look like a very pleasant integral. There are now FOUR pieces to this thing: the square root pole, the full pole, the $\sec^3(\theta/2)$ inversion factor, and now a $\sin\theta$ area factor. All factors have to be there. I think I have at least got this general functional form correct.

Let's make a few definitions right here:

$$A = \tan(\alpha/2) \qquad C = \tan(\theta_c/2)$$

Then we have

$$q_1/Q_2 = +(1/2\pi)\sqrt{1+C^2}\sqrt{A^2-C^2}\int_{\alpha}^{\pi} \sin\theta\, d\theta\, \sec^3(\theta/2)/\left[\sqrt{\tan^2(\theta/2)-A^2}\,(\tan^2(\theta/2)-C^2)\right] \quad (*)$$

$$= +(1/2\pi)\sqrt{1+C^2}\sqrt{A^2-C^2}\; I$$

So we now want to consider:

$$I = \int_{\alpha}^{\pi} \sin\theta\, d\theta\, \sec^3(\theta/2)/\left[\sqrt{\tan^2(\theta/2)-A^2}\,(\tan^2(\theta/2)-C^2)\right]$$

At this point, in Appendix E, for a specific valid of α and θ_c, I numerically integrate the integral I shown above, and I thus compute q_1/Q_2 . My result agrees precisely with the Smythe claim which is

$$q_1/Q_2 = (2/\pi)\sin^{-1}[\cos(\alpha/2)/\cos(\theta_c/2)]$$

Therefore, I know that my expression (*) above for q_1/Q_2 is correct. I have done the physics correctly, and now it is "only" a matter of doing the integral. Specifically: (1) I have applied the reciprocity rule correctly; (2) I have done the inversion correctly; (3) I have correctly computed the charge density $\sigma(\theta)$ on the bowl; (4) I written the correct integral for the total charge. (4) I have correctly computed both q_1 and Q_2 so to speak. Again, all the physics is done!

3. Transforming the integral to a standard form

Note: I managed to not mix up A = tan(α/2) with A = radius of bowl, ouch, bad notation.

My first inclination is to set // $A = \tan(\alpha/2)$ $C = \tan(\theta_c/2)$ $A > C$

$$x = \tan(\theta/2) \qquad dx = \sec^2(\theta/2)\,(1/2)\,d\theta \qquad \Rightarrow \qquad \sec^2(\theta/2)\,d\theta = 2\,dx$$

Endpoints in x are

$$\text{lower} = \tan(\alpha/2) = A \qquad \text{upper} = \tan(\pi/2) = \infty$$

$$x^2 = \tan^2(\theta/2) = (1 - \cos\theta)/(1+\cos\theta)$$

$$\Rightarrow \qquad \cos\theta = (1-x^2)/(1+x^2)$$
$$\sin\theta = 2x/(1+x^2)$$

$$\sec^2(\theta/2) = 1 + x^2 \qquad \Rightarrow \qquad \sec(\theta/2) = \sqrt{x^2+1}$$

So let's install these changes into our integral:

$$I = \int_{\alpha}^{\pi} \sin\theta \; d\theta \; \sec^2(\theta/2) \; \sec(\theta/2)/\left[\sqrt{\tan^2(\theta/2) - A^2}\,(\tan^2(\theta/2) - C^2)\right]$$

$$= \int_{A}^{\infty} 2x/(1+x^2) \; * \; 2\,dx \; * \; \sqrt{x^2+1} \; / \left[\sqrt{x^2 - A^2}\,(x^2 - C^2)\right]$$

$$= 4 \int_{A}^{\infty} dx\, x/\left[\sqrt{x^2+1}\,\sqrt{x^2 - A^2}\,(x^2 - C^2)\right] = 4\,J$$

We have then [in this integral, x > A and we know A > C so $x^2 - C^2 > 0$, integral J is positive]

$$J(A,C) = \int_{A}^{\infty} dx\, x/\left[(x^2 - C^2)\sqrt{x^2+1}\,\sqrt{x^2 - A^2}\right] \qquad \text{// all red-checked}$$

The presence of xdx tells us to change variables to $y = x^2$ and $dy = 2xdx$ so that

$$J(A,C) = (1/2) \int_{A^2}^{\infty} dy / [(y - C^2) \sqrt{y+1} \sqrt{y - A^2}]$$

Before going on, let's recap to this point:

$$q_1/Q_2 = + (1/2\pi) \sqrt{1+C^2} \sqrt{A^2-C^2} * I = + (1/2\pi) \sqrt{1+C^2} \sqrt{A^2-C^2} * (4J)$$

$$\Rightarrow q_1/Q_2 = (2/\pi) \sqrt{1+C^2} \sqrt{A^2-C^2} J$$

where J is our integral. But Smythe is claiming that

$$q_1/Q_2 = (2/\pi) \sin^{-1} [\cos(\alpha/2) /\cos(\theta_c/2)] = (2/\pi) \sin^{-1} [\sec(\theta_c/2) /\sec(\alpha /2)]$$

$$= (2/\pi) \sin^{-1} [\sqrt{C^2+1} /\sqrt{A^2+1}] \quad \text{// using our sec}(\theta/2) \text{ result above}$$

so we hope to show this result:

$$\sqrt{1+C^2} \sqrt{A^2-C^2} J(A,C) = \sin^{-1}[\sqrt{C^2+1} /\sqrt{A^2+1}]$$

which is to say, we want to show that

$$J(A,C) = [\sqrt{1+C^2} \sqrt{A^2-C^2}]^{-1} \sin^{-1}[\sqrt{C^2+1} /\sqrt{A^2+1}]$$

4. Looking up the integral, success!

This is one of those funny integrals that GR does not directly present, but here is one way:

$$J = (1/2) \int_{A^2}^{\infty} dx / [\sqrt{x-C^2} \sqrt{x-C^2} \sqrt{x+1} \sqrt{x - A^2}]$$

and they have stuff on that. You can do the definite integral according to this: (GR7 p 103) n=1

See also 2.252

2.281[3]
$$\int \frac{dx}{(x+p)^n \sqrt{R}} = -\int \frac{t^{n-1}\,dt}{\sqrt{c + (b-2pc)t + (a-bp+cp^2)\,t^2}} \qquad \left[t = \frac{1}{x+p} > 0\right]$$

But today, the fastest way to do this is the Wolfram Integrator. Maple gives some limit mess, so go with this:

$$J = (1/2) \int_{A^2}^{\infty} dx / [(x - C^2) \sqrt{x+1} \sqrt{x - A^2}]$$

and

$$\int \frac{1}{(x-c^2)\sqrt{x+1}\sqrt{x-a^2}}\,dx =$$

$$\frac{2 \tan^{-1}\left(\frac{\sqrt{x+1}\sqrt{c^2-a^2}}{\sqrt{-c^2-1}\sqrt{x-a^2}}\right)}{\sqrt{-c^2-1}\sqrt{c^2-a^2}}$$

In our case, A > C so we write

$$J = (1/2)\, 2\, [\sqrt{C^2+1}\sqrt{A^2-C^2}]^{-1} \tan^{-1}[\,(\sqrt{A^2-C^2}/\sqrt{C^2+1})(\sqrt{x+1}/\sqrt{x-A^2})\,]\,|_{A^2}^{\infty}$$

$$= [\sqrt{C^2+1}\sqrt{A^2-C^2}]^{-1}\{\,\tan^{-1}(\infty) - \tan^{-1}(\sqrt{A^2-C^2}/\sqrt{C^2+1})\}$$

$$= [\sqrt{C^2+1}\sqrt{A^2-C^2}]^{-1}\{\,\pi/2 - \tan^{-1}(\sqrt{A^2-C^2}/\sqrt{C^2+1})\,\}$$

Draw a triangle with angle $\psi = \tan^{-1}(\sqrt{A^2-C^2}/\sqrt{C^2+1})$ and you see that

$$\{\,\pi/2 - \tan^{-1}(\sqrt{A^2-C^2}/\sqrt{C^2+1})\,\} = \{\pi/2 - \psi\} = \sin^{-1}(\sqrt{C^2+1}/\sqrt{A^2+1})$$

so we conclude that

$$J = [\sqrt{C^2+1}\sqrt{A^2-C^2}]^{-1} \sin^{-1}(\sqrt{C^2+1}/\sqrt{A^2+1})$$

which we compare with our desired result noted above:

$$J(A,C) = [\sqrt{1+C^2}\sqrt{A^2-C^2}]^{-1} \sin^{-1}[\sqrt{C^2+1}/\sqrt{A^2+1}]$$ QED!

Part 4: Junk left over from making an error converting from θ to x.

****
****
****

All the work below was done because I made one small algebra mistake in the above work when I did it the first time. I omitted a factor of x from the transformed integral, that was it (shown in red above). This error caused the integral to be a messy elliptic function deal which was completely wrong and led me on a wild goose chase. I did not know where I was making the error until I found it, so I was debugging by doing numerical integrations to see at what point I was picking up the error. I narrowed it down to the transforming of the integral from θ to x. Even though I had checked this transformation about 4 times, I did not detect the error until I knew for sure it was in this transformation, then I found it. So it goes.

****
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****

In Section 3.157 of GR7 page 284 we find exactly this integral apart from an overall sign:

5. $$\int_b^u \frac{dx}{(p-x^2)\sqrt{(a^2+x^2)(x^2-b^2)}} = \frac{1}{p(p-b^2)\sqrt{a^2+b^2}}\left\{b^2\,\Pi\left(\varepsilon, \frac{p}{p-b^2}, s\right) + (p-b^2)\,F(\varepsilon, s)\right\}$$
$$[u > b > 0, \quad p \neq b^2] \qquad \text{BY (211.14)}$$

which fits our mold if we set

p = C^2 a = 1 b = A u = ∞

The quantities ε and s from earlier in GR7 :

$$\varepsilon = \arccos\frac{b}{u}, \qquad s = \frac{a}{\sqrt{a^2+b^2}},$$

So in our case we have

ε = $\cos^{-1}(A/\infty) = \cos^{-1}(0) = \pi/2$

s = $1/\sqrt{1+A^2}$

Filling in, then, we have (recall that A > C)

-J = $[\, C^2\,(C^2-A^2)\,(1+A^2)^{1/2}]^{-1}\,\{\; A^2\,\Pi\,(\pi/2,\, C^2/(C^2-A^2),\, 1/\sqrt{1+A^2}\,) + (C^2-A^2)F(\pi/2,\, 1/\sqrt{1+A^2}\,)\,\}$

which is pretty ugly looking. This is my first-ever application of the elliptic integral of the third kind. I quote the exact forms GR uses for these two elliptic integrals:

$$\Pi(\varphi, n, k) = \int_0^{\varphi} \frac{d\alpha}{(1-n\sin^2\alpha)\sqrt{1-k^2\sin^2\alpha}} = \int_0^{\sin\varphi} \frac{dx}{(1-nx^2)\sqrt{(1-x^2)(1-k^2x^2)}}$$
BY (110.04)

$$F(\varphi,k)=\int_0^{\varphi}\frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}=\int_0^{\sin\varphi}\frac{dx}{\sqrt{(1-x^2)\,(1-k^2x^2)}}$$

We can go with the GR7 usage and say

$$1.\qquad \boldsymbol{K}(k)=F\left(\frac{\pi}{2},k\right)=\boldsymbol{K}'\,(k')$$

Our integral evaluation then involves the complete elliptic integrals of the first and the third kind! Here is a little snippet to confirm the naming of this third kind object, where we are duly warned about that fact that there is a sign convention issue with the "n" argument. Morgan Ward is probably embarrassed that his very first and main equation for his article contains a typo: the radical should contain $\sin^2\varphi$.

THE CALCULATION OF THE COMPLETE ELLIPTIC INTEGRAL OF THE THIRD KIND

MORGAN WARD, California Institute of Technology

1. Introduction. The difficulty of computing an elliptic integral of the third kind is well known; even a computation of the complete integral

$$(1.1)\qquad \int_0^{\frac{1}{2}\pi}\frac{d\phi}{(1+\nu\sin^2\phi)\sqrt{(1-k^2\sin\phi)}}$$

for numerical values of its parameters ν and k is a time-consuming operation. Apparently it is only very recently that any systematic tabulation has been made. I develop here a very efficient way to compute this integral when the parameters ν and k are real. The procedure is well adapted to a digital computer, and is so easily programmed that no tables should be necessary in the future.

If the complete Π integral could be written as some simpler functions, surely he would tell us. None of my sources shows and "simpler results" for the third kind elliptic integral when it is "complete".

Status. So as usual my ship has run aground. Either I have made an error, or the integral of the charge on the bowl involves some horrible elliptic integral of the third kind, and we will never get the simple result quoted in the problem. I was hopeful on this second attempt at problem 40, but it has just crashed.

Status 5 PM. I have done a full and complete red check on the above, and it is all correct, to the best of my abilities, such as they are. Let's look first at my claimed result

$$V(\theta_c) = (q_1/Q_2)\,V_0 = +\,(1/2\pi)\,\sqrt{1+C^2}\,\sqrt{A^2-C^2}\;\;I\;\;V_0$$

$$= +\,1/(2\pi)\,*\,\sqrt{1+C^2}\,\sqrt{A^2-C^2}\;\;(4)\;J(A,C)\;V_0$$

and here is Smythe's claimed result

$$V(\theta_c) = (2/\pi)\ \sin^{-1}(\cos\alpha./\cos\theta_c)\ V_0$$

So the claim he makes is this:

$$(2/\pi)\ \sin^{-1}(\cos\alpha./\cos\theta_c) = 1/(2\pi) * \sqrt{1+C^2}\sqrt{A^2-C^2}\ \ (4)\ J(A,C)$$

$$4\ \sin^{-1}(\cos\alpha./\cos\theta_c) = \sqrt{1+C^2}\sqrt{A^2-C^2}\ \ (4)\ J(A,C)$$

$$\sin^{-1}(\cos\alpha./\cos\theta_c) = \sqrt{1+C^2}\sqrt{A^2-C^2}\ J(A,C)$$

$$J(A,C) = 1/[\sqrt{1+C^2}\sqrt{A^2-C^2}\ ]\ \sin^{-1}[\ \{\ (1-A^2)/(1+A^2)\ \}\ /\ \{\ (1-C^2)/(1+C^2)\}\ ]$$

which agrees with my earlier statement of this same "wish" above.

This very well could be correct and I just took a convoluted path to doing the integral which gave an obscure answer that equals the above, but that is just not very obvious.

Let's take another look at our answer for J

$$-J = [\ C^2\ (C^2-A^2)\ (1+A^2)^{1/2}]^{-1}\ \{\ A^2\ \Pi\ (\pi/2,\ C^2/(C^2-A^2),\ 1/\sqrt{1+A^2}\ ) + (C^2-A^2)F(\pi/2,\ 1/\sqrt{1+A^2}\ )\ \}$$

Notice that both Π and F are "complete", and also that both have the same k value.

Appendix A: Working backwards from Smythe's Result

Consider the following little development:

$$\sin^{-1} a = \int_0^a dx/\sqrt{1-x^2} \qquad \text{// Maple OK}$$

$$\sin^{-1}(u/v) = \int_0^{u/v} dx/\sqrt{1-x^2}$$

Now rescale the variable so that

$$y = x\ (v/u) \qquad dy = dx\ (v/u) \qquad x = uy/v$$

Then we have

$$\sin^{-1}(u/v) = \int_0^1 (u/v)dy\ /\sqrt{1-(uy/v)^2} = \int_0^1 (u)dy\ /\sqrt{v^2-(uy)^2} = u\ \int_0^1 dy\ /\sqrt{v^2-(uy)^2}$$

So we have this little "integral representation" :

$\sin^{-1}(u/v) = u \int_0^1 dy / \sqrt{v^2 - u^2 y^2})$ // Maple OK

In particular, we can say

$\sin^{-1}(\cos\alpha./ \cos\theta_c) = \cos\alpha \int_0^1 dy / \sqrt{\cos^2(\theta_c) - \cos^2(\alpha)\, y^2})$

So this suggests that maybe I should try to recast my integral into a $\int_0^1$ type integral.

Appendix B: Another Hint from a Specialty Paper on Elliptic Integrals.

$$(\sin^{-1} x)/x = R_C(1 - x^2, 1), \qquad -1 \le x \le 1;$$

$$R_C(a, b) = R_F(a, b, b)$$

$$R_F(a, b, c) = \frac{1}{2} \int_0^\infty (t + a)^{-1/2} (t + b)^{-1/2} (t + c)^{-1/2} \, dt$$

This R stuff is called a Carlson canonical form. There are several of these. The upshot is this:

$(\sin^{-1}x)/x = R_F(1-x^2,1,1) = (1/2) \int_0^\infty dt\, (t + (1-x^2))^{-1/2} (t+1)^{-1}$

In our case, we have $x = \cos\alpha/\cos\theta_c$ so you could put this in to get

$\sin^{-1}(\cos\alpha/\cos\theta_c) = (\cos\alpha/\cos\theta_c) * (1/2) \int_0^\infty dt\, (t + (1-(\cos\alpha/\cos\theta_c)^2))^{-1/2} (t+1)^{-1}$

Now suppose we let

$A = \tan(\alpha/2)$ $C = \tan(\theta_c/2)$

Then we can say that

$\cos\alpha = (1-A^2)/(1+A^2)$

$\cos\theta_c = (1-C^2)/(1+C^2)$ $\sec\theta_c = (1+C^2)/(1-C^2)$

so the last integral shown above would become

$\int_0^\infty dt\, (t+1)^{-1} (t + (1-(\cos\alpha/\cos\theta_c)^2))^{-1/2}$

$$= \int_0^\infty dt\,(t+1)^{-1}\,(t + (1-\cos^2\alpha\,\sec^2\theta_c))^{-1/2}$$

$$= \int_0^\infty dt\,(t+1)^{-1}\,[(t + 1) - \cos^2\alpha\,\sec^2\theta_c]^{-1/2}$$

$$= \int_0^\infty dt\,(t+1)^{-1}\,[(t + 1) - (1-A^2)^2\,(1+C^2)^2 / ((1+A^2)^2\,(1-C^2)^2)]^{-1/2}$$

$$= ((1+A^2)^2\,(1-C^2)^2)^{1/2} \int_0^\infty dt\,(t+1)^{-1}\,[(t + 1)\,(1+A^2)^2\,(1-C^2)^2 - (1-A^2)^2\,(1+C^2)^2\,]^{-1/2}$$

$$= (1+A^2)\,(1-C^2) \int_0^\infty dt\,(t+1)^{-1}\,[(t + 1)\,(1+A^2)^2\,(1-C^2)^2 - (1-A^2)^2\,(1+C^2)^2\,]^{-1/2}$$

So at least this is something written in terms of our half angle tangents that seem to appear naturally when we set up this problem. So let's go back here:

$$J = \int_A^\infty dx\; 1/\,[\sqrt{x^2+1}\,\sqrt{x^2 - A^2}\;\,(C^2 - x^2)\,]$$

Suppose we now try

$y = x^2-A^2$ => endpoints become 0 and ∞ $x^2 = y+A^2$

$dy = 2xdx$ => $dx = (1/2)dy / \sqrt{y+A^2}$

Then we have

$$J = \int_0^\infty (1/2)dy\,(1/[\,\sqrt{y+A^2}\,)\;1/[\,\sqrt{y+A^2+1}\,\sqrt{y}\;\,(C^2-A^2-y)]$$

Now rescale so that

$y = (C^2-A^2)z$ $dy = (C^2-A^2)dz$

Then we have

$$J = (1/2)\,(C^2-A^2) \int_0^\infty dz\;\,(1/[\,\sqrt{(C^2-A^2)z + A^2}\,)\;1/[\,\sqrt{(C^2-A^2)z + A^2+1}\,\sqrt{(C^2-A^2)z}\;\,(C^2-A^2-(C^2-A^2)z)]$$

$$= (1/2)(1/\sqrt{(C^2-A^2)}\,) \int_0^\infty dz\;\,(1/[\,\sqrt{(C^2-A^2)z + A^2}\,)\;1/[\,\sqrt{(C^2-A^2)z + A^2+1}\,\sqrt{z}\;\,(1-z)]$$

No good! This does not look like the t integral above although I roughly have the pole and range OK.

Appendix C: Combining the complete integrals.

On scratch paper I am able to show this fact:

$$J = [\, C^2 (C^2 - A^2) (1+A^2)^{1/2}]^{-1} \{\; A^2 \Pi (\pi/2, C^2/(C^2-A^2), 1/\sqrt{1+A^2}\,) + (C^2-A^2) F(\pi/2, 1/\sqrt{1+A^2}\,) \}$$

$$= [\, (C^2 - A^2) (1+A^2)^{1/2}]^{-1} \int_0^{\pi/2} d\alpha \; (1/\sqrt{1-k^2 \sin^2\alpha}\,) \cos^2\alpha \,/\, (1-n \sin^2\alpha)$$

where $n = C^2/(C^2-A^2)$ and $k = 1/\sqrt{1+A^2}$. So there was some significant combining of things. We can then say

$$x = \sin\alpha \qquad dx = \cos\alpha \, d\alpha$$

$$J = [\, (C^2 - A^2) (1+A^2)^{1/2}]^{-1} \int_0^1 dx \; (1/\sqrt{1-k^2x^2}\,) \sqrt{1-x^2} \,/\, (1-nx^2)$$

$$= [\, (C^2 - A^2) (1+A^2)^{1/2}]^{-1} (1/2) \int_0^1 dy \; (1/\sqrt{y}\,)(1/\sqrt{1-k^2y}\,) \sqrt{1-y} \,/\, (1-ny)$$

which is still of the form "a pole and three square roots" in the denominator.

Appendix D: try some numbers.

I don't know whether I have made a mistake in setting up the problem, or whether I just don't know how to get the integral in the correct form. It might be good to try some numbers. Here is the thing we would want to check:

$$\sin^{-1}[\, \cos(\alpha/2)/ \cos(\theta_c/2)\,] = \sin^{-1}[\sqrt{C^2+1}\; /\sqrt{A^2+1}\,] = \sqrt{1+ C^2}\, \sqrt{A^2-C^2}\; J(A,C)$$

where

$$A = \tan(\alpha/2) \qquad C = \tan(\theta_c/2)$$

$$-J = [\, C^2 (C^2 - A^2) (1+A^2)^{1/2}]^{-1} \{\; A^2 \Pi (\pi/2, C^2/(C^2-A^2), 1/\sqrt{1+A^2}\,) + (C^2-A^2)\, F(\pi/2, 1/\sqrt{1+A^2}\,)) \}$$

$$J = [\, C^2 (A^2 - C^2) (1+A^2)^{1/2}]^{-1} \{\; A^2 \Pi (\pi/2, -C^2/(A^2-C^2), 1/\sqrt{1+A^2}\,) - (A^2-C^2)\, F(\pi/2, 1/\sqrt{1+A^2}\,) \}$$

In this last form, I have always shown A^2-C^2 which we know is positive. We also have this fact from earlier

$$\sin^{-1}(\, \cos\alpha./ \cos\theta_c) = \sin^{-1}[\sqrt{C^2+1}\; /\sqrt{A^2+1}\,]$$

So here is what we need to show:

$$\sin^{-1}[\sqrt{C^2+1}\ /\sqrt{A^2+1}\ ] = \sqrt{1+C^2}\,\sqrt{A^2-C^2}\ * [\ C^2\ (A^2-C^2)\ (1+A^2)^{1/2}]^{-1}\ \ \{\ \}$$

$$= \sqrt{1+C^2}\,\sqrt{A^2-C^2}\ /\ [\ C^2\ (A^2-C^2)\,\sqrt{1+A^2}\ ]\ \ \{\}$$

$$= \sqrt{1+C^2}\ /\ [\ C^2\,\sqrt{A^2-C^2}\ \ \sqrt{1+A^2}\ ]\ \{\ \ A^2\ \Pi\ (\pi/2,\ -C^2/(A^2-C^2),\ 1/\sqrt{1+A^2}\ ) - (A^2-C^2)\ F(\pi/2,\ 1/\sqrt{1+A^2}\ )\ \}$$

I will now attempt to see if "numbers" from Maple agree in some test cases for A and C.

But first, let's see how Maple defines its elliptic integrals and compare with GR. First, I cut and paste the Maple definitions to see what they look like

$$\int_0^z \frac{1}{\sqrt{1-t^2}\,\sqrt{1-k^2\,t^2}}\,dt \qquad \int_0^z \frac{1}{(1-\nu\,t^2)\,\sqrt{1-t^2}\,\sqrt{1-k^2\,t^2}}\,dt$$

so comparing with GR above we see that

F(φ,k) = EllipticF(sinφ,k)

Π(φ,n,k) = EllipticPi(sinφ,n,k)

For the Maple functions replace the π/2 of first argument with sin(π/2) = 1. So define

f1 = $\sin^{-1}[\sqrt{C^2+1}\ /\sqrt{A^2+1}\ ]$

f2 = $\{\ \ A^2\ \Pi\ (\pi/2,\ -C^2/(A^2-C^2),\ 1/\sqrt{1+A^2}\ ) - (A^2-C^2)K(1/\sqrt{1+A^2}\ )\ \}$ = t1 + t2

f3 = $\sqrt{1+C^2}\ /\ [\ C^2\,\sqrt{A^2-C^2}\ \ \sqrt{1+A^2}\ ]$

f4 := f3 * f2

Then we want to see if

f1 = f4

In my first example with $\alpha = \pi/8$ and $\theta_c = \pi/10$, the two do NOT agree:

```
evalf(f1);
```

1.452497859

```
evalf(f4);
```

5.667447608

```
evalf(f4)/evalf(f1);
```

3.901862969

I have checked that I entered the above equations correctly in Maple. Changing the sign of "n" makes things worse, so that does not fix it. So there are several possibilities:

(1) I made an error in transcribing the integral from θ to x. In Appendix E below, I have convinced myself that my θ integral is correct. I include here the possibility that I have made an error in the various constants that go along with the integral, not just in the integrand active elements.

(2) GR7's integral is incorrect, certainly possible.

(3) Maple makes an error in computing elliptic integrals.

Appendix E: try some numbers at an earlier point

Way back early in our development, we got this evaluation for our charge ratio:

$$q_1/Q_2 = +(1/2\pi)\sqrt{1+C^2}\sqrt{A^2-C^2}$$

$$\int_{\alpha}^{\pi} \sin\theta\, d\theta\, \sec^3(\theta/2)/\left[\sqrt{\tan^2(\theta/2) - A^2}\,(\tan^2(\theta/2) - C^2)\right]$$

I will now try a numerical integral with the same numbers of my example above. Start integration at 11:25 after saving the Maple file just in case it crashes. This is where I could use a faster machine. It does the above integral in about 3 minutes

```
II := evalf(Int(sin(theta)* sec(theta/2)^3 * (  tan(theta/2)^2 -
A^2)^(-1/2)*(tan(theta/2)^2-C^2)^(-1),theta = alpha..Pi));
```

$$II := 47.68740112$$

```
Int(sin(theta)* sec(theta/2)^3 * (  tan(theta/2)^2 -
A^2)^(-1/2)*(tan(theta/2)^2-C^2)^(-1),theta = alpha..Pi);
```

$$\int_{\frac{1}{8}\pi}^{\pi} \frac{\sin(\theta)\sec\left(\frac{1}{2}\theta\right)^3}{\sqrt{\tan\left(\frac{1}{2}\theta\right)^2 - \tan\left(\frac{1}{16}\pi\right)^2}\left(\tan\left(\frac{1}{2}\theta\right)^2 - \tan\left(\frac{1}{20}\pi\right)^2\right)}\, d\theta$$

where I am also showing that the integral was entered correctly. I then compute the ratio:

```
ratio1 := evalf((2*Pi)^(-1)*sqrt(1+C^2)*sqrt(A^2-C^2)*II);
```

$$ratio1 := .9246888540$$

Now according to Smythe, this same ratio should be

$$q_1/Q_2 = (2/\pi)\ \sin^{-1}(\cos(\alpha/2)/\cos(\theta_c/2))$$

```
ratio2 := evalf( (2/Pi)*arcsin(cos(alpha/2)/cos(thetac/2)));
```

$$ratio2 := .9246888536$$

and I am suddenly a happy camper! I have precisely verified the above expression for q_1/Q_2 against the result Smythe gives, so I can "proceed" with the problem solution.

Appendix F: try numerical integration of the integral after transformed from θ to x

First, here is the integral in question

$$J(A,C) = \int_A^\infty dx\ 1/\left[(x^2 - C^2)\sqrt{x^2+1}\ \sqrt{x^2 - A^2}\ \right]$$

Second, here is what is supposed to be true:

$$\sqrt{1+C^2}\ \sqrt{A^2-C^2}\ \ J(A,C) = \sin^{-1}\left[\sqrt{C^2+1}\ \ /\sqrt{A^2+1}\ \right]$$

I first enter the integral into Maple

```
Int(1/((x^2-C^2)*sqrt(x^2+1)*sqrt(x^2-A^2)),x=A..infinity);
```

$$\int_A^\infty \frac{1}{(x^2 - C^2)\sqrt{x^2+1}\ \sqrt{x^2 - A^2}}\,dx$$

The numerical integration complains about the singularity. The lower end is convergent, however. So I try to fake it like this:

```
evalf(Int(1/((x^2-C^2)*sqrt(x^2+1)*sqrt(x^2-A^2)),x=A+.000000001..infinity));
```

$$46.51063458$$

The result is roughly stable as I change number of 0's. I then compute the two sides of my supposed equality above:

$$\sqrt{1+C^2}\ \sqrt{A^2-C^2}\ \ J(A,C)$$

```
g1 := evalf(sqrt(1+C^2)*sqrt(A^2-C^2)*J);
```

$$g1 := 5.666620150$$

$$\sin^{-1}[\sqrt{C^2+1}\ /\sqrt{A^2+1}\ ]$$

```
f1 := evalf(arcsin( sqrt(C^2+1)/sqrt(A^2+1)));
```

$$f1 := 1.452497859$$

This is the same discrepancy as in Appendix D. This tells me that the GR integral is correct. Therefore, I must have done the transformation incorrectly !!!! The debug process is narrowing it down.

****

Here are some debug notes from another doc I started, but then deleted. Just part of the trouble-shooting effort.

Debug Problem 40 Problems PhL 6.30.10

The main problem is that I cannot get Smythe's answer! So here I will try to recap the flow.

1. Following Smythe's hints, using the Green's reciprocation theorem, I was able to show that

$$V(\theta_c) = (q_1/Q_2)\ V_0$$

where q_1 and Q_2 are the total charges on the bowl and ring which appear in R space after you "invert" the iris and ring in R' space. Could this result itself be wrong? I just don't think so. The recip relation is linear, the only error could be a sign, there are no constant factors, nothing. I defined my S and S" situations carefully. It all fits perfectly with the hints. I just don't think this can be wrong.

2. The charge density on the iris is given by this

$$\sigma'(\rho') = -\ (q/2\pi^2)\ (\sqrt{B^2-S^2}\ /\ \sqrt{\rho'^2 - B^2}\ )\ /\ (\rho'^2-S^2)$$

where we know what B, S and ρ' and q all mean. q is the total charge on the ring, and -q is the total charge on the iris. This result I derived while doing problem 39, and it agrees with Smythe's claimed result, so this thing cannot possibly be wrong. It is the one-side charge, both sides are the same.

3. Doing the inversion itself is very simple. We have V = 0 on the iris because Smythe says it is "earthed" in problem 38

38. A charge q is at a distance b from the center of a circular hole of radius a in an infinite earthed flat conducting sheet and lies in the same plane. Show that the charge density induced on the sheet at a distance r from the charge and c from the center of the hole is $-q(a^2 - b^2)^{\frac{1}{2}}/[2\pi^2 r^2 (c^2 - a^2)^{\frac{1}{2}}]$.

The iris stays "earthed" when we superpose to form a ring. As we superpose 1000 point charges in a ring, for each case we have V = 0, so there is no Red Flag superposition problem.

So, the iris has V = 0 and has some charge σ'. The inversion result is a bowl touching the origin with V = 0, and a ring on the complementary surface. There are no "constant potentials or point charges" required in this inversion. It is a textbook simple case. There are only two facts we need to use:

(1) $\sigma(\mathbf{r}) = (a/r)^3\,\sigma'(\mathbf{r}')$ $= (r'/a)^3\,\sigma'(\mathbf{r}')$ $rr' = a^2$ $a/r = r'/a$

(2) $q_i = q_i'(a/r_i')$ => $q_G = q_{hole}\,(a/\sqrt{d^2+S^2})$ => $Q_2 = q\,(a/\sqrt{d^2+S^2})$

Is it possible that I have the ratio wrong in one or both of these claims?

Concerning (1): When I did my "first inversion" of the offset disk in R' space to get a bowl in R space, while doing problem 38, I used this inversion transformation and was successful

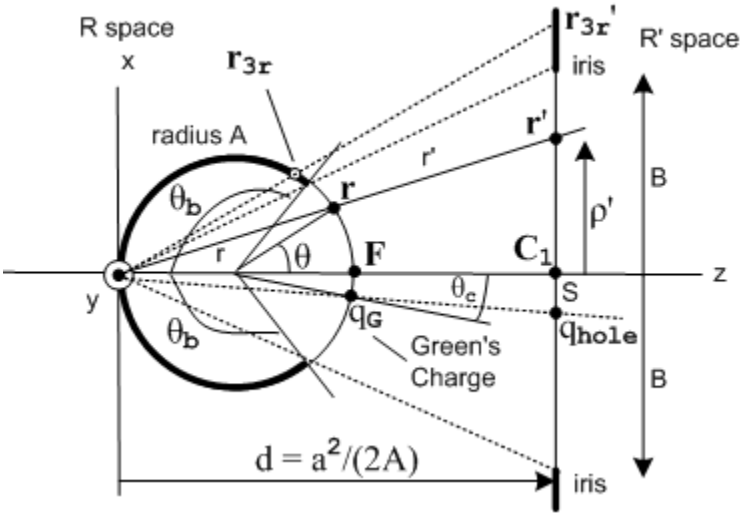
$\sigma(\mathbf{r}) = (a/R)^3\,\sigma'(\mathbf{r}')$ $R = \sqrt{x^2+y^2+z^2}$ $\mathbf{r}' = (a^2/R^2)\,\mathbf{r}$

where R was |**r**| in inversion coordinates. This is exactly what I am claiming in (1), so it must be right.

Concerning (2): this agrees with Jackson page 35 where he says q' = (a/r) q so q = (r/a)q' = q'(a/r'). This general formula is definitely correct, and I will check on the r' geometry below.

So I don't think I have any conceptual or expression errors in my inversion of the iris to the bowl.

4. Picture geometry. Here is the picture, where $\alpha = \pi - \theta_b$.



and here are ALL the claims I make regarding this picture:

$\rho'/d = \tan(\theta/2)$ $d/r' = \cos(\theta/2)$ $B/d = \tan(\alpha/2)$ $S/d = \tan(\theta_c/2) > 0$

$\rho' = d\,\tan(\theta/2)$ $B = d\,\tan(\alpha/2)$ $S = d\,\tan(\theta_c/2)$

The half angle rule is trivial based on the inscribed isosceles triangle