Smythe Problem 41 PhL 6.30.10

The problem is to find the capacitance of a spherical bowl. The claimed answer is C = (A/π){β +sinβ} in Jackson units where β is the polar angle of the bowl of radius A. The full bowl of course has C = A.

41. Starting with the preceding result find the first term in the spherical harmonic expansion for the potential of a freely charged spherical bowl all elements of which are at a distance a from the origin where its radius subtends an angle β. From this show that its capacitance is $4\epsilon a(\beta + \sin\beta)$.

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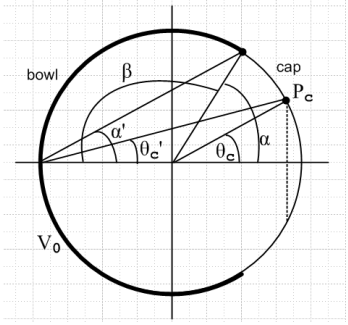
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Overview (1 page, written 11.26.10)

Problem 40 showed that, for a spherical bowl of radius A and angle α charged to potential V_0, the potential on the cap is given by $V(\theta_c) = V_o\ (2/\pi)\ \sin^{-1}[\ \cos(\alpha/2)\ /\cos(\theta_c/2)\ ]$. Since $V = V_0$ on the bowl, we thus have a full closed-surface Dirichlet boundary condition and we can use it to compute the potential everywhere either inside or outside the bowl sphere. By looking at just the leading term of the exterior solution, we are able to obtain the bowl capacitance.



In **Section 1** we do the usual Smythian fit $V(r,\theta,\varphi) = \Sigma_{n=0}^{\infty} E_n\ r^{-n-1}\ P_n(\cos\theta)$ for the external potential, and going far away this becomes $V(r) = E_0/r$, so the charge on the bowl must be $Q = E_0$. The capacitance of the bowl is then $C = Q/[V_0 - 0] = Q/V_0 = E_0/V_0$. If we can compute the coefficient E_0, we have the answer to our problem which is then $C = E_0/V_0$. We expect E_0 will be proportional to V_0 so the answer will be a function of geometry and won't depend on V_0.

In **Section 2** we use the usual orthogonality of the P functions to get an expression for any coefficient E_n

$$E_n = V_0\ A^{n+1}\ (2n+1)(1/2)$$

$$\{(2/\pi)\ \int_0^{\alpha} d\theta\ \sin\theta\ P_n(\cos\theta)\ \sin^{-1}[\ \cos(\alpha/2)/\cos(\theta/2)\ ])\ +\ \int_{\alpha}^{\pi} d\theta\ \sin\theta\ P_n(\cos\theta)\ \}$$

where the first term comes from the cap, and the second from the bowl. We don't do these integrals for the general case, but setting n = 0 we find

$$E_0 = V_0\ A\ (1/2)$$
$$\{(2/\pi)\ \int_0^{\alpha} d\theta\ \sin\theta\ \sin^{-1}[\ \cos(\alpha/2)/\cos(\theta/2)\ ])\ +\ \int_{\alpha}^{\pi} d\theta\ \sin\theta\ \}$$

The second integral is 1+cos(α). The first integral is $I \equiv \int_0^{\alpha} d\theta\ \sin\theta\ \sin^{-1}[\ \cos(\alpha/2)/\cos(\theta/2)\ ])$ which I transform into the form $4\ \int_1^{1/a} dx\ \sin^{-1}(ax)/\ x^3$. Maple can do this integral, and after some fiddling we find that $(2/\pi)\ I\ = 1 - \cos\alpha\ - (2/\pi)\alpha + (2/\pi)\sin\alpha$. Adding the second integral 1+cos(α), the cosα terms cancel and we get $\{..\} = 2 - (2/\pi)\alpha + (2/\pi)\sin\alpha\ = (2/\pi)[\ \pi - \alpha + \sin\alpha\ ]$ so we find that

$$E_0 = V_0\ A\ (1/\pi)\ \{\pi\ - \alpha + \sin\alpha\ \} \qquad \Rightarrow \qquad C = (A/\pi)\ \{\pi\ - \alpha + \sin\alpha\ \}$$

In our picture α is the polar angle of the cap. If we define β = π-α to be the polar angle of the bowl, we get sinβ = sinα and the result takes a simpler form, which is the result Smythe gives as the answer:

$$C = (A/\pi)\ \{\beta + \sin\beta\} \qquad // \quad \partial_\beta C = (A/\pi)[\ 1 + \cos\beta\ ] > 0$$ so C increases with β

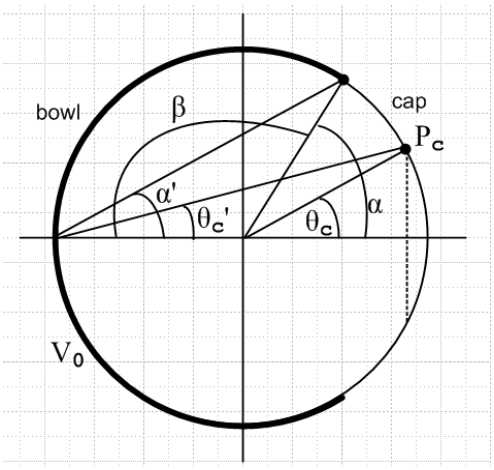
In **Section 3** we summarize all the results. Notice that we are able to obtain C by computing only a *single* coefficient E_0 which involves "not too bad" integrals. Assuming we can compute *all* the E_n, we then have a full solution to the charged bowl problem right here! We know the potential outside as expanded above. Inside it is the same thing with r^n in place of r^{-n-1}. We could then compute $\partial_n V$ at the inner and outer surface of the bowl to get σ_{inner} and σ_{outer}. All results would be in expressed in a Legendre P expansion. I don't think I have seen the results expressed this way. Smythe in the next problems leads us to a full bowl solution where we don't need any expansions like this. [In fact, I don't think I have seen the *potential* of the charged bowl expressed anywhere in any terms whatsoever, so our solution here is maybe the best I have. Somewhere I may have done the charged bowl in toroidals. The charged bowl is also the inversion of the axial Green's function for a disk, something I also have never computed successfully. There is also the solution using dual integral equations a la Sneddon or Canonical.]

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1. Introduction

In problem 40 we learned the potential on a charged bowl, but only at a point on the complementary surface. Here was the picture

$$\alpha' = \alpha/2 \qquad \theta_c' = \theta_c/2$$



and here was our result, where I replace θ_c by θ,

$$V(\theta) = V_o\,(2/\pi)\,\sin^{-1}\,[\,\cos(\alpha/2)\,/\cos(\theta/2)\,] \qquad \text{where } \theta \text{ is on the cap}$$

I put the $\hat{\mathbf{z}}$ axis to the right so we have polar angles relative to it. The expansion for the potential outside the sphere must be this

$$V(r,\theta,\varphi) = \Sigma_{n=0}^{\infty}\, E_n\, r^{-n-1}\, P_n(\cos\theta)$$

because we can see that that the potential is azisym. If we evaluate our Smythian form on the sphere at point P we get

$$V(A,\theta,\varphi) = \Sigma_{n=0}^{\infty}\, E_n\, A^{-n-1}\, P_n(\cos\theta) \;= V_o\,(2/\pi)\,\sin^{-1}\,[\,\cos(\alpha/2)\,/\cos(\theta/2)\,] \qquad \theta \le \alpha$$

$$V(A,\theta,\varphi) = V_0 \qquad \alpha \le \theta \le \pi$$

We can then use orthogonality to compute the E_n coefficients. Imagine we have done that. If then go to large r, we find that

$$V(r,\theta,\varphi) = \Sigma_{n=0}^{\infty}\, E_n\, r^{-n-1}\, P_n(\cos\theta) \;\approx\; E_0\, P_0(\cos\theta)\, r^{-1} = \; E_0\, r^{-1} \;=\; Q/r$$

so we identify coefficient E_0 as Q, the total charge on the bowl. We will of course find that E_0 is proportional to V_0

$$E_0 \;= V_0\, f(\alpha)$$

then the capacitance is given by

$$Q = CV \Rightarrow \qquad C = Q/V_0 = f(\alpha)$$

and then we will have done the problem.

2. Computing the coefficients.

We start with this,

$$V(A,\theta,\varphi) = \Sigma_{n=0}^{\infty} E_n A^{-n-1} P_n(\cos\theta) \qquad (*)$$

Go look up Legendre orthogonality from Schaum p 147

$$\int_0^{\pi} P_n(\cos\theta) P_m(\cos\theta) \sin\theta \, d\theta = \delta_{mn} \, 2/(2n+1)$$

Now apply $P_m(\cos\theta) \sin\theta$ to both sides of the equation (*) and integrate $d\theta$

$$\Sigma_{n=0}^{\infty} E_n A^{-n-1} \int_0^{\pi} P_m(\cos\theta) P_n(\cos\theta) \sin\theta \, d\theta = \int_0^{\pi} P_m(\cos\theta) \sin\theta \, V(A,\theta,\varphi) \, d\theta$$

$$\Sigma_{n=0}^{\infty} E_n A^{-n-1} \delta_{mn} \, 2/(2n+1) = \int_0^{\pi} P_m(\cos\theta) \sin\theta \, V(A,\theta,\varphi) \, d\theta$$

$$E_m A^{-m-1} \, 2/(2m+1) = \int_0^{\pi} P_m(\cos\theta) \sin\theta \, V(A,\theta,\varphi) \, d\theta$$

Now replace m by n to avoid future confusion

$$E_n A^{-n-1} \, 2/(2n+1) = \int_0^{\pi} P_n(\cos\theta) \sin\theta \, V(A,\theta,\varphi) \, d\theta$$

$$E_n = A^{n+1} (2n+1)(1/2) \int_0^{\pi} P_n(\cos\theta) \sin\theta \, V(A,\theta,\varphi) \, d\theta$$

Now, finally, we insert our potential and the integral breaks into two terms

$$E_n = A^{n+1} (2n+1)(1/2)$$
$$\{ \int_0^{\alpha} d\theta \sin\theta \, P_n(\cos\theta) \, (V_o \, (2/\pi) \sin^{-1} [\cos(\alpha/2) / \cos(\theta/2)]) + \int_{\alpha}^{\pi} d\theta \sin\theta \, P_n(\cos\theta) V_0 \}$$

or

$$E_n = V_0 \, A^{n+1} (2n+1)(1/2)$$
$$\{(2/\pi) \int_0^{\alpha} d\theta \sin\theta \, P_n(\cos\theta) \sin^{-1} [\cos(\alpha/2) / \cos(\theta/2)]) + \int_{\alpha}^{\pi} d\theta \sin\theta \, P_n(\cos\theta) \}$$

In general this is pretty nasty looking integral, but for n = 0 we get

$$E_0 = V_0 \, A \, (1/2)$$
$$\{(2/\pi) \int_0^{\alpha} d\theta \sin\theta \, \sin^{-1} [\cos(\alpha/2) / \cos(\theta/2)]) + \int_{\alpha}^{\pi} d\theta \sin\theta \}$$

Maple quickly tells us that

$$\int_{\alpha}^{\pi} d\theta \sin\theta = 1 + \cos(\alpha)$$

but then we have to study this integral

$$I = \int_{0}^{\alpha} d\theta \sin\theta \sin^{-1}[\cos(\alpha/2)/\cos(\theta/2)])$$

in terms of which we have

$$E_0 = V_0 A (1/2) \{(2/\pi) I + 1 + \cos\alpha\}$$

Even this is non-trivial. Let's try

$$x = \sec(\theta/2)$$

$$dx = \sec(\theta/2)\tan(\theta/2)(1/2)d\theta = x\sqrt{x^2-1}(1/2)d\theta$$

$$\Rightarrow \quad d\theta = 2dx/[x\sqrt{x^2-1}]$$

$$\theta = 0 \Rightarrow x = 1 \qquad \theta=\alpha \Rightarrow x = \sec(\alpha/2)$$

$$\sin\theta = 2\sin(\theta/2)\cos(\theta/2) = 2\sqrt{x^2-1}/x^2 \qquad \text{// scratch}$$

so we then have

$$I = \int_{0}^{\alpha} \sin^{-1}[\cos(\alpha/2)/\cos(\theta/2)]\sin\theta\, d\theta$$

$$= \int_{1}^{\sec(\alpha/2)} \sin^{-1}(ax) * 2\sqrt{x^2-1}/x^2 * 2dx/[x\sqrt{x^2-1}] \qquad a \equiv \cos(\alpha/2)$$

$$= 4\int_{1}^{\sec(\alpha/2)} dx \sin^{-1}(ax)/x^3$$

$$= 4\int_{1}^{1/a} dx \sin^{-1}(ax)/x^3$$

We invoke Maple

```
Int(arcsin(a*x)/x^3,x=1..1/a);
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$$\int_{1}^{\frac{1}{a}} \frac{\arcsin(a\,x)}{x^3}\,dx$$

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int(arcsin(a*x)/x^3,x=1..1/a);
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$$-\frac{1}{4}\pi\,a^2+\frac{1}{2}\arcsin(a)+\frac{1}{2}\sqrt{1-a^2}\,a$$

So we then have

$I = -\pi a^2 + 2\sin^{-1}a + 2a\sqrt{1-a^2}$

Now insert $a = \cos(\alpha/2)$ and we get

$I = -\pi\cos^2(\alpha/2) + 2\sin^{-1}[\cos(\alpha/2)] + 2\cos(\alpha/2)\sin(\alpha/2)$

Draw a triangle to see that

$\psi = \sin^{-1}[\cos(\alpha/2)] = (\pi/2 - \alpha/2)$

so that

$I = -\pi\cos^2(\alpha/2) + (\pi-\alpha) + 2\cos(\alpha/2)\sin(\alpha/2)$

$= -\pi(1+\cos\alpha)/2 + (\pi-\alpha) + \sin(\alpha)$

$= \pi/2 - (\pi/2)\cos\alpha - \alpha + \sin(\alpha)$

This says that

$(2/\pi)\,I = (2/\pi)[\pi/2 - (\pi/2)\cos\alpha - \alpha + \sin(\alpha)]$

$= 1 - \cos\alpha - (2/\pi)\alpha + (2/\pi)\sin\alpha$

and recall from above that

$E_0 = V_0\,A\,(1/2)\,\{(2/\pi)\,I + 1 + \cos\alpha\}$

$= V_0\,A\,(1/2)\,\{1 - \cos\alpha - (2/\pi)\alpha + (2/\pi)\sin\alpha + 1 + \cos\alpha\}$

$= V_0\,A\,(1/2)\,\{2 - (2/\pi)\alpha + (2/\pi)\sin\alpha\}$

$= V_0\ A\ (1/2)(2/\pi)\ \ \{\pi\ - \alpha + \sin\alpha\ \}$

$= V_0\ A\ (1/\pi)\ \ \{\pi\ - \alpha + \sin\alpha\ \}$

The capacitance of the bowl is then

$C = A\ (1/\pi)\ \ \{\pi\ - \alpha + \sin\alpha\ \}$

Test: suppose $\alpha = 0$, then $C = A$ which is correct for a sphere.

Smythe's answer is this

$C = 4\varepsilon A(\beta + \sin\beta)$

As usual, Smythe is hazy about the meaning of angle β . "the radius of a bowl subtends an angle β"

Just for fun, instead of using α, suppose I use $\kappa = \pi-\alpha$ which would be the polar angle of the bowl (measured from the sphere center) . Then my answer becomes, using $\alpha = \pi-\kappa$,

$C = A\ (1/\pi)\ \ \{\pi\ - \alpha + \sin\alpha\ \}$

$= A\ (1/\pi)\ \ \{\kappa + \sin(\pi-\kappa)\ \}$

$= A\ (1/\pi)\ \ \{\kappa + \sin\kappa\}$

And there it is. So his β is indeed the half polar angle of the bowl. $\varepsilon = 1/4\pi$ in his convention if we want to match Jackson convention.

3. Conclusions:

(1) The capacitance of a bowl subtending polar angle β is

$C = (A/\pi)\ (\beta + \sin\beta)$ $\qquad\qquad\qquad

and I only computed E_0 in this doc. I suspect these would not be easy to compute!

(3) The potential inside the bowl is given by

$$V(r,\theta,\varphi) = \Sigma_{n=0}^{\infty} E_n A^{-n-1} (r/A)^n P_n(\cos\theta)$$

so that at the surface of the bowl, the two agree.

(4) I don't know the charge density on the bowl's surfaces. You could have to compute all the E_n and do a radial derivative on the two sides. These σ values are not related in any obvious way to those in the S situation discussed in Problem 40.