Smythe Problem 42 PhL 7.1.10

Problem 42: Find σ_i and σ_o on a charged bowl.

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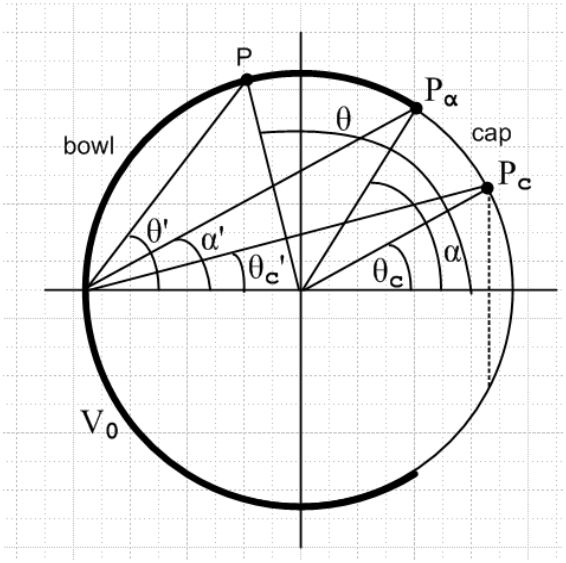
Overview (4 pages, added 11.24.10)

Section 1: statement of the problem

Here I draw a picture of a spherical bowl and its cap, just to define coordinates the way I like them. (the half angle relations follow from simple geometry)

$\theta' = \theta/2$ $\alpha' = \alpha/2$ $\theta_c' = \theta_c/2$



Problem 42 asks: what is the charge density σ on each surface of an isolated charged metal spherical bowl of radius A and polar angle α which is at of potential V_0. Smythe *claims* these answers:

$$\sigma_{inner}(\theta) = V_0/(4\pi^2 A)\ \{\ \sin(\alpha')/\sqrt{\sin^2(\theta') - \sin^2(\alpha')}\ -\ \sin^{-1}[\sin(\alpha')/\sin(\theta')]\ \}$$

$$\sigma_{outer} = V_o/(4\pi A) + \sigma_{inner}$$

It is our task to come up with these results! I might add that these are very famous results and were first obtained by Kelvin in 1869, but not published by him until 1872 as paper XV in his collected papers.

Section 2: How to make a cap of uniform σ_0

In Smythe Problem 40, we figured out the following obscure fact. Suppose we start with a metal spherical bowl as described above, but one that was grounded to potential V=0. We then add a ring of total charge $dQ_2(\theta_c)$ at angle θ_c on the *cap* of the bowl. That ring of charge causes an induced charge density $d\sigma_b(\theta)$ at all points on the grounded bowl as follows (the charge comes in from infinity on the bowl's grounding wire)

$$d\sigma_b(\theta) = dQ_2(\theta_c)\, f(\theta;\theta_c)$$

where f is the following fairly messy expression,

$$f(\theta;\theta_c) = -(1/[8\pi^2 A^2])\sec^3(\theta/2)\,(\sqrt{1+\tan^2(\theta_c/2)}\,)$$

$$(\sqrt{\tan^2(\alpha/2)-\tan^2(\theta_c/2)}\,/\,\sqrt{\tan^2(\theta/2)-\tan^2(\alpha/2)}\,)\,/\,(\tan^2(\theta/2)-\tan^2(\theta_c/2))$$

This was the conclusion of Problem 40, so we accept it here as being true. The charge density by the way is found to be the same $d\sigma_b(\theta)$ on the inner and outer surfaces of the bowl.

I then show that if we were to *superpose* a continuum of charge rings on the cap, where each ring had a total charge given as follows,

$$dQ_2(\theta_c) = 2\pi A^2\,\sigma_0\,\sin\theta_c d\theta_c$$

then those charge rings when superposed would result in a cap of constant charge density σ_0. Moreover, using the same superposition, we could compute the total charge density induced on the grounded bowl as

$$\sigma_b(\theta) = \int_0^{\alpha} d\theta_c\, d\sigma_b(\theta;\theta_c) = \int_0^{\alpha} d\theta_c\,[dQ_2(\theta_c)\, f(\theta;\theta_c)] = 2\pi A^2\,\sigma_0 \int_0^{\alpha} d\theta_c \sin\theta_c\, f(\theta;\theta_c)$$

This then is the charge density induced on the grounded bowl by a uniform cap of density σ_o. At this point, we have this formidable looking result for the charged induced on the bowl by this uniform cap:

$$\sigma_b(\theta) = -2\pi A\,\sigma_0\,(1/[8\pi^2 A^2])\sec^3(\theta/2)\,(1/\sqrt{\tan^2(\theta/2)-\tan^2(\alpha/2)}\,)\ *$$

$$\int_0^{\alpha} d\theta_c \sin\theta_c\sqrt{1+\tan^2(\theta_c/2)}\,[\,\sqrt{\tan^2(\alpha/2)-\tan^2(\theta_c/2)}\ /\,(\tan^2(\theta/2)-\tan^2(\theta_c/2))\,]$$

Section 3: transform the integral I(θ)

Here I study the integral shown above on the second line, which I call I. I am able to transform the integral into this form, after some amount of work,

$$I(\theta) = 2 \int_0^{a^2} dy \ \sqrt{a^2-y} \ / \ [(t^2 - y)\,(1+y)^{3/2}] \qquad a \equiv \tan(\alpha/2) \qquad t \equiv \tan(\theta/2)$$

I do not at this time attempt to evaluate the integral and I just call it I(θ) so our $\sigma_b(\theta)$ above is:

$$\sigma_b(\theta) = -2\pi A \sigma_0 \ (1/[8\pi^2 A^2]) \sec^3(\theta/2) \ (1/\sqrt{\tan^2(\theta/2) - \tan^2(\alpha/2)}\,) \ I(\theta) \qquad (*)$$

Section 4: the magic superposition

Now comes the very clever part; later we shall do the integral. Suppose onto the situation described above (grounded metal bowl plus uniform charged cap) we superpose a metal sphere of charge density $-\sigma_o$ lying on its outer surface (notice that such a metal sphere cannot have charge on its inner surface). I show in a reference doc listed below that this superposition is "legal" and satisfies my "Red Flag Theorem". The superposed metal sphere is placed exactly coincident with the bowl, has the same radius, etc. So here is what we have

	Problem #1	grounded sphere with $\sigma_b(\theta)$ on both surfaces and σ_0 on the cap
+	Problem #2	metal sphere of charge density $-\sigma_o$ on outer surface, for which situation we have $V_0 = -(4\pi A^2)\sigma_o/A$ on the sphere ($= -4\pi A\sigma_0$)
	Problem #3	bowl at potential V_0 and no charge on the cap

In the superposition problem #3, the bowl potential is V_0 which is the sum of 0 for problem #1 and V_0 for problem #2. For problem #3 then we have

$$\sigma_i = \sigma_b(\theta)$$
$$\sigma_o = \sigma_b(\theta) - \sigma_0 = \sigma_b(\theta) + V_0/(4\pi A)$$

But it is Problem #3 that we have been trying to solve: find the charge density on the two surfaces of a charged bowl at potential V_0. So all we need to do is compute our integral I(θ) to get $\sigma_b(\theta)$.

Section 5: Evaluate integral I(θ) and give final results

In this section I evaluate the rather painful integral I(θ) shown above. When I first did this, I got a sign error which is shown wrong as red in the raw notes below. My error was that I got a + for the $\sin^{-1}$ term instead of a minus. I found the problem, and here then is the correct integral evaluation:

$$I(\theta) = 4 \tan(\alpha/2) \cos^2(\theta/2) \ - \ 4 \cos^3(\theta/2) \sqrt{\tan^2(\theta/2) - \tan^2(\alpha/2)} \ \sin^{-1} [\sin(\alpha/2)/\sin(\theta/2)]$$

When this integral is installed in (*) above and when σ_0 is set to $-V_0/(4\pi A)$, we find that

$$\sigma_b(\theta) = V_0/(4\pi^2 A) \{ \sin(\alpha/2)/\sqrt{\sin^2(\theta/2) - \sin^2(\alpha/2)} - \sin^{-1}[\sin(\alpha/2)/\sin(\theta/2)] \}$$

and then

$$\sigma_i = \sigma_b(\theta)$$
$$\sigma_o = \sigma_b(\theta) + V_0/(4\pi A)$$

and these results are seen to match Smythe's claims as stated above.

In **Section 6** I still had my **sign error** and was trying to see if I was wrong or Smythe was wrong. I tried looking at the **limit** where the bowl has a very small polar angle and so the bowl becomes a small disk. But I found after much work that the second $\sin^{-1}$ term makes no contribution at all in this limit, and therefore the limit could not help me see whose sign was right!

In **Section 7** I specialize my results for a bowl that is a **hemisphere**. This was a problem I had long wondered about and had problems doing. The half sphere bowl results are

$$\sigma_i(\theta) = V_0/(4\pi^2 A) \{ (1/\sqrt{2}) / \sqrt{\sin^2(\theta/2) - 1/2} - \sin^{-1}[(1/\sqrt{2})/\sin(\theta/2)] \}$$

$$\sigma_o = V_o/(4\pi A) + \sigma_i$$

Back in Problem 41 we computed the Capacitance of a spherical bowl to be

$$C = A\,(1/\pi)\ \{\pi - \alpha + \sin\alpha\} \qquad \text{// C for cap of a spherical bowl of polar angle } \alpha$$

and this then becomes, for the half spherical bowl,

$$C = A\ \{1/2 + 1/\pi\} \qquad \text{// } C = A\ \{1/2 + 1/2\} \text{ for full sphere}$$

In **Section 8** I looked at outside sources and found **confirmation** of Smythe's minus sign in a Kirkland paper which quoted Kelvin's result, so I knew by then that I had made a mistake. While looking at this stuff, I was able to convert the above $\sigma_b(\theta)$ into what I call "**the Kelvin form**", namely

$$\sigma_b(\theta) = V_0/(4\pi^2 A) \{ \sqrt{1-\cos\alpha} / \sqrt{\cos\alpha - \cos\theta} - \tan^{-1}[\sqrt{1-\cos\alpha}/\sqrt{\cos\alpha - \cos\theta}] \}$$

$$\sigma_b(\theta) = V_0/(4\pi^2 A) \{ \sin(\alpha/2)/\sqrt{\sin^2(\theta/2) - \sin^2(\alpha/2)} - \sin^{-1}[\sin(\alpha/2)/\sin(\theta/2)] \} \qquad \text{// above}$$

where I show in the second line my and Smythe's "form" for the result. Kelvin's is nicer.

External Doc References

1. The sign-of-the-second-term question was finally resolved in this document

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algebraic integrals.doc

The correct sign is obtained when one properly handles the square roots of negative quantities.

2. When I do the superposition in the raw notes below, I used a "math sticky sphere" of charge, and I had to explain why you put it just outside and not just inside the metal bowl. When I replaced that math sphere with a real metal sphere, it became clear that the charge of the sphere would contribute to the charge on the outside of the bowl. A metal sphere with charge on the inner surface would not be a valid problem #2 so you can rule that out for superposition. And if you used an inner math sphere of sticky charge, that sticky sphere would still exist as a non-equilibrium surface in the final problem, but that is not the final problem we want. This subject is discussed here, and this doc includes the Red Flag theorem mentioned above.

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Superposition in electrostatics.doc

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1. Statement of the problem and changing from Smythe angle names to PhL angle names.

Units: Smythe consistently uses ε in this sense (page 179 for example)

$$V = \frac{q}{4\pi\epsilon R}$$

This we have:

$\varepsilon = \varepsilon_0$	mks units
$\varepsilon = 1/4\pi$	green Jackson cgs units.

This is the last in the series of 5 connected problems which mainly make use of inversion to obtain results that would be very hard to obtain by brute force methods. Here is the problem: (the $-\delta_1$ should be $-\sigma_1$)

42. Invert the result of problem 39 and find the charge density induced on a spherical earthed conducting bowl by a uniform charge distribution $-\delta_1$ over the remainder of the sphere. By superimposing a uniform spherical surface charge of density σ_1 show that the charge density at P on the inside or outside, respectively, of a freely charged conducting spherical bowl at potential V_0 is

$$\sigma_i = \frac{\epsilon V_0}{\pi a}\left[\frac{\sin\alpha}{(\sin^2\theta - \sin^2\alpha)^{\frac{1}{2}}} - \sin^{-1}\frac{\sin\alpha}{\sin\theta}\right], \qquad \sigma_o = \frac{\epsilon V_0}{a} + \sigma_i$$

where the angles are defined as in problem 40.

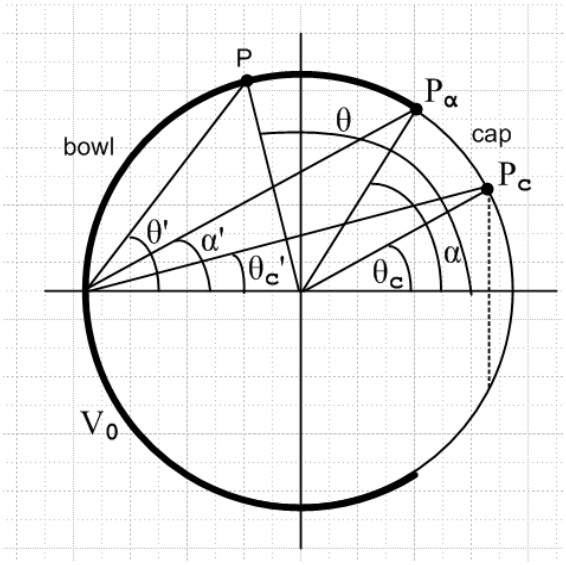
Smythe's angles are, it turns out, all angles relative to the inversion origin which is at the pole of the bowl, picture below. In terms of my names for these angles, and using $\varepsilon = 1/4\pi$, his claimed results for this problem can be restated as follows: (A is the radius of the bowl)

$$\sigma_i(\theta) = V_0/(4\pi^2 A)\ \{\ \sin(\alpha')/\sqrt{\sin^2(\theta') - \sin^2(\alpha')}\ -\ \sin^{-1}[\sin(\alpha')/\sin(\theta')]\ \}$$

$$\sigma_o = V_o/(4\pi A) + \sigma_i$$

Due to my very long history for doing this problem and earlier ones in the series, I have special names for angles that I want to stick with. There are three points of interest on the sphere, and for each point, we can express its "polar" angle relative to two origins. In each case, the "inversion angle" is half the "spherical angle", and in each case I just add a prime to the spherical angle name to get the inversion origin angle:

$\theta' = \theta/2$ $\alpha' = \alpha/2$ $\theta_c' = \theta_c/2$



I will be using the spherical angles only, and notice that these are polar angles in (0,π) range. But in fact only the "half angles" will ever appear, for example tan(θ/2) will appear, so the half angles will range from (0,π/2). These are of course the inversion angles shown above. Notice that

$$\theta > \alpha > \theta_c$$

$$\tan(\theta/2) > \tan(\alpha/2) > \tan(\theta_c) \qquad t \equiv \tan(\theta/2) \qquad a \equiv \tan(\alpha/2)$$

$$t > a$$

where I define two constants a and t which will be used in what follows.

The metal bowl shown in heavy dark is called "the bowl", while the complementary part of the sphere is called "the cap", just to have easy names. We are going to fill the cap region with uniform charge. Luckily, we can think of the c on θ_c as standing for "cap".

2. Arranging a set of ring charges on the complementary surface such that $\sigma_c = \sigma_0$.

We already know how to put a ring of charge on the cap, so the idea is to somehow "add up a bunch of rings" to cause a uniform σ to appear on this cap.

My first stop is at the Red Flag superposition theorem station. I went there and wrote Example 4, and it says that you can superpose (bowl + ring1) + (bowl + ring2) = (bowl + ring1 + ring2) because the bowl is grounded in all three "problems". So our superposition of rings of sticky charge is valid.

Now here are some quotes from Problem 40: (notice that only half-spherical angles appear!)

$$\sigma_b(\theta) = - \; Q_2 \, (1/[8\pi^2 A^2]) \sec^3(\theta/2) \, (\sqrt{1+ \tan^2(\theta_c/2)} \,)$$

$$(\sqrt{\tan^2(\alpha/2)-\tan^2(\theta_c/2)} \,/\, \sqrt{\tan^2(\theta/2) - \tan^2(\alpha/2)} \,) \,/\, (\tan^2(\theta/2)-\tan^2(\theta_c/2))$$

The $\sigma_b(\theta)$ shown above is the σ on each surface of the grounded bowl induced by a charge ring at θ_c (off the bowl) which ring has total charge Q_2. Let's compress this notation down a bit and say (treat α and A as constants)

$$\sigma_b(\theta; \theta_c) = Q_2(\theta_c) \, f(\theta; \theta_c)$$

The "ring" considered in Problem 40 was a 1D surface, but now we want to imagine the ring to be a 2D surface having some slight angular extent $d\theta_c$. Then we would write the above as

$$d\sigma_b(\theta; \theta_c) = dQ_2(\theta_c) \, f(\theta; \theta_c)$$

If we were to have two rings 5 and 6, then according to our just-validated superposition theorem we should have this as our charge on the bowl

$$d\sigma_b(\theta; \theta_{c5}, \theta_{c6}) = dQ_2(\theta_{c5}) \, f(\theta; \theta_{c5}) \; + dQ_2(\theta_{c6}) \, f(\theta; \theta_{c6})$$

Now here is our question: Suppose you want σ on the cap to be a constant. How do you then relate the two dQ_2's shown here? Assume rings 5 and 6 have angular width $d\theta$. Then if ring 5 had a charge density $\sigma_c(\theta_{c5})$ [σ_c = cap, as opposed to σ_b = on the bowl] then we can write for these two rings at difference θ_c angles:

$$dQ_2(\theta_{c5}) = \sigma_c(\theta_{c5}) \, 2\pi A^2 \sin\theta_{c5} \, d\theta \qquad\qquad d(\text{area}) = (A d\varphi)(A \sin\theta d\theta)$$

$$dQ_2(\theta_{c6}) = \sigma_c(\theta_{c6}) \, 2\pi A^2 \sin\theta_{c6} \, d\theta$$

=>

$$[dQ_2(\theta_{c5})/ \, dQ_2(\theta_{c6})] = [\sigma_c(\theta_{c5})/ \, \sigma_c(\theta_{c6})] \, [\sin\theta_{c5}/ \, \sin\theta_{c6}]$$

So, if we want $\sigma_c(\theta_{c5}) = \sigma_c(\theta_{c6})$ in order to obtain a "uniform charge distribution" on the complementary sphere, we need to relate the two total charges Q_2 in this way

$$[dQ_2(\theta_{c5})/ \, dQ_2(\theta_{c6})] = [\sin\theta_{c5}/ \, \sin\theta_{c6}]$$

Now if we want to superpose a continuum of rings with θ_c as the variable, this really says

$$dQ_2(\theta_c) \csc(\theta_c) = \text{constant, independent of } \theta_c \;\; = \sigma_c(\theta_c) \, 2\pi A^2 \, d\theta_c$$

Let's then require that (I go with $+\sigma_o$, Smythe went with $-\sigma_1$)

$\sigma_c(\theta_c) = \sigma_0$ = our target constant σ on the complementary surface.

Then we have

$$dQ_2(\theta_c)\csc(\theta_c) = 2\pi A^2 \sigma_0 \, d\theta_c$$

$$dQ_2(\theta_c) = 2\pi A^2 \sigma_0 \sin\theta_c d\theta_c$$

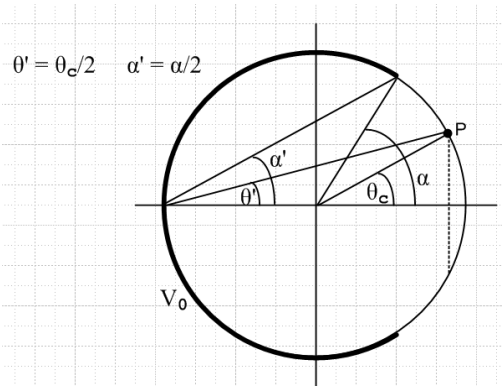
So imagine we have this one ring at θ_c. The σ_b it creates is

$$d\sigma_b(\theta; \theta_c) = dQ_2(\theta_c) f(\theta; \theta_c) = 2\pi A^2 \sigma_0 \sin\theta_c \, f(\theta; \theta_c) \, d\theta_c$$

If we then want to superpose all these rings such that the complementary surface is filled with charge density σ_0, we will find this charge distribution on the bowl:

$$\sigma_b(\theta) = \int_0^{\alpha} d\theta_c \, d\sigma_b(\theta; \theta_c) = 2\pi A^2 \sigma_0 \int_0^{\alpha} d\theta_c \sin\theta_c \, f(\theta; \theta_c)$$

where here is our picture stolen from Problem 41:



In full then, we have

$$\sigma_b(\theta) = -2\pi A^2 \sigma_0 \int_0^{\alpha} d\theta_c \sin\theta_c \; (1/[8\pi^2 A^2]) \sec^3(\theta/2) \left(\sqrt{1+\tan^2(\theta_c/2)}\right) *$$

$$\left(\sqrt{\tan^2(\alpha/2)-\tan^2(\theta_c/2)} \,/\, \sqrt{\tan^2(\theta/2)-\tan^2(\alpha/2)}\right) / \left(\tan^2(\theta/2)-\tan^2(\theta_c/2)\right)$$

$$= -2\pi A \sigma_0 \; (1/[8\pi^2 A^2]) \sec^3(\theta/2) \left(1/\sqrt{\tan^2(\theta/2)-\tan^2(\alpha/2)}\right) *$$

$$\int_0^{\alpha} d\theta_c \sin\theta_c \sqrt{1+\tan^2(\theta_c/2)}\ [\ \sqrt{\tan^2(\alpha/2)-\tan^2(\theta_c/2)}\ /\ (\tan^2(\theta/2)-\tan^2(\theta_c/2))\ ]$$

I have checked this once after doing it, it seems right.

As discussed in my superposition doc example, the potential of the bowl will be $V_0 = Q/A$ and we have $Q = (-\sigma_o)4\pi A^2$, so we find $V_0 = Q/A = (-\sigma_o)4\pi A^2/A = (-\sigma_o)4\pi A$ which says

$$-\sigma_o = V_0/(4\pi A)$$

3. Transformation of the Integral

Notice in passing that in Problem 40 we integrated $\sigma_b(\theta;\ \theta_c)$ due to one ring at θ_c over angle θ in order to get the total charge q_1 on the bowl due to that one ring. Here, we are instead integrating $\sigma_b(\theta;\ \theta_c)$ over θ_c to get the sum effect of a bunch of rings at different θ_c, and θ on the bowl is held constant. The integral is of course different from that in Problem 40. So let

$$I = \int_0^{\alpha} d\theta_c \sin\theta_c \sqrt{1+\tan^2(\theta_c/2)}\ [\ \sqrt{\tan^2(\alpha/2)-\tan^2(\theta_c/2)}\ /\ (\tan^2(\theta/2)-\tan^2(\theta_c/2))\ ]$$

I will try to stay close to the problem 41 notation, even doing some copy and paste

My first inclination is to set // $a \equiv \tan(\alpha/2)$ $t \equiv \tan(\theta/2)$

$x = \tan(\theta_c/2)$ $dx = \sec^2(\theta_c/2)\ (1/2)\ d\theta_c$ => $d\theta_c = 2dx\ /\ \sec^2(\theta_c/2)$

Endpoints in x are

lower = 0 upper = $\tan(\alpha/2) = a$

$x^2 = \tan^2(\theta_c/2) = (1 - \cos\theta_c)/(1+\cos\theta_c)$ // copied from problem 40 or 41....

=> $\cos\theta_c = (1-x^2)/(1+x^2)$
$\sin\theta_c = 2x/(1+x^2)$

$\sec^2(\theta_c/2) = 1 + x^2$ => $d\theta_c = 2dx\ /\ (1+x^2)$

So let's install these changes into our integral:

$$I = \int_0^{\alpha} d\theta_c \sin\theta_c \sqrt{1+\tan^2(\theta_c/2)}\ [\ \sqrt{\tan^2(\alpha/2)-\tan^2(\theta_c/2)}\ /\ (\tan^2(\theta/2)-\tan^2(\theta_c/2))\ ]$$

$$I = \int_0^{a} [2dx/(1+x^2)\ ]\ \ [2x/(1+x^2)]\ \sqrt{x^2+1}\ \ [\sqrt{a^2-x^2}\ /\ (t^2 - x^2)\ ]$$

$$= 4 \int_0^a [x\,dx/(1+x^2)^2]\ \sqrt{x^2+1}\ [\sqrt{a^2-x^2}/(t^2-x^2)]$$

$$= 4 \int_0^a dx\, x\sqrt{a^2-x^2}/[(t^2-x^2)(1+x^2)^{3/2}]$$

Since we have xdx, we at once want to replace

$$y = x^2 \qquad dy = 2xdx \qquad \Rightarrow \qquad xdx = (1/2)dy$$

so

$$I = 4 \int_0^{a^2} (1/2)dy\ \sqrt{a^2-y}/[(t^2-y)(1+y)^{3/2}]$$

$$= 2 \int_0^{a^2} dy\ \sqrt{a^2-y}/[(t^2-y)(1+y)^{3/2}]$$

and this certainly falls into the world of "ugly integrals". Now would be a good time to recheck everything above before sailing off to GR7 land and other places. // Done, found one error and fixed it. // Did another check and found *another* error, the integral is simpler now.

4. Assume for the moment we can do the integral, and move on in the problem

Note that θ is on the *bowl* (whereas θ_c was on the cap) so $\theta > \alpha$, and both these angles are in the range $(0,\pi)$, see the picture above (but polar angle θ to a point on the bowl is not shown).

$$a \equiv \tan(\alpha/2) \qquad t \equiv \tan(\theta/2)$$

$$I(a,t) = 2 \int_0^{a^2} dy\ \sqrt{a^2-y}/[(t^2-y)(1+y)^{3/2}]$$

Assume we have looked up this integral and we know $I(a,t) = I(\theta)$. We then have

$$\sigma_b(\theta) = -2\pi A^2 \sigma_0\ (1/[8\pi^2A^2])\sec^3(\theta/2)\ (1/\sqrt{\tan^2(\theta/2) - \tan^2(\alpha/2)})\ * I(\theta)$$

$$= -2\pi A^2 \sigma_0\ (1/[8\pi^2A^2])\sec^3(\theta/2)\ (1/\sqrt{\tan^2(\theta/2) - a^2})\ * I(\theta)$$

$$= -\sigma_0\ (1/(4\pi))\sec^3(\theta/2)\ (1/\sqrt{\tan^2(\theta/2) - a^2})\ * I(\theta)$$

Where do we go from here? We take the above situation as Problem 1 of a superposition, and we take as Problem 2 a full metal sphere (same radius as bowl) with outer surface charge density $-\sigma_0$. This superposition is studied in detail as Example 6 of "superposition in electrostatics.doc". The conclusion reached is that we end up with an isolated charged bowl of potential $V_0 = Q/A$ where $Q = 4\pi A^2(-\sigma_0)$, and the surface charges on the bowl surfaces are these:

$\sigma_{inner} = \sigma_b(\theta)$

$\sigma_{outer} = \sigma_{inner} + V_0/4\pi A$

This general form agrees with Smythe, and our task is now to do the integral I(θ) and see if we can obtain his result for $\sigma_b(\theta)$.

So let's now restate **what is happening in this problem:** (1) we know the bowl σ situation for a **point** charge on the cap. Bowl is grounded, so this is a Green's Function situation. (2) we then do an integral in effect (done in an earlier problem via the iris pathway) to learn the bowl σ situation for a **ring** charge on the cap. (3) In this problem we do another integral and learn the bowl σ situation for a **uniform** cap of charge σ_0. Bowl is still grounded.

(4) We then superpose onto this problem a second problem which is just a uniform sphere at V_0 with uniform external charge $-\sigma_0$. This makes a connection between σ_o and V_0. In this superposition, the cap charge is exactly canceled and we end up with a bowl at $0+V_0 = V_0$. That is, we replace variable σ_0 in our result (3) with the variable V_0. But this resulting third problem is found to be exactly that of a charged bowl at potential V_0.

(5) So, in (3) we had $\sigma_{in} = \sigma_{out}$ for the problem with the uniform cap charge. That means for the third problem σ_{in} does not change, but σ_{out} gets the constant problem 2 charge adder $-\sigma_0$. We just take our (3) result for uniform cap and replace σ_0 by expression of V_0 as just outlined, and this becomes the final σ_{in}.

Our Smythe voyage to the iris and back again could have been omitted. We could have just started with the result of (1) above and done a double integral to get our uniform cap and then move to step (4). This is how Kirk outlines things, and is perhaps how Kelvin did it.

5. Doing or looking up the integral

$$I(a,T) = 2 \int_0^{a^2} dy \ \sqrt{a^2-y} \ / \ [(t^2 - y)\,(1+y)^{3/2}\,]$$

$a \equiv \tan(\alpha/2)$ $\quad t \equiv \tan(\theta/2)$ $\quad \theta > \alpha$ $\quad t > a$

Notice that each factor is real at every point in the integration, so nothing tricky is happening here. The first factor does vanish at the upper endpoint.

Maple can actually *do* the thing as an indefinite integral, so we have something promising. But when you put the a^2 endpoint, its result is in terms of a limit $y \rightarrow a^2-$ of a Big Mess. Perhaps it needs to know something about the size of t, but all I know is $t^2 > a^2 > 0$. No amount of "assuming" causes Maple to make a better answer.

When you ask Maple for the indefinite integral, you get a mess with two different arctanh objects. Here is verification of the Maple entry of the form. (t=T)

$$f := \int \frac{\sqrt{a^2 - x}}{(T^2 - x)(1+x)^{\left(\frac{3}{2}\right)}}\, dx$$

But when you ask Wolfram, you get a slightly more concise answer:

$$\int \frac{\sqrt{a^2 - x}}{(t^2 - x)(1+x)^{3/2}}\, dx =$$

$$-\frac{\frac{2\sqrt{a^2-t^2}\,\tan^{-1}\left(\frac{\sqrt{x+1}\,\sqrt{a^2-t^2}}{\sqrt{-t^2-1}\,\sqrt{a^2-x}}\right)}{\sqrt{-t^2-1}} + \frac{2\sqrt{a^2-x}}{\sqrt{x+1}}}{t^2+1}$$

Alpha gives the same thing, but *won't evaluate endpoints* if I give them, but does the indefinite, though it gives the above with the two terms written out (which I checked against the above, first term is +)

Alternate form of the integral:

$$\frac{2\sqrt{-t^2-1}\,\sqrt{a^2-t^2}\,\tan^{-1}\left(\frac{\sqrt{x+1}\,\sqrt{a^2-t^2}}{\sqrt{-t^2-1}\,\sqrt{a^2-x}}\right)}{(t^2+1)^2} - \frac{2\sqrt{a^2-x}}{(t^2+1)\sqrt{x+1}}$$

So I will assume this integral is correct and "work with it". I have in mind that t>a, so I will make compensating flip-arounds [ouch!] of things inside radicals so everything looks positive and real. Specifically, I rewrite the above alternate form of the indefinite integral this way:

$$+ 2\sqrt{t^2+1}\,\sqrt{t^2-a^2}\,\tan^{-1}\left([\sqrt{x+1}\,\sqrt{t^2-a^2}\,]/[\sqrt{t^2+1}\,\sqrt{a^2-x}\,]\right) / (t^2+1)^2 \; - 2\left(\sqrt{a^2-x}\,/\sqrt{x+1}\,\right) / \; (t^2+1)$$

NOTE ADDED 7.3.10: In fact, the sign of the first term here should be - instead of +, but I did not know that when I was writing this doc, so I won't fix the sign below. This led me to think harder about whether Smythe's answer was right or wrong, and led me to do the disk limit, which it turned out did not help to settle the sign question. The reason for the minus sign is explained in "algebraic integrals.doc" and arises when you do a correct handling of radicals of negative quantities. So we shall now proceed as if we did not know about this sign change, and here then is the original text. But I have highlighted the incorrect sign in red throughout the development.

We now wish to evaluate this at the two endpoints:

$x = a^2$: (notice that we really do get $+\infty$ and not $-\infty$ inside the arctan in our limit $x \to a^2$ from below)

$$+ 2\sqrt{t^2+1}\sqrt{t^2-a^2}\ \tan^{-1}(+\infty) / (t^2+1)^2 \ - 0 \ = \ 2\sqrt{t^2+1}\sqrt{t^2-a^2}\ (+\pi/2) / (t^2+1)^2$$

$x = 0$:

$$+ 2\sqrt{t^2+1}\sqrt{t^2-a^2}\tan^{-1}[\sqrt{t^2-a^2}/(a\sqrt{t^2+1})] / (t^2+1)^2 \ - \ 2a / (t^2+1)$$

We can subtract the endpoints to get:

$$I(a,t)/2 \ = + [2\sqrt{t^2+1}\sqrt{t^2-a^2}/(t^2+1)^2 \ ] (\pi/2- \psi) + 2a / (t^2+1)$$

where $\psi = \tan^{-1}[\sqrt{t^2-a^2}/(a\sqrt{t^2+1})]$

Aside: The argument of $\tan^{-1}$ is a positive number ranging from 0 to $+\infty$, so it seems pretty reasonable to assume that ψ is an angle that ranges from 0 to $\pi/2$. Therefore, $(\pi/2- \psi)$ is positive.

If you draw the usual triangle picture, it is easy to show by simple geometry that

$$\sin(\pi/2 - \psi) \ = (a\sqrt{t^2+1})/(t\sqrt{a^2+1}) \qquad => \qquad (\pi/2- \psi) = \sin^{-1}[(a\sqrt{t^2+1})/(t\sqrt{a^2+1})]$$

where all quantities shown are positive numbers, so we write our integral result again:

$$I(a,t)/2 \ = + 2a / (t^2+1) \ + \ [2\sqrt{t^2+1}\sqrt{t^2-a^2}/(t^2+1)^2 \ ] \ \sin^{-1}[(a\sqrt{t^2+1})/(t\sqrt{a^2+1})]$$

This is the sum of two positive numbers! At this point let's examine the arcsin and see what happens in there

$$a \equiv \tan(\alpha/2) \qquad t \equiv \tan(\theta/2)$$

$$\text{arcsin arg} \ = \ [(a\sqrt{t^2+1})/(t\sqrt{a^2+1})] \ = \ [(\tan(\alpha/2)\sec(\theta/2))/(\tan(\theta/2)\sec(\alpha/2))]$$

$$= \ [(\tan(\alpha/2)\cos(\alpha/2))/(\tan(\theta/2)\cos(\theta/2))]$$

$$= \ [\sin(\alpha/2)/\sin(\theta/2)] \qquad \text{// which I like a lot....}$$

Next, we look at

$$[2\sqrt{t^2+1}\sqrt{t^2-a^2}/(t^2+1)^2 \ ] \ = \ 2\sec(\theta/2)\sqrt{\tan^2(\theta/2) - \tan^2(\alpha/2)} / \sec^4(\theta/2)$$

$$= 2\cos^3(\theta/2)\sqrt{\tan^2(\theta/2) - \tan^2(\alpha/2)}$$

So once again, with vigor,

$$I(a,t)/2 = 2\tan(\alpha/2)\cos^2(\theta/2)$$

$$+ [2\cos^3(\theta/2)\sqrt{\tan^2(\theta/2) - \tan^2(\alpha/2)}]\ \sin^{-1}[\sin(\alpha/2)/\sin(\theta/2)]$$

We now go back and install this where it goes

$$\sigma_b(\theta) = -\sigma_0\ (1/(4\pi))\sec^3(\theta/2)\ (1/\sqrt{\tan^2(\theta/2) - a^2})\ * I(\theta)$$

$$= -\sigma_0\ (1/(4\pi))\sec^3(\theta/2)\ (1/\sqrt{\tan^2(\theta/2) - \tan^2(\alpha/2)})\ * I(a,t)$$

$$= -\sigma_0\ (1/(2\pi))\sec^3(\theta/2)\ (1/\sqrt{\tan^2(\theta/2) - \tan^2(\alpha/2)})\ * I(a,t)/2$$

$$= -\sigma_0\ (1/(2\pi))\sec^3(\theta/2)\ (1/\sqrt{\tan^2(\theta/2) - \tan^2(\alpha/2)})\ *$$

$$\{ 2\tan(\alpha/2)\cos^2(\theta/2) + [2\cos^3(\theta/2)\sqrt{\tan^2(\theta/2) - \tan^2(\alpha/2)}]\ \sin^{-1}[\sin(\alpha/2)/\sin(\theta/2)]\ \}$$

$$= -\sigma_0\ (1/(\pi))\ (1/\sqrt{\tan^2(\theta/2) - \tan^2(\alpha/2)})\ *$$

$$\{\ \tan(\alpha/2)\sec(\theta/2) + [\sqrt{\tan^2(\theta/2) - \tan^2(\alpha/2)}]\ \sin^{-1}[\sin(\alpha/2)/\sin(\theta/2)]\ \}$$

$$= -\sigma_0(1/\pi)\ \{\ \tan(\alpha/2)\sec(\theta/2)/\sqrt{\tan^2(\theta/2) - \tan^2(\alpha/2)} + \sin^{-1}[\sin(\alpha/2)/\sin(\theta/2)]\ \}$$

Now recall from earlier that

$$-\sigma_o = V_0/(4\pi A) \qquad \Rightarrow \qquad -\sigma_0(1/\pi) = V_0/(4\pi^2 A)$$

so our result takes this form

$$\sigma_b(\theta) = V_0/(4\pi^2 A)\ \{\ \tan(\alpha/2)\sec(\theta/2)/\sqrt{\tan^2(\theta/2) - \tan^2(\alpha/2)} + \sin^{-1}[\sin(\alpha/2)/\sin(\theta/2)]\ \}$$

The second term agrees with Smythe except for the sign. The first factor can be written like this

$$\tan(\alpha/2)\sec(\theta/2)/\sqrt{\tan^2(\theta/2) - \tan^2(\alpha/2)} = \sin(\alpha/2)\ [\sec(\alpha/2)\sec(\theta/2)/\sqrt{\tan^2(\theta/2) - \tan^2(\alpha/2)}]$$

The bracket I did on scratch. First replace $\Delta\tan^2$ by $\Delta\sec^2$ in the radical. Then $\sec^2 = 1/\cos^2$ then rationalize inside the radical and get $1/\text{sqrt}(c^2\text{-}c^2)$, but then reverse or and put sines. The result:

$$[\sec(\alpha/2)\sec(\theta/2)/\sqrt{\tan^2(\theta/2) - \tan^2(\alpha/2)}] = 1/\sqrt{\sin^2(\theta/2) - \sin^2(\alpha/2)}$$

So here then is **our final result**:

$$\sigma_b(\theta) = V_0/(4\pi^2 A)\ \{\ \sin(\alpha/2)/\sqrt{\sin^2(\theta/2) - \sin^2(\alpha/2)} + \sin^{-1}[\sin(\alpha/2)/\sin(\theta/2)]\ \}$$

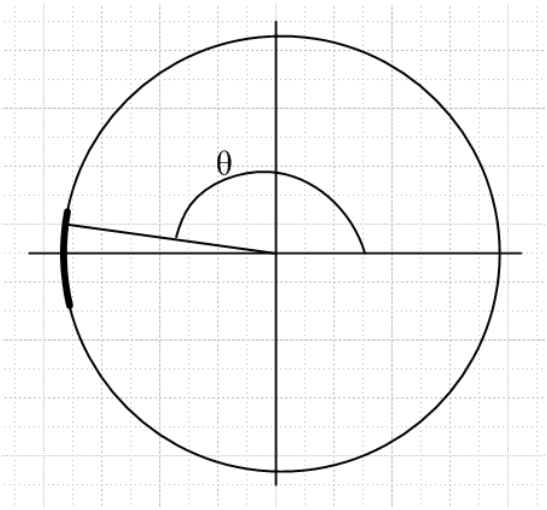
If we set $\varepsilon = 1/4\pi$ in Smythe's result, our outside factor matches his. My result matches his perfectly, except I get a + sign for the second term, he gets a minus sign. I don't see my error, and I have looked pretty hard to find it. [see red note above]

6. How could we test the sign? Doing the disk limit.

My result is this:

$$\sigma_b(\theta) = V_0/(4\pi^2 A)\ \{\ \sin(\alpha/2)/\sqrt{\sin^2(\theta/2) - \sin^2(\alpha/2)}\ +\ \sin^{-1}[\sin(\alpha/2)/\sin(\theta/2)]\ \}$$

If $\alpha \to 0$ we have the full sphere, the two terms each vanish, no charge on the inside, so that limit is not going to help us. But as $\alpha \to \pi$, we get a point charge or we get a disk if you play your cards right. Here is this disk shown on the left, ie, the bowl becomes a disk:



Let's define complementary (add to π) angles this way (both of which are small)

$\theta'' = \pi - \theta$ $\qquad$ $\theta'' < \alpha''$ $\qquad$ $\theta = [\pi - \theta'']$
$\alpha'' = \pi - \alpha$

The quantity usually called ρ for the disk is then

$$\rho = A\sin\theta''$$

and if we call the radius of the disk B, we have

$$B = A\sin\alpha''$$

Now consider:

$$\sin(\theta/2) = \sin([\pi - \theta'']/2) = \sin(\pi/2 - \theta''/2) = \cos(\theta''/2)$$

$$\sin(\alpha/2) = \sin([\pi-\alpha"]/2) = \sin(\pi/2 - \alpha"/2) = \cos(\alpha"/2)$$

We can then, with no approximation, write our $\sigma_b(\theta)$ in this way:

$$\sigma_b(\theta) = V_0/(4\pi^2 A) \{ \sin(\alpha/2)/\sqrt{\sin^2(\theta/2) - \sin^2(\alpha/2)} + \sin^{-1}[\sin(\alpha/2)/\sin(\theta/2)] \}$$

$$= V_0/(4\pi^2 A) \{ \cos(\alpha"/2)/\sqrt{\cos^2(\theta"/2) - \cos^2(\alpha"/2)} + \sin^{-1}[\cos(\alpha"/2)/\cos(\theta"/2)] \}$$

But again with no approximation, we rewrite as

$$= V_0/(4\pi^2 A) \{ \cos(\alpha"/2)/\sqrt{\cos^2(\theta"/2) - \cos^2(\alpha"/2)} + \sin^{-1}[\cos(\alpha"/2)/\cos(\theta"/2)] \}$$

where now we have all our angles being our "small angles". Now let's remove the cluttery double primes and redefine our small angles for the upcoming local calculation:

$$= V_0/(4\pi^2 A) \{ \cos(\alpha/2)/\sqrt{\cos^2(\theta/2) - \cos^2(\alpha/2)} + \sin^{-1}[\cos(\alpha/2)/\cos(\theta/2)] \}$$

Here is a small angle expansion in case it is needed later:

$$\cos(\theta/2) = 1 - (\theta/2)^2/2 + (\theta/2)^4/4! = 1 - \theta^2/8 + \theta^4/(2^4 4!) + \ldots$$

$$\cos^2(\theta/2) = (1 - \theta^2/8 + \theta^4/(2^4 4!))(1 - \theta^2/8 + \theta^4/(2^4 4!))$$

$$= 1 - \theta^2/4 + \theta^4\{ (2/(2^4 4!) + (1/64) \}$$

$$= 1 - \theta^2/4 + \theta^4/72$$

Now let's just do the naive approximation: [$\rho = A\sin\theta \approx A\theta$ $B = A\sin\alpha \approx A\alpha$]

$$\cos^2(\theta/2) = 1 - \theta^2/4$$

$$\cos^2(\theta/2) - \cos^2(\alpha/2) = (1 - \theta^2/4) - (1 - \alpha^2/4) = (\alpha^2/4 - \theta^2/4) = ((B/2A)^2 - (\rho/2A)^2)$$

$$= (1/2A)^2 (B^2 - \rho^2)$$

Then we have

$$\sigma_b(\theta) = V_0/(4\pi^2 A) \{ \cos(\alpha/2)/\sqrt{\cos^2(\theta/2) - \cos^2(\alpha/2)} + \sin^{-1}[\cos(\alpha/2)/\cos(\theta/2)] \}$$

$$\approx V_0/(4\pi^2 A) \{ 1/\sqrt{(1/2A)^2 (B^2 - \rho^2)} + \sin^{-1}[1/1] \}$$

$$= V_0/(4\pi^2 A) \{ (2A)/\sqrt{(B^2 - \rho^2)} + \pi/2 \}$$

$$= V_0/(2\pi^2)\ \{\ 1/\sqrt{(B^2-\rho^2)}\ + (\pi/4A)\ \}$$

Now the disk has capacitance $C = (2/\pi)\,B$ and $q = CV$ so $V_0 = q/C = (q/B)(\pi/2)$ then

$$= (q/B)(\pi/2)/(2\pi^2)\ \{\ 1/\sqrt{(B^2-\rho^2)}\ + (\pi/4A)\ \}$$

$$= (q/(4\pi B))\ \{\ 1/\sqrt{(B^2-\rho^2)}\ + (\pi/4A)\ \}$$

q = total charge on disk (both sides)

The first term is consistent with Jackson page 93 for charge on one side. So we are claiming that

$$\sigma_i = (q/(4\pi B))\ \{\ 1/\sqrt{(B^2-\rho^2)}\ + (\pi/4A)\ \}$$

Then we have

$$\sigma_o = V_o/(4\pi A) + \sigma_i = (q/B)(\pi/2)/(4\pi A) + \sigma_i = (q/2)(B/A) + \sigma_i$$

We can argue that $B << A$, so $(B/A) << 1$ and then we just have $\sigma_o = \sigma_i$ which we like. The idea here is that V_0 is very small because the disk is very small (I guess), or A is "very large".

So what are we to make of the "extra term"? Well rewrite:

$$\sigma_i = [q/(4\pi B)]\ \{\ 1/\sqrt{(B^2-\rho^2)}\ + (\pi/4A)\ \}$$

$$= \{\ [q/(4\pi B)]/\sqrt{(B^2-\rho^2)}\ + [q/(4\pi B)]\ (\pi/4A)\ \}$$

$$= \{\ [q/(4\pi B)]/\sqrt{(B^2-\rho^2)}\ + (q/AB)\ \}$$

$$= (q/B^2)\ \{\ (1/4\pi)B/\sqrt{(B^2-\rho^2)}\ + (B/A)\ \}$$

where now we have done the scaling properly. We find that the first term is order 1, so to speak, while the second term is order $(B/A) << 1$, so we can ignore the second term.

Conclusions: When we take the disk limit of our result

(1) we get the correct Jackson result for the disk surface charge.
(2) the second term makes no contribution, so we are unable to test its sign!

7. Result for the hemispherical bowl.

Our general results above (with my sign) are:

$$\sigma_i(\theta) = V_0/(4\pi^2 A)\ \{\ \sin(\alpha/2)/\sqrt{\sin^2(\theta/2) - \sin^2(\alpha/2)}\ +\ \sin^{-1}[\sin(\alpha/2)/\sin(\theta/2)]\ \}$$

$\sigma_o = V_o/(4\pi A) + \sigma_i$

From problem 41 we learned the capacitance of our bowl

$C = A\,(1/\pi)\ \{\pi - \alpha + \sin\alpha\}$

Fine. For a hemispherical bowl, we have $\alpha = \pi/2$, $\sin\alpha = 1$, $\sin(\alpha/2) = 1/\sqrt{2}$. Then the above results become:

$\sigma_i(\theta) = V_0/(4\pi^2 A)\ \{\ (1/\sqrt{2}\ ) / \sqrt{\sin^2(\theta/2) - 1/2}\ +\ \sin^{-1}[(1/\sqrt{2}\ )/\sin(\theta/2)\ ]\ \}$

$\sigma_o = V_o/(4\pi A) + \sigma_i$

$C = A\,(1/\pi)\ \{\pi/2 + 1\ \} = A\ \{1/2 + 1/\pi\ \}$ // $C = A\ \{\ 1/2 + 1/2\}$ for full sphere

$q = CV_0$

This case does of course test the sign of the second term.

Note added 12.8.10. On scratch I shows that $\sqrt{\sin^2(\theta/2) - 1/2} = \sqrt{-(\cos\theta)/2}$ and $\sin^{-1}[(1/\sqrt{2}\ )/\sin(\theta/2)\ ] = \tan^{-1}(1/\sqrt{-\cos\theta}\ )$. If we define θ" as the polar angle measured from the center of the bowl (as Kelvin did), we have θ" = π-θ and then cosθ" = -cosθ. Let's refer to cosθ" as just z, the usual variable name and drop the primes. Then we can write the charge density on the inner surface of the charged hemispherical bowl as: (putting in now the corrected sign, not the red one)

$\sigma_i(\theta) = V_0/(4\pi^2 A)\ \{\ (1/\sqrt{z}\ ) -\ \tan^{-1}(1/\sqrt{z}\ )\ \}$ $z = \cos\theta"$ bowl center = +z axis

The "edge" of the bowl is at z = 0 and very close to this value of z the second term is -π/2. As expected, the σ is infinite at this edge. At bowl center we have z = 1 and we get {} = { 1 - π/4) = 1-.785.

I have not found direct confirmation of this result for the half-bowl, but it is just a special case of the formulas that I do have confirmation for.

8. Evidence that Smythe's sign is correct

I found a nice PDF by Kirk McDonald of Princeton that treats the "orifice in a spherical shell" problem , from which I just quote a few results in various notations. First, this is "Green's result" which is valid for small angle bowl:

$$\sigma_-(\theta) \approx \frac{V_0}{4\pi^2 a}\left(\frac{\sqrt{2}\theta_0/2}{\sqrt{1-\cos\theta}} - \tan^{-1}\frac{\sqrt{2}\theta_0/2}{\sqrt{1-\cos\theta}}\right). \qquad (30)$$

The claim is that Kelvin got this result by inversion (σ_- is what is on the inner surface). He was the first to prove it, did it in 1847, but published it in 1872

$$\sigma_-(\theta) = \frac{V_0}{4\pi^2 a}\left(\sqrt{\frac{1-\cos\theta_0}{\cos\theta_0-\cos\theta}} - \tan^{-1}\sqrt{\frac{1-\cos\theta_0}{\cos\theta_0-\cos\theta}}\right). \qquad (36)$$

which is supposed to be valid for large angles as well as small angles (ie, for any bowl). In these formulas, θ_o is "the half angle of the bowl" as measured from the center of the cap, and is therefore certainly the same as my α. The author does not at this point present Kelvin's inversion analysis, just quotes the result. Also, the angle θ agrees with my θ, measured from cap center.

Note added 12.8.10: Here is an exact quote from Kelvin's page which appears on page 185 of the collection (the contents shows this paper at the wrong page location!) Kelvin's polar angle relative to metal bowl center is called η (his angle COP), and for a point on the lip of the bowl he calls this same angle α. Note that f is the diameter of the sphere, as Kelvin states on page 184, which is my 2A.

Hence we have, in conclusion,

$$\rho = \frac{V}{2\pi^2 f}\left\{\sqrt{\frac{\cos\alpha+1}{\cos\eta-\cos\alpha}} - \tan^{-1}\sqrt{\frac{\cos\alpha+1}{\cos\eta-\cos\alpha}}\right\} \ldots (18),$$

So both Kelvin's angles are measured from bowl center, not cap center. The connection to me and Kirk is $\eta = \pi-\theta$ and $\alpha = \pi-\alpha$ (sorry, same symbol), and this causes all cosines to negate, and then the above becomes my result below which agrees with Kirk and which I copy up here:

$$\sigma_i(\theta) = V_0/(4\pi^2 A)\ \{\ \sqrt{1-\cos\alpha}\ /\ \sqrt{\cos\alpha - \cos\theta}\ \ -\ \ \tan^{-1}\left[\sqrt{1-\cos\alpha}\ /\sqrt{\cos\alpha - \cos\theta}\ \right]\ \}$$

Kelvin goes on to say, regarding a geometric version (17) of his (18) shown above,

exactly the results stated above in § 220. Twenty-two years ago these and the very simple formula (17) were communicated by me to M. Liouville without proof, and were published in his Journal. From that time till now they have not been proved, or even noticed, so far as I am aware, by any other writer.

His paper is dated Jan 1869, so I guess the comm took place in 1847.

My point here is that the second term has a minus sign, not a plus sign, which is supportive of Smythe's result, which result is this (Smythe's minus sign)

$$\sigma_b(\theta) = V_0/(4\pi^2 A) \{ \sin(\alpha/2)/ \sqrt{\sin^2(\theta/2) - \sin^2(\alpha/2)} - \sin^{-1} [\sin(\alpha/2)/\sin(\theta/2)] \}$$

Now would be a good time to **convert this to the Kelvin form**. The first term works like this:

$$\sin(\alpha/2) = \sqrt{1-\cos\alpha} /\sqrt{2}$$

$$\sqrt{\sin^2(\theta/2) - \sin^2(\alpha/2)} = \sqrt{(1-\cos\theta)/2 - (1-\cos\alpha)/2} = \sqrt{\cos\alpha - \cos\theta} / \sqrt{2}$$

$$\Rightarrow \quad \sin(\alpha/2)/ \sqrt{\sin^2(\theta/2) - \sin^2(\alpha/2)} = \sqrt{1-\cos\alpha} / \sqrt{\cos\alpha - \cos\theta}$$

and this gives Kelvin's first term. We can write our second term this way

$$\sin^{-1} [\sin(\alpha/2)/\sin(\theta/2)] = \sin^{-1} [\sqrt{1-\cos\alpha} /\sqrt{1-\cos\theta}]$$

and this in itself gives a nice result:

$$\sigma_i(\theta) = V_0/(4\pi^2 A) \{ \sqrt{1-\cos\alpha} / \sqrt{\cos\alpha - \cos\theta} - \sin^{-1} [\sqrt{1-\cos\alpha} /\sqrt{1-\cos\theta}] \}$$

If we draw a triangle for $\psi = \sin^{-1}$, the horizontal edge has length $\sqrt{\cos\alpha - \cos\theta}$, so we can write

$$\sin^{-1} [\sqrt{1-\cos\alpha} /\sqrt{1-\cos\theta}] = \tan^{1} [\sqrt{1-\cos\alpha} /\sqrt{\cos\alpha - \cos\theta}]$$

then Smythe's result becomes

$$\sigma_i(\theta) = V_0/(4\pi^2 A) \{ \sqrt{1-\cos\alpha} / \sqrt{\cos\alpha - \cos\theta} - \tan^{-1} [\sqrt{1-\cos\alpha} /\sqrt{\cos\alpha - \cos\theta}] \}$$

The nice thing about this form is that the quantity $\sqrt{1-\cos\alpha} / \sqrt{\cos\alpha - \cos\theta}$ appears in each term, and this then agrees with the Kelvin (Thomson) result (as quoted by Kirk)