Why you cannot invert iris to get charged bowl PhL 6.29.10

Notes added 12.7.10

In my work on Smythe's hole-in-plate stuff, I came up with a strange physical situation. I started with an infinite plane of constant surface charge σ_o and I knew the potential of that, a Λ shape. It we make the top of the Λ be V=0, then at infinity on both sides of the plane we have V = -∞ (the lower tips of Λ). It is of course arbitrary where you set V = 0. I then "punched a hole" in the plane and got a potential solution for the resulting situation, which is sort of a Λ with distortion near the hole.

Question: what happens if you "invert" the infinite charged plane? Let's just ponder that and forget about the case with the hole. We think of the infinite charge plane as being our "usual inversion plane" in R' space, and we know that it inverts into our "usual sphere" in R space. That much is certainly true.

Secondly, if we set V = 0 on our plane (which was by the way a metal plane), then we must have V = 0 on the sphere in the image space. That much is certainly true from the inversion formalism. Thus, we can regard both the plane and the sphere as being made of metal.

What will σ look like on this sphere? We have

$$\sigma(\mathbf{r}) = (a/r)^3\, \sigma'(\mathbf{r'}) = (r'/a)^3\, \sigma'(\mathbf{r'})$$

so

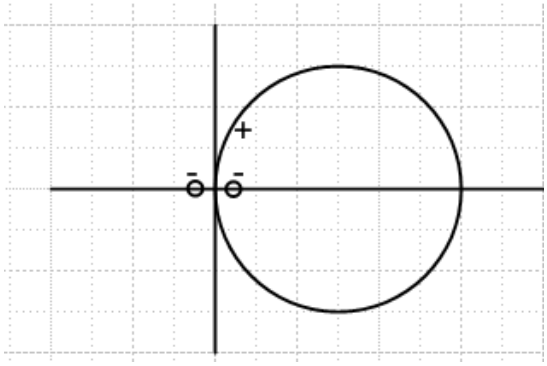
$$\sigma(\mathbf{r}) = (r'/a)^3\, \sigma_0$$

As we go to distance parts of our infinite plane, r' is very large, and this takes us to toward the opposite pole on the sphere and there we will fined $\sigma \to \infty$. Fine.

We know the total charge on the plane is ∞. It seems pretty likely that is also true for our sphere, though not totally obvious since the divergence of σ there could in theory be integrable. Suppose that were the case and suppose the sphere had a *finite* charge on it. In that case, we end up in R space with a sphere having V = 0 and finite charge Q. If this were the case, we would know that V(∞) = -Q/C which could be some finite negative value and this would be the value everywhere on the great sphere in R space. This great sphere in R space maps to a sphere around the inversion origin in R' space where indeed we will have a finite potential in R's space, so there is no violent contradiction here. So I suppose a finite Q is possible for our sphere.

So our next question: is this charged V = 0 sphere in R space of any use to us? Can we regard it as a charged sphere in isolation? That we know is not true, since that would have a uniform σ. So what exactly is going on with this sphere?

Here is a possible answer. Our plane in R' space of positive σ_0 charge density "must be accompanied by" two other planes each carrying $-\sigma_0/2$ and being infinitely far off to the left and right. Let's imagine they are at a large but finite distance away. This is where the field lines from the center plane land. So we have to invert these two planes as well as our sphere. Here is what we then have in R space:



The two distant negatively charge planes map into two tiny spheres of charge as shown. Again, I suspect the charges on these little spheres → -∞, but no matter. The point is that THIS is the true picture of what we have in R space! The + charge on the sphere is very large at the origin pole because it is induced to be large by those two very close "point negative charges" . In the limit, the two tiny spheres become point charges which pinch the sphere and that is why σ on the sphere goes infinite. So we do NOT have a sphere in isolation here, we have a sphere pinched by two equal point charges.

Now if we redo all this with a hole in the central plane so it becomes an iris, then our sphere above does not close at the left pole, but those two point charges are still there and influence things. So we do NOT end up with an isolated bowl. We end up with the Green's function for a bowl with point charge at the pole. So this suggests that the Green's function for a bowl is the inversion of a V=0 iris which has σ_0 at distant points along with those two other planes. If you throw out those planes, you no longer have your point charge so there is no longer a Green's function.

So I guess the upshot is that doing this inversion does not produce anything useful.

Here are my earlier comments on this same subject:

Older notes probably 6.29.10 (with [today comments])

Pie in the Sky. This is one of those "too good to be true" ideas. The idea was this: take the solution I found in the Smythe hole in isolated plate doc, and just invert it into the bowl. The claim is that the iris is made of metal, formed by making a hole in an infinite metal plane of uniform charge. Since the iris is at V = 0, it will invert into a bowl of V = 0, and no point charges are required, and we are done! [but done with what? I guess my hope was that we were ending up with an isolated charged bowl. That really would be 'too good to be true".]

Counter-argument #1: If these were true, the infinite plane at V = 0 would invert to a full sphere at V = 0, but the sphere would have some all-same-sign σ on it that inverts from the plane, so sphere must have V > 0. Also, this σ would be non-uniform, which we know is wrong. [Well, we can get around V>0 by changing the zero point for V. And yes, σ would be non-uniform, and that says we cannot have just an isolated sphere, agreed.]

Counter-argument #2: If this were to invert to a charged bowl, the bowl would not be at V = 0, which makes you at once wonder about the iris's situation. What would the potential be for an infinite plane of uniform charge density on both sides? Is this even a viable situation, at least in theory? How can you

model it? Perhaps a metal sphere with an equatorial metal plane, then take limit as $R \to \infty$. But at finite R, what happens? Could charge sit on the plane? Each half sphere is a closed boundary of metal with V = constant on the entire boundary, and there are no separate charges inside this half sphere. So Laplace says that V = constant everywhere inside. Near the surface on the inside, V = constant, so cannot have any $\partial_n V$ to support a surface charge density, so there can be no surface charge density on any inside surface, and this includes the equatorial plane. Then take limit $R \to \infty$ and the plane remains with no σ. If you put some σ on it, it will run off to the great sphere and transfer to the outside of the great sphere.

This same counter argument is valid for an infinite iris that has any shaped hole or holes, as long as there are no islands of separated metal inside the great sphere. So it is valid for the iris with round hole. Therefore, a situation with an infinite plane with uniform charge distribution is not possible, it does not satisfy Laplace. So you cannot really talk about putting a hole in such an infinite charged plane, as I did in my problem.

Approximation to infinite charged plane: But suppose you had a very large disk with some large charge on it Q, and at some potential V > 0. In a region surrounding the center of the disk, maybe you could approximate things as a uniform charge distribution. We know that $\sigma = (1-(\rho/a)^2)^{-1/2}$. Here is a plot for a = 100 units where things are normalized to σ = 1 at the origin:

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plot(1/sqrt(1-(rho/100)^2),rho = 0..100, view = 0..10, numpoints = 1000);
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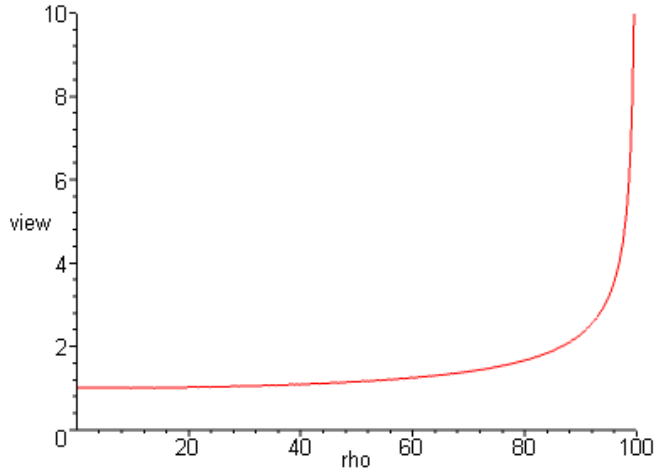

So yes, the notion of a region of constant σ near the center is completely reasonable, this picture says it all. If the rim of the disk is "miles away", then what happens when you make a hole at the center is correctly described by my solution to the hole in charged plate problem. It's just that you have to view it in this "large disk context".

The correct charge density is in fact $\sigma = (Q/2\pi a^2)(1-(\rho/a)^2)^{-1/2}$ so at the center is $\sigma_o = (Q/2\pi a^2)$, an interesting fact in itself. It is half what you would get with a uniform distribution. As long as we have a large but finite disk, we can talk about $\sigma_0 > 0$. On the disk we have $V = V_0 = Q/C = (\pi/2)Q/a$. If we try to take the limit as Q and a $\to \infty$ together to maintain a constant σ_o at the center, we find that

$$V_0 = Q/C = (\pi/2)Q/a = (\pi/2)2\pi a^2\sigma_o/a = \pi^2 a\, \sigma_0$$

so in this last-hope limit, $V_0 \rightarrow \infty$, not 0. If you want to have $\sigma_0 > 0$, you cannot have $V_0 = 0$.

In my solution to the problem in Smythe notes, I never said anything about V = 0 on this hole in plate. I just said uniform σ_0.