Iris Green's Function attempt using the Stak integral equation method. PhL 1.17.10

Most of this doc is in polar coordinates, but Section 2(c) does a little with oblate spheroidals. It was not until well after I wrote this that I learned a simple form for F_n.

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Overview (5 pages, written 4.15.10, updated 12.6.10)

A. Comments -- Note the very early date of this doc, January 2010 !

Here I note that I probably went down the same path as Kelvin and Green with my naive efforts on the disk and iris using cylindrical coordinates (polar + z). I note that it is OK to have a cylindrical origin on a piece of metal, though not OK in sphericals. I note that my integral equation *seems* to be 2D, but really this is a 3D problem. I thought at first that my $1/\sqrt{a - b\cos(\theta)}$ expansion had really messy Fourier coefficients F_n, but later I found that they are just $Q_{n-1/2}$ functions, so not so bad after all.

B. Problem 1: In-hole Green's function for a metal ring.

(a) I consider here a wire ring with a point charge located somewhere inside the ring (ie, in the plane of the ring). This is a Green's Function problem. I recall that the "image method for the sphere" tells you how, given one charge, you can add a second "image charge" such that the total potential of these two charges vanishes on a sphere. I realize I can make that sphere align with the ring, so, at first blush, we seem to have a solution of this wire Green's Function problem: it is the sum of the potentials of the original in-ring charge and an image charge outside the ring of the right size and position. I then claim since I have found a solution, it must be unique.

The fallacy of this argument is this: the "bounding surface" σ must be of one less dimension than the region R in which you are operating. Somehow Stakgold's entire theory is predicated on this idea (see Section 5.9) of the boundary σ being a hypersurface of dimension n-1. My image "solution" to the ring problem does meet the BC's on the ring and at ∞. If somehow this ring were 2D, then "the Dirichlet Theorem" would tell us that if you find a solution that meets the BC's, then (1) that solution is in fact a

solution to your problem, and (2) it is the only solution. Since the ring is 1D, statement (1) is not true, and in fact the image solution is NOT a solution to the ring problem. For example, in the correct solution you would not have V = 0 on the entire sphere containing the ring.

So this little example has opened a door to a room one does not normally enter, and Stak was quiet about it. To get a solution to this problem, you have to think of the ring as a 3D object with a 2D surface and then try to take a limit. For example, it could be a toroid of some tiny cross section. Of course once you make it a 3D object, it no longer matches the image solution BC of V=0 on the sphere. It does in the limit, but the limit in fact does not exist! As the ring cross section → 0, the ring goes away, and there is only the point charge left. As of 12.5.10 I have not solved this problem with a toroid. This is a fascinating problem that would be good to ponder in a class. I discuss this in some other doc which I have lost track of at the moment.

(b) Despite the error of part (a) above, part (b) is essentially correct. Suppose we think of the ring as a tiny toroid and we find the exact solution where cross section << all other dimensions. So V = 0 on this ring. I then realize that if you try to form a disk or iris by superposing rings (that is what I was all about here), you have a problem Houston. This problem later became my **Red Flag Theorem** for superposing situations involving metal objects, see superposition folder. If you consider just two such rings, you see that although problem 1 has V=0 on ring 1, and problem 2 has V = 0 on ring 2, when you superpose to get problem 3, you will have V = 0 on neither ring. So my conclusion here is entirely correct in that regard. This idea is further, and correctly, discussed in the Problem 3 section below.

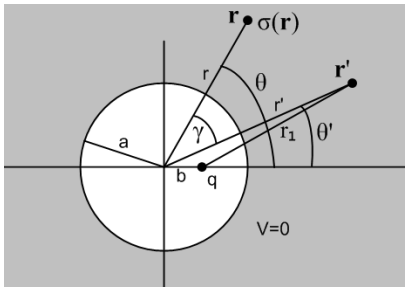
C. Problem 2: In-hole Green's function for a metal iris.

This is of course motivated by Smythe Problem 38.

In **Problem 2 part (a)** I examine the following equation together with its picture, where I use r and θ as two of the three cylindrical coordinates, which we might call "polar" coordinates:

$$V(r',\theta') = q/r_1(r',\theta') + \int_a^\infty r\,dr \int_0^{2\pi} d\theta\ \sigma(r,\theta)/R(r,\theta;r',\theta') = 0 \qquad r' \geq a$$

$$r_1^2 = r'^2 + b^2 - 2br'\cos\theta' \qquad R^2 = r^2 + r'^2 - 2rr'\cos(\theta-\theta')$$



The "fact" is surely true -- the potential everywhere on the iris is V=0, and we can think of this as being the superposition of the potentials of all charges. My approach here was to regard this as an **integral equation** for the quantity σ(r,θ) which is what I want to know in order to compare with Smythe's result.

I now see how this is connected to the discussion in Stakgold about the computation of Green's functions in general using the "integral equation method", as reviewed on page 14 of my Stak Chap 6 meta meta notes. The Stak method is this: [v is potential of just the induced charge]

A. First, solve this integral equation for I(ξ|t): $E(\mathbf{s}|\xi) = \int_\sigma dS_t\, E(s|t)\, I(\xi|t)$ // 147 A

B. Insert that result into this equation: $v(\mathbf{x}|\xi) = -\int_\sigma dS_t\, I(\xi|t) E(x|t)$

C. The Green's function is then: $g(x|\xi) = E(x|\xi) + v(x|\xi)$

In the first line -- a first kind Fredholm integral equation for I -- the Green's charge is located as position ξ, while **s** is a point on the grounded metal, and the point **t** wanders over the metal surface (called σ, the iris) in the integration. Quantity $I(\xi|t) = -\sigma(\mathbf{t}; \xi)$ is the negative of the surface charge at point **t** on the metal, induced by the Green's charge at ξ. The E function is just 1/4πR where I don't care about 4π's at this point, my equation above uses the 4π→1 convention for q. So, my equation written above in polar coordinates is precisely Step A of the Stak algorithm shown above. If you can solve step A, then you insert your computed charge density into Step B to obtain the potential everywhere of the induced charge, and then in step C you add in the potential of the point charge to get the complete Green's function. So my document here is concerned only with Step A.

We **could** go on to say that a derivative of the potential is zero in the hole because there is no charge there, and express our mixed BC problem as some kind of **dual** integral equation system. But in this doc, I am just looking at our above Fredholm equation all by itself. The "integral equation method" of Stakgold is really an alternative to the "dual integral equation method" Jackson uses for the charged disk.

My first (and only) gambit is to try the expansion $\sigma(r,\theta) = \Sigma_{n=0}^{\infty} \sigma_n(r) \cos(n\theta)$ motivated by the fact that cos(nθ) are partial atomic forms for Laplace solutions in our polar/cylindrical coordinate system. So here I am just trying to get this atomic function cos(nθ) "into play". I then expand 1/R and $1/r_1$ in these same cos(nθ) basis functions and this causes our equation above to look like this:

$$(q/b)\ [c_0/2 + \Sigma_{n=1}^{\infty} c_n \cos(n\theta')] + \Sigma_{n=0}^{\infty} \cos(n\theta') \int_a^{\infty} dr\, \sigma_n(r)\, F_n(f) = 0$$

where we know what the c_n and the $F_n(f)$ are (they are ugly, however). Using all these cosine expansions, we have achieved one of our goals in that we can now "set each partial wave coefficient to 0" in terms of the θ dimension of the problem. We then get

$$(q/b)\ [c_0/2] + \int_a^{\infty} dr\, \sigma_0(r)\, F_0(f) = 0 \qquad n = 0$$

$$(q/b)\ c_n(g) + \int_a^{\infty} dr\, \sigma_n(r)\, F_n(f) = 0 \qquad n = 1,2,3....$$

Yes, this is an infinite number of 1D integral equations, but at least we have gotten rid of our 2D integral equation doing the cosine expansion trick. We can think really of the above as a single integral equation,

and if only we could solve it for $\sigma_n(r)$ with n as a bystander parameter, we would be in like flint. Quantity f = r'/r and g = r'/b, so we really have r' as a "parameter" in our integral equation (and r' ≥ a). The $F_n(f)$ is some very horrible combination of Bessel K and E functions that I am not even able to write down except as a sum of F functions. The point I guess is that F is so horrible that we don't stand an iceberg's chance in hell of solving our 1D integral equation. [but maybe that horrible mess is the new-found Q function...]

In **Problem 2 part (b),** I then tried to take what Smythe Problem 38 claims is the "known solution" to this problem for surface charge σ,

$$\sigma(r,\theta) = - (q/2\pi[r^2 + b^2 - 2br\cos\theta]) / \sqrt{r^2-a^2} \qquad r \geq a$$

and I try to put that in the above and at least try to "verify" that it is right. After all, this should be simpler than actually *solving* the integral equation! After much manipulation, I end up with this result: our 2D integral equation is verified if we can show that

$$\int_d^\infty dh\, (h/\sqrt{h^2-d^2}\,) \int_0^{2\pi} d\theta * 1/[\ (1 + h^2 - 2h\cos\theta)\ \sqrt{h^2 + \rho'^2 - 2\rho' h\cos(\theta-\theta')}\ ] \quad (*)$$

$$= (\ 2\pi / \sqrt{d^2 - 1}\ ) * 1/ \sqrt{1 + \rho'^2 - 2\rho'\cos\theta'}\ ] \qquad d = a/b \qquad \rho' = r'/b$$

But I cannot even begin to do even one of the two integrals shown here, so I am not able to verify anything. I suppose I could examine some limits, but why bother.

Having done all this, one gets the feeling that expanding on cos(nθ) was a "bad thing to do".

However: In the last section of this doc, I try again at showing the validity of equation (*) above, and I come very close to showing it is true, and Q functions are involved.

In **Problem 2 part (c)** I ponder, probably for the first time, how the iris problem might be solved in **oblate** spheroidal coordinates. I conjecture a *form* for the solution potential which, amazingly, is exactly correct, as I later found out when I solved the Green's function for an open hyperboloid a month after writing this document. At this point, though, I just had a general form with some unknown coefficients. I could then compute the charge density σ on the iris in this form (first line; Smythe's Problem 38 result is on the second line)

$$\sigma(\zeta,\theta) \sim \Sigma_{mn}\ C_{nm}\, P_n^m(\xi_0)\, q_n^m(\zeta) \cos(m\theta)$$

$$\sigma(\rho,\theta) = - (q/2\pi[\rho^2 + b^2 - 2b\rho\cos\theta]) / \sqrt{\rho^2-a^2}$$

where here I cheat a little and steal from the open bloid solution. In the above, θ of course is the azimuthal location on the iris, and ζ encodes the radial distance $\rho = c_1\sqrt{1+\zeta^2}$, and finally ξ_0 encodes the location b of the Green's point charge in the hole of the iris. At this point, I could not see, and I still cannot see, how this complicated looking double sum shown above on the first line could possibly reduce to the simple elementary function shown on the second line. Some day I hope to understand this.

[I don't think I have ever done a Fourier analysis of 1/(a+bcosθ) which is what you would want for the above comparison starting off. This GR integral looks appropriate

3.613

$$1.^6 \qquad \int_0^\pi \frac{\cos nx\,dx}{1+a\cos x} = \frac{\pi}{\sqrt{1-a^2}}\left(\frac{\sqrt{1-a^2}-1}{a}\right)^n \qquad [a^2<1, \quad n\geq 0]$$

This would remove the θ part of the comparison I think. You would end up with something like this:

$\Sigma_n\, C_{nm}\, P_n^m(\xi_0)\, q_n^m(\zeta) = f^m(\rho,b,a)$ where f^m is some elementary function]

D. Problem 3: Green's for a disk as superposition of rings?

In **Problem 3** I take a very brief look at the "inverse" problem of a point charge in the plane of a metal disk. In some sense I feel this is the inverse of the iris problem. All I do here is comment that you cannot treat this as a superposition of rings because adding a second ring affects the charge on the first. See comments above in Problem 1 part (b)

E. Last Section: try again verifying the Stak integral equation with the known Problem 38 σ.

Here I make a *valiant* effort to verify that equation (*) above is true. My method is to expand the $1/\sqrt{a+b\cos\theta}$ on each side of this equation, using expansions I "discovered", then do partial waves in cos(nθ). These expansions involve Q functions. I get it down to a final statement that I think I could verify if I wanted to spend the time.

There is an implicit lesson here I think. In the end, things always have to "work out", but sometimes what you have to show is "exceedingly obscure". By doing things in a certain manner or order, you force the math to have to do triple back flips, but in the end it will do all those flips for you.

A. Comments:

In this document, I am going **down a path** that doubtless Green and Kelvin and others went down, flailing away trying to find the charge induced onto a *flat disk or iris* by a point charge. It is quite amazing how hard this problem is to solve with "the usual" methods. Recall that finding the very simple charge distribution on a charged disk is similarly hard and requires certain tricks, such as Jackson's cylindrical atomic expansion coupled with a beforehand knowledge of how to solve a strange pair of dual integral equations, or MF's oblate spheroidal coordinate limit.

In my approach here, I do the obvious thing. I select a set of polar coordinates centered on the disk or iris. If I set the origin somewhere else, such as on the point charge, then the integration limits become unpleasant, since we are no longer centered on the disk or iris. I start with the 2D integral equation which says V = 0 on the iris, and I Fourier-expand the charge density. I think it is because the point charge breaks the θ symmetry that things don't diagonalize and the Fourier analysis just ends up producing an infinite set of radial integral equations for the $\sigma_n(r)$ polar components of the charge σ. So the original θ integration is replaced by the infinite n sum.

This problem is really just a standard Green's Function problem, but for a particular class of location of the point charge. Here that point charge lies in the hole of the iris or the plane of the disk.

Although I briefly comment on oblate coordinates below, I think that is probably the right way to solve this kind of problem and I will resume that work in another document. [I did this and got a solution, but it is a double sum of p and q functions.]

(0) Note added 1.25.10. I now know that you can't put a spherical or polar origin at disk center or anywhere on the metal disk and still have an expansion in spherical or polar atoms. (!!) [see next note]

(0') Above note updated 4.15.10. It is true that you cannot put a spherical origin at disk center and have the near-field potential be expressed in spherical atoms. But it is OK to do this for cylindrical coordinates, and thus for our polar coordinates. See "charged disk by Smythe Fit method.doc" last section(s).

(1) Although the geometry is 2D, the potential and gradients of it really exist in 3D space, so this really is a 3D problem, despite the *illusion* that it is a 2D problem. We have 3D Laplace. Using a 2D coordinate system to solve this 3D problem might then be expected to cause trouble. You could say you are using 3D cylindrical coordinates I suppose, and green Jackson does get some traction with such coordinates for the charged disk. [Red Jackson does the distant E field and iris in cylindricals; Smythe did it in oblates.] So I think one could pursue this successfully in cylindrical coordinates, I have not done anything with that.

(2) The issue is that this is a 3D problem, but you are working with a singular 2D surface. It might be simpler to work with non-singular 3D surface, such as the oblate spheroid, and then work on the problem as a fully 3D problem, and only in the end take the limit of the disk or iris.

(3) One indication of the trouble here is the complexity of the Fourier coefficients you get when you Fourier expand the famous 1/R factor where R is the distance between two points in the integration. Although you can *describe* both points simply in (r,θ) format, the 1/R expansion is quite horrible:

$$\text{Coefficients of } 1/R = 1/|\mathbf{r}-\mathbf{r}'| = (1/r)\, 1/\sqrt{a - b\cos(\theta)} \qquad a=1+f^2 \quad b = 2f \quad f = r'/r$$

$$= (1/r) \Sigma_n F_n(f) \cos(n\theta)$$

$$F_n(f) = (2/|1-f|) \Sigma_{s=0}^{n} (-1)^s \, (2n, 2n-2s)\, B[\, s+1/2, n-s+1/2\,]\, F\,[s+1/2, 1/2; n+1; -4f/(1-f)^2\,]$$

In a true 2D problem, the potential is ~ ln(1/R) instead of 1/R, and then the expansion is much nicer and is often used by Stakgold in solving problems.

[Note added 5.2.10: I now know that

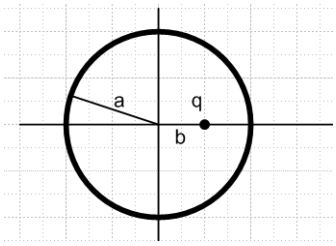
$$F_n(a,b) = (1/\pi)\sqrt{2/b}\ \varepsilon_n\ Q_{n-1/2}(a/b) = (1/\pi)\sqrt{1/f}\ \varepsilon_n\, Q_{n-1/2}[(1+f^2)/2f]$$

$$= (1/\pi)\sqrt{r/r'}\ \varepsilon_n\, Q_{n-1/2}[(r^2 + r'^2)/(2rr')] \qquad]$$

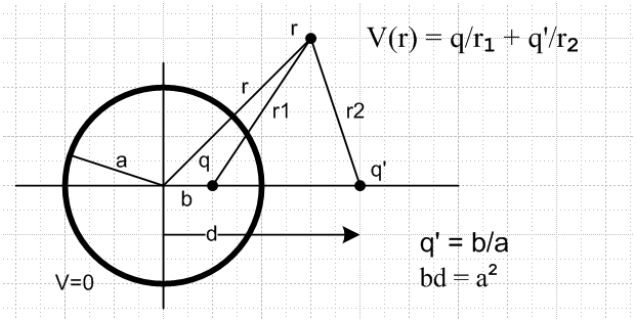
B. Problem 1. A point charge inside a grounded metal ring. (3D)

See Overview concerning the Big Error which exists in this section.

This is just a preliminary problem, the point of which is that you cannot construct a disk with V=0 from rings which are at V=0, where everything has slippery induced charge on it. I comment on this again in the last section of this document. So here is the ring situation:



We know that if put an image charge in a certain place outside the ring, we get V = 0 on the entire sphere which contains this ring. So that will certainly make V = 0 on the ring. I guess if we think of the entire boundary as a very thin but solid metal ring and the infinity sphere, then this solution is unique. What is the Green's Function for this problem? I think it is just V(**r**) as given here: [wrong]



Comment: If you superpose two such rings, concentric with radii a_1 and a_2, superposition sum "Problem 3" will *not* have V = 0 on the two rings. There *is* a solution of a point charge plus two rings both at V = 0, but it won't be the superposition of Problem 1 and Problem 2. This says basically that we cannot solve a disk problem by superposing ring problems! Too bad. [probably correct]

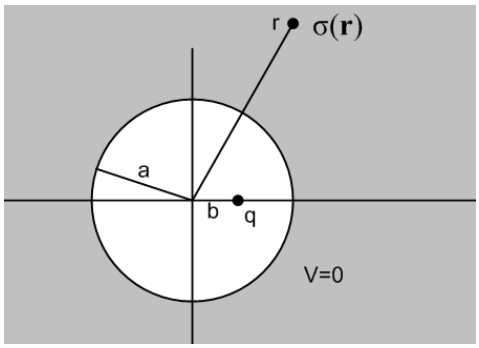
Note added 3.9.10. We have at least solved one problem: the Green's function in 3D for a metal ring where the Green's charge lies in the plane of the ring. This Green's function happens to vanish on an entire sphere which contains the ring. [this note is wrong]

C. Problem 2. Find σ on an grounded metal iris due to point charge inside the hole:

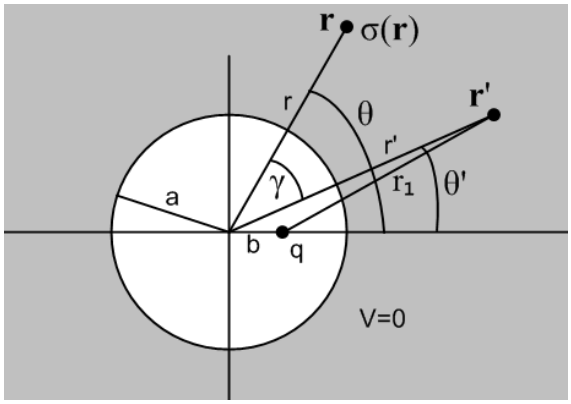
This section is the main act of this little paper. the polar coordinates can be regarded two of the cylindrical coordinates.

(a) Try doing this in polar coordinates.

Setting up the problem as a 2D integral equation



So here is an instrumentation of the problem



We know that the following must be true:

$$V(\mathbf{r}') = q/r_1 + \int_{\mathbf{a}}^{\infty} r\,dr \int_{\mathbf{0}}^{\mathbf{2\pi}} d\theta\ \sigma(r,\theta) / |\mathbf{r} - \mathbf{r}'| = 0 \qquad r' \geq a$$

Fourier expand the charge density.

Our setup has $\sigma(r,\theta)$ being even in θ so we can expand

$$\sigma(r,\theta) = \Sigma_{n=0}^{\infty}\ \sigma_n(r)\cos(n\theta)$$

At the same time, we know that

$$|\mathbf{r}-\mathbf{r}'|^2 = r^2 + r'^2 - 2rr'\cos(\theta-\theta')$$

so our condition above can be written

$$V(\mathbf{r}') = q/r_1 + \int_{\mathbf{a}}^{\infty} r\,dr \int_{\mathbf{0}}^{\mathbf{2\pi}} d\theta\ \{\Sigma_{n=0}^{\infty}\ \sigma_n(r)\cos(n\theta)\} / \sqrt{r^2 + r'^2 - 2rr'\cos(\theta-\theta')} = 0$$

$$= q/r_1 + \Sigma_{n=0}^{\infty} \int_a^{\infty} r\, dr\, \sigma_n(r) \int_0^{2\pi} d\theta \cos(n\theta) / \sqrt{r^2 + r'^2 - 2rr' \cos(\theta-\theta')} \quad = 0$$

This is a famous integral family I have run into many times (and I now have a doc on it in my new "integrals" directory). It is some kind of elliptic integral. Assume we can do this integral and we get as a result $I_n(r; r',\theta')$. We then are left with

$$V(r',\theta') = q/r_1 + \Sigma_{n=0}^{\infty} \int_a^{\infty} r\, dr\, \sigma_n(r)\, I_n(r; r',\theta') = 0$$

We are then left with an **unpleasant integral equation** for $\sigma_n(r)$, Stakgold style. But let's go ahead and process it some more. Pull out 1/r to get dimensionless:

$$V(\mathbf{r}') = q/r_1 + \Sigma_{n=0}^{\infty} \int_a^{\infty} dr\, \sigma_n(r) \int_0^{2\pi} d\theta \cos(n\theta) / \sqrt{1^2 + f^2 - 2f\cos(\theta-\theta')}$$

$$I \equiv \int_0^{2\pi} d\theta \cos(n\theta) / \sqrt{1^2 + f^2 - 2f\cos(\theta-\theta')}$$

where $f = r'/r$. Now change variable to $\theta'' = \theta-\theta'$,

$$I = \int_0^{2\pi} d\theta''\, \mathrm{cosn}(\theta'+\theta'') / \sqrt{1^2 + f^2 - 2f\cos(\theta'')} = \int_0^{2\pi} d\theta\, \mathrm{cosn}(\theta'+\theta) / \sqrt{1^2 + f^2 - 2f\cos(\theta)}$$

and use

$$\mathrm{cosn}(\theta'+\theta) = \mathrm{cosn}\theta'\ \mathrm{cosn}\theta - \mathrm{sinn}\theta'\ \mathrm{sinn}\theta$$

$$I = \cos(n\theta') \int_0^{2\pi} d\theta\, \mathrm{cosn}\theta / \sqrt{1^2 + f^2 - 2f\cos(\theta)} - \sin(n\theta') \int_0^{2\pi} d\theta\, \mathrm{sinn}\theta / \sqrt{1^2 + f^2 - 2f\cos(\theta)}$$

$$= \cos(n\theta') \int_0^{2\pi} d\theta\, \mathrm{cosn}\theta / \sqrt{1^2 + f^2 - 2f\cos(\theta)} = \cos(n\theta')\, F_n(f)$$

In the second integral on the first line above, integrand is odd, range is even if done $-\pi,\pi$, integral is 0. So in our little Fourier analysis, only the cos integral is needed.

$$I(f) = \cos(n\theta')\, F_n(f) \qquad f = r'/r$$

$$F_n(f) = \int_0^{2\pi} d\theta\, \mathrm{cosn}\theta / \sqrt{1^2 + f^2 - 2f\cos(\theta)}$$

$$V(r',\theta') = q/r_1 + \Sigma_{n=0}^{\infty} \cos(n\theta') \int_a^{\infty} dr\, \sigma_n(r)\, F_n(f).$$

Let $a = 1+f^2 > 1$ and $b = 2f > 0$

$$F_n(a,b) = \int_0^{2\pi} dx\, \cos(nx)/\sqrt{a - b\cos(x)}$$

Compute the projection integral $F_n(a,b)$

[Note added 5.2.10: I now know (from "cos(nx) over sqrt...") that

$$F_n(a,b) = 2 \int_0^{\pi} dx\, \cos(nx)/\sqrt{a - b\cos(x)} = 2\sqrt{2/b}\; Q_{n-1/2}(a/b)$$

so in our case we have $a = 1+f^2$ and $b = 2f$ so result is

$$F_n(a,b) = 2\sqrt{1/f}\; Q_{n-1/2}[(1+f^2)/2f] = 2\sqrt{r/r'}\; Q_{n-1/2}[(1+(r'/r)^2)/(2r'/r)]$$

$$= 2\sqrt{r/r'}\; Q_{n-1/2}[(r^2+r'^2)/(2r'r)]$$

$$= \int_0^{2\pi} d\theta\, \cos n\theta/\sqrt{1^2 + (r'/r)^2 - 2(r'/r)\cos(\theta)}$$

$$= r\int_0^{2\pi} d\theta\, \cos n\theta/\sqrt{r^2 + r'^2 - 2rr'\cos(\theta)}$$

Therefore we can write

$$\int_0^{2\pi} d\theta\, \cos n\theta/\sqrt{r^2 + r'^2 - 2rr'\cos(\theta)} = 2\sqrt{1/rr'}\; Q_{n-1/2}[(r^2+r'^2)/(2r'r)]$$

where both sides are symmetric in $r \leftrightarrow r'$.]

I spent an entire day getting this integral understood and have a doc just on it. The integral is the Fourier projection (on a circle, ie, on one of the polar coordinates) of the function shown, and here is the result:

$$F_n(a,b) = (2/\sqrt{a - b}\,)\, \Sigma_{s=0}^{n}\, (-1)^s\, (2n, 2n-2s)\, B[\, s+1/2,\, n-s+1/2\,]\, F\,[s+1/2,\, 1/2;\, n+1;\, -2b/(a-b)\,]$$

where B is the beta function and F the HGF. It is a real mess. For each n, you can write $F_n(a,b)$ as a linear combination of the complete E and K elliptic integrals, but it is a nasty ladder situation with no closed result for all n in that form. So the result above is really the best you can do. Numerically at least we could compute these coefficients for the lowest n.

The argument of the F is usually called k^2 which is the k^2 you see in elliptic integrals. So we have

$k^2 = -2b/(a-b) = -4f/(1-f)^2 < 0$ $\quad a = 1+f^2 > 1$ $\quad b = 2f > 0$ $\quad f = r'/r$

$(a-b) = (1-f)^2 \geq 0$ so $a \geq b$

$(a+b) = (1+f)^2 > 0$

Meanwhile, we can write (both expressions obvious from staring at the instrumented picture above)

$$r_1^2 = (r'\cos\theta' - b)^2 + (r'\sin\theta')^2 = r'^2 + b^2 - 2br'\cos\theta'$$

So once again, here is where we sit in our analysis of this problem:

$$V(r',\theta') = q/r_1 + \Sigma_{n=0}^{\infty} \cos(n\theta') \int_a^{\infty} dr\, \sigma_n(r)\, F_n(f) = 0 \qquad r' \geq a \qquad f = r'/r$$

$$F_n(f) = (2/|1-f|)\, \Sigma_{s=0}^{n} (-1)^s\, (2n, 2n-2s)\, B[\, s+1/2, n-s+1/2\,]\, F[\, s+1/2, 1/2; n+1; -4f/(1-f)^2\,]$$

$$r_1^2 = r'^2 + b^2 - 2br'\cos\theta'$$

Fourier analysis leads to an infinite set of 1D integral equations.

We have basically done a Fourier analysis of the potential in the angle of polar coordinates relative to the hole center. We picked this origin so we could have tractable limits on our integration. Our next step really should be a Fourier analysis of the q/r_1 term, then we can set the total Fourier coefficient of each term to 0, and that is about the best we can do.

So write

$$r_1^2 = r'^2 + b^2 - 2br'\cos\theta' = b^2\,[r'^2/b^2 + 1 - 2r'/b\cos\theta'] = b^2\,[1 + g^2 - 2g\cos\theta'] \qquad g \equiv r'/b$$

Now define $A = 1+g^2$ and $B = 2g$ in analogy to the above. Then we have

$$r_1^2 = b^2\,[A - B\cos\theta'] \qquad q/r_1 = (q/b)\, 1/\sqrt{A - B\cos\theta'}$$

We look out our integral document which shows how to do Fourier on this:

$$f(x) = 1/\sqrt{a - b\cos(x)} = a_0/2 + \Sigma_{n=1}^{\infty} a_n \cos(nx) \qquad \text{// expansion}$$
$$I = \pi a_n = \int_0^{2\pi} dx\, \cos(nx)/\sqrt{a - b\cos(x)} \qquad \text{// projection}$$

so we therefore write

$$q/r_1 = (q/b)\,[c_0/2 + \Sigma_{n=1}^{\infty} c_n \cos(n\theta')]$$

$$c_n = (1/\pi) \int_0^{2\pi} dx\, \cos(nx)/\sqrt{A - B\cos\theta'}$$

and we know that

$$\pi c_n = I = (2/\sqrt{A - B}\,)\, \Sigma_{s=0}^{n} (-1)^s\, (2n, 2n-2s)\, B[\, s+1/2, n-s+1/2\,]\, F[\, s+1/2, 1/2; n+1; -2B/(A-B)\,]$$

Continuing our analogy we write

$K^2 = -2B/(A-B) = -4g/(1-g)^2 < 0$ $\quad A = 1+g^2 > 1$ $\quad B = 2g > 0$ $\quad g = r'/b$

$(A-B) = (1-g)^2 \geq 0$ so $A \geq B$

$(A+B) = (1+g)^2 > 0$

At this point we have

$$V(r',\theta') = q/r_1 + \Sigma_{n=0}^{\infty} \cos(n\theta') \int_a^{\infty} dr\, \sigma_n(r)\, F_n(f) = 0$$

$$= (q/b)\ [c_0/2 + \Sigma_{n=1}^{\infty} c_n \cos(n\theta')] + \Sigma_{n=0}^{\infty} \cos(n\theta') \int_a^{\infty} dr\, \sigma_n(r)\, F_n(f) = 0$$

We group by our θ' basis functions

$$= (q/b)\ [c_0/2] + \int_a^{\infty} dr\, \sigma_0(r)\, F_0(f)$$

$$+ \Sigma_{n=1}^{\infty} [(q/b)\, c_n(g) + \int_a^{\infty} dr\, \sigma_n(r)\, F_n(f)\ ] \cos(n\theta')$$

where we have written the n=0 term on a separate line. So here is where we want to be, the best we can hope for in this approach I think. We then identify:

$(q/b)\ [c_0/2] + \int_a^{\infty} dr\, \sigma_0(r)\, F_0(f) = 0$ $\qquad n = 0$

$(q/b)\, c_n(g) + \int_a^{\infty} dr\, \sigma_n(r)\, F_n(f) = 0$ $\qquad n = 1,2,3....$

Our observation point is **r'** so we can think of r' and b and thus g as specified numbers. So we should rewrite the above in this way:

$\int_a^{\infty} dr\, \sigma_0(r)\, F_0(f) = -\ (q/b)\, c_0(g)/2$ $\qquad n=0$ $\qquad g \equiv r'/b$

$\int_a^{\infty} dr\, \sigma_n(r)\, F_n(f) = -\ (q/b)\, c_n(g)$ $\qquad n = 1,2,3$

So now we have an infinite set of 1D integral equations that we need to solve for the $\sigma_n(r)$! We might be tempted to insert our known expressions for the F_n and the c_n and require term by term equality. That would be incorrect, however, because the terms in the expansion are known not to be independent. For example, we know that each F_n is going to be some linear combination of the elliptic K and E functions, when we add the various HG F functions together.

Discussion: Our starting point was this:

$$V(\mathbf{r}') = q/r_1 + \int_a^\infty r\,dr \int_0^{2\pi} d\theta\ \sigma(r,\theta) / |\mathbf{r} - \mathbf{r}'| = 0 \qquad r_1^2 = r'^2 + b^2 - 2br'\cos\theta'$$

This is a 2D integral equation for σ(r,θ). By doing a Fourier analysis on the θ variable, we were able to replace this with an infinite set of 1D integral equations, those shown just above. It was a good exercise in "maintaining control" in a complex (for me) situation, with lots of symbols flying by. But basically, we are no closer to the solution than when we started. [This is what happens in any of the integral equation problems I have tried in the past, you just get a mess with some K functions flying around.]

So somehow, this was not the right approach. Live and learn.

(b) Attempt to show that Smythe's simple expression for σ solves our 2D integral equation

How is it that Smythe gets such a simple answer to this problem (Smythe p 203)

38. A charge q is at a distance b from the center of a circular hole of radius a in an infinite earthed flat conducting sheet and lies in the same plane. Show that the charge density induced on the sheet at a distance r from the charge and c from the center of the hole is $-q(a^2 - b^2)^{\frac{1}{2}}/[2\pi^2 r^2 (c^2 - a^2)^{\frac{1}{2}}]$.

I will rephrase this for my picture: "Show that the charge density induced on the sheet at a distance r_1 from the charge and r' from the center of the hole is $-q(a^2 - b^2)^{1/2} / [\,2\pi^2 r_1^2 (r'^2 - a^2)^{1/2}]$. "

So he is claiming that the answer is this, for my picture,

$$\sigma(r',\theta') = -\,(q/2\pi r_1^2)\sqrt{a^2-b^2}\,/\sqrt{r'^2-a^2} \qquad r_1^2 = r'^2 + b^2 - 2br'\cos\theta'$$

Can I "check" to see if this result works? The above is stated for $\mathbf{r}' = (r',\theta')$ being any point on the disk, so I should be able to write

$$\sigma(r',\theta') = -\,(q/2\pi[r'^2 + b^2 - 2br'\cos\theta'])\sqrt{a^2-b^2}\,/\sqrt{r'^2-a^2}$$

$$\sigma(r,\theta) = -\,(q/2\pi[r^2 + b^2 - 2br\cos\theta])\sqrt{a^2-b^2}\,/\sqrt{r^2-a^2}$$

$$= -(q/2\pi b^2)\sqrt{a^2-b^2}\ [\,1 + h^2 - 2h\cos\theta]^{-1} / \sqrt{r^2-a^2} \qquad h = r/b > 1$$

We could further write

$$r^2-a^2 = b^2\,[\,h^2 - d^2] \qquad d = (a/b) = \text{a constant fraction} > 1$$
$$\sqrt{r^2-a^2} = b\sqrt{h^2-d^2}$$

$$\sigma(r,\theta) = -(q/2\pi b^3)\sqrt{a^2-b^2}\,[\,1 + h^2 - 2h\cos\theta]^{-1} / \sqrt{h^2-d^2} = \sigma(h,\theta)$$

so we have replaced the r variable with the dimensionless h variable.

Our integral condition right at the start above was

$$V(\mathbf{r}') = q/r_1 + \int_{\mathbf{a}}^{\infty} r\,dr \int_{\mathbf{0}}^{\mathbf{2\pi}} d\theta\ \sigma(r,\theta) / |\mathbf{r} - \mathbf{r}'| = 0$$

where

$$|\mathbf{r}-\mathbf{r}'|^2 = r^2 + r'^2 - 2rr' \cos(\theta-\theta') = r^2 [1^2 + f^2 - 2f \cos(\theta-\theta')] \qquad r = r'/r$$

$$|\mathbf{r}-\mathbf{r}'| = r \sqrt{1^2 + f^2 - 2f \cos(\theta-\theta')}$$

so rewrite as

$$V(\mathbf{r}') = q/r_1 + \int_{\mathbf{a}}^{\infty} dr \int_{\mathbf{0}}^{\mathbf{2\pi}} d\theta\ \sigma(r,\theta) / \sqrt{1^2 + f^2 - 2f \cos(\theta-\theta')} = 0$$

Now let's try "installing" the claimed solution charge denstiy

$$V(\mathbf{r}') = q/r_1 + \int_{\mathbf{a}}^{\infty} dr \int_{\mathbf{0}}^{\mathbf{2\pi}} d\theta\ \sigma(r,\theta) / \sqrt{1^2 + f^2 - 2f \cos(\theta-\theta')} \qquad f = r'/r$$

Before installing, lets replace dr with dh since h = r/b so

$$\int_{\mathbf{a}}^{\infty} dr = b \int_{\mathbf{d}}^{\infty} dh$$

$$V(\mathbf{r}') = q/r_1 + b \int_{\mathbf{d}}^{\infty} dh \int_{\mathbf{0}}^{\mathbf{2\pi}} d\theta\ \sigma(h,\theta) / \sqrt{1^2 + f^2 - 2f \cos(\theta-\theta')}$$

$$\sigma(h,\theta) = -(q/2\pi b^3) \sqrt{a^2-b^2}\ [\ 1 + h^2 - 2h \cos\theta]^{-1} / \sqrt{h^2-d^2}$$

Now we do the installation:

$$V(\mathbf{r}') = q/r_1 - b(q/2\pi b^3) \sqrt{a^2-b^2} \ * \qquad d = a/b$$

$$\int_{\mathbf{d}}^{\infty} dh \int_{\mathbf{0}}^{\mathbf{2\pi}} d\theta\ [\ 1 + h^2 - 2h \cos\theta]^{-1} / [\ \sqrt{h^2-d^2}\ \sqrt{1^2 + f^2 - 2f \cos(\theta-\theta')}\ ]$$

In the above we think of h = r/b so

$$f = r'/r = (r'/b)\ (b/r) = (r'/b) / h$$

and we need to process the f square root a bit:

$$1 + f^2 - 2f \cos(\theta-\theta') = 1 + (r'/b)^2\ h^{-2} - 2\ (r'/b)h^{-1} \cos(\theta-\theta')$$

$= h^{-2} [h^2 + (r'/b)^2 - 2 (r'/b)h \cos(\theta-\theta')]$

But we already have a symbol $g \equiv r'/b$ but let's change that to $\rho' = r'/b$ so we get

$= h^{-2} [h^2 + \rho'^2 - 2 \rho'h \cos(\theta-\theta')]$

Then we have

$$\sqrt{1^2 + f^2 - 2f \cos(\theta-\theta')} = \sqrt{h^2 + \rho'^2 - 2 \rho'h \cos(\theta-\theta')} / h$$

$$1/\sqrt{1^2 + f^2 - 2f \cos(\theta-\theta')} = h / \sqrt{h^2 + \rho'^2 - 2 \rho'h \cos(\theta-\theta')}$$

So our result above becomes

$$V(\mathbf{r}') = q/r_1 - b(q/2\pi b^3) \sqrt{a^2-b^2} \quad * \qquad d = a/b$$

$$\int_d^\infty dh\, h \int_0^{2\pi} d\theta \, [1 + h^2 - 2h \cos\theta]^{-1} / [\sqrt{h^2-d^2} \sqrt{h^2 + \rho'^2 - 2 \rho'h \cos(\theta-\theta')}]$$

$= V(\rho',\theta') = 0 \qquad \rho' = r'/b \qquad h = r/b$

Meanwhile

$r_1^2 = b^2 [1 + \rho'^2 - 2\rho' \cos\theta'] \qquad g \equiv r'/b = \rho'$

Rewrite the above thing putting stuff on the other side:

$$\int_d^\infty dh\, h \int_0^{2\pi} d\theta \, [1 + h^2 - 2h \cos\theta]^{-1} / [\sqrt{h^2-d^2} \sqrt{h^2 + \rho'^2 - 2 \rho'h \cos(\theta-\theta')}]$$

$$= (q/r_1) [b(q/2\pi b^3) \sqrt{a^2-b^2}]^{-1} = (q/r_1) (1/b) (2\pi b^3/q) 1/ \sqrt{a^2-b^2}$$

$$= (1/r_1)(2\pi b^2) / \sqrt{a^2-b^2}$$

$$= 2\pi b^2 / \sqrt{a^2-b^2} * 1/b * 1/\sqrt{1 + \rho'^2 - 2\rho' \cos\theta'}$$

$$= 2\pi b / [\sqrt{a^2-b^2} \sqrt{1 + \rho'^2 - 2\rho' \cos\theta'}]$$

Restate:

$$\int_d^\infty dh\, h \int_0^{2\pi} d\theta \, [1 + h^2 - 2h \cos\theta]^{-1} / [\sqrt{h^2-d^2} \sqrt{h^2 + \rho'^2 - 2 \rho'h \cos(\theta-\theta')}]$$

$$=? = \ 2\pi b / [\ \sqrt{a^2-b^2}\ \sqrt{1 + \rho'^2 - 2\rho' \cos\theta'}\ ]$$

Now write using $d = a/b > 1$,

$$a^2 - b^2 = b^2(d^2 - 1) \qquad \Rightarrow \qquad \sqrt{a^2 - b^2} = b\sqrt{d^2 - 1}$$

So in order to show that Smythe's charge density really works, we have to show this:

$$\int_d^\infty dh\ (h/\sqrt{h^2-d^2}\ )\ \int_0^{2\pi} d\theta\ * 1/ [\ (1 + h^2 - 2h\cos\theta)\ \sqrt{h^2 + \rho'^2 - 2\rho' h\cos(\theta-\theta')}\ ]$$

$$= \ (\ 2\pi / \sqrt{d^2 - 1}\ ) * 1/ \sqrt{1 + \rho'^2 - 2\rho' \cos\theta'}\ ]$$

Both sides are dimensionless. Apart from algebra errors I no doubt have made, this integral no doubt comes out exactly as above. But the complexity of even the single integrals within this double integral is a testimony to the *Badness* of the "coordinate system" in which I have just been studying this problem!

BUT, see last section below where I basically do both integrals and pretty much verify the result!! (5.2.10)

(c) Is Smythe doing this somehow in oblate spheroidal coordinates??

If I tried to do a "form fit" to this problem, what would it look like? Outside the hole

$$V_o(\zeta,\xi,\varphi) \ = \ \Sigma_{n,m}\, A_{nm}\ Q_n^m(i\zeta)P_n^m(\xi)e^{im\varphi}$$

where in ξ I keep only the P because I know the problem includes in $\xi = \pm 1$ where the Q would blow up. We could then compute the charge density in this way: $[\partial_n = (1/h_1)\,\partial_{q1}]$

$$\partial_\xi V_o(\zeta,\xi,\varphi) \ = \ \Sigma_{n,m}\, A_{nm}\ Q_n^m(i\zeta)P'^m_n(\xi)\ e^{im\varphi}$$

$$\sigma \ = \partial_n V_o(\zeta,\xi=0,\varphi) \ = (h_\xi)^{-1}\ \partial_\xi V_o(\zeta,\xi=0,\varphi) \ = \ (h_\xi)^{-1}\ \Sigma_{n,m}\, A_{nm}\ Q_n^m(i\zeta)P'^m_n(0)\ e^{im\varphi}$$

The coordinate ζ tells you where you are on the iris.

[Note added 4.16.10: The above conjectured form for the potential is in fact correct!! This doc is dated 1.17.10, and a month later, circa 2.21.10, I solved the iris problem in oblate spheroidals in "oblate bloid with p and q attempt 4.doc" and got the following result for a Green's charge q in the hole of an iris:

$$V(\zeta,\xi,\varphi) \ = \Sigma_{nm}\, B_{nm}\, [\sqrt{\pi}\ 2^m\ h(n,m)]\ q_n^m(\zeta)\ \ P_n^m(\xi)\cos(m\varphi) \qquad \text{// upper and lower}$$

$$B_{nm} \ = \ (2-\delta_{m,0})\ (q/[4\pi c_1 \varepsilon])\ \ P_n^m(\xi_0)\ \sec[(n+m)\pi/2]\ f(n,-m)^2\ (2n+1) \qquad q_n^m(\zeta) \equiv (-i)^{2m}\ (\pm i)^{n+1}\ Q_n^m(i\zeta)$$

where parameter ξ_0 encodes the position of the Green's charge in the hole, and where ζ and ξ encode a location on the iris.

Recall that

$$x = c_1\zeta\xi \tag{4}$$

and eliminating x gives

$$\rho = c_1[(1+\zeta^2)(1-\xi^2)]^{\frac{1}{2}} \tag{5}$$

so in our case we have $\xi = 0$ on the iris so $\rho = c_1\sqrt{1+\zeta^2}$ going radially out on the iris.

$\sigma = (h_\xi)^{-1} \Sigma_{n,m} A_{nm} Q_n^m(i\zeta)P'^m_n(0) e^{im\varphi}$

Now I have to go "hunting" in Smythe for these zero values. Here is one hit,

$$(n+m \text{ even}) \qquad P_n^m(0) = (-1)^{\frac{1}{2}(n-m)}\frac{1\cdot 3\cdot 5\cdots(n+m-1)}{2\cdot 4\cdot 6\cdots(n-m)}$$

$$(n+m \text{ odd}) \qquad P_n^m(0) = 0$$

$$\left[\frac{d^r}{d\mu^r}P_n^m(\mu)\right]_{\mu=0} = P_n^{m+r}(0)$$

So this would say that

$P'^m_n(0) = P_n^{m+1}(0)$ = see above

The simplest you could possibly hope for here is that only the n=0,m=0 term survives. Then you would have

$\sigma = (h_\xi)^{-1} A_{00}$

$$h_1 = c_1^2\xi\left[\frac{x^2}{(-c_1^2\xi^2)^2} + \frac{\rho^2}{c_1^4(1-\xi^2)^2}\right]^{\frac{1}{2}} = c_1\left(\frac{\xi^2+\zeta^2}{1-\xi^2}\right)^{\frac{1}{2}} \tag{6}$$

This h_1 is h_ξ for Smythe. But this says then that $(h_\xi)^{-1} = 0$ everywhere on the iris. [The iris has angle coordinate $\xi = 0$. Then $h_1 = c_1\zeta$. I think ζ then takes values c_1 through ∞ on the iris, so maybe conclusion wrong. But surly we don't just get a 00 term since the point charge is off center.]

So OK, I did some due diligence [pretty minimal] , I don't see how oblates could possibly get such a simple result! [This is a full blown Green's Function problem. I guess you would do it with the oblate not crushed somehow, then in the end crush down to the iris. But I have never tried the on-plane Green's function problem for an oblate spheroid!]

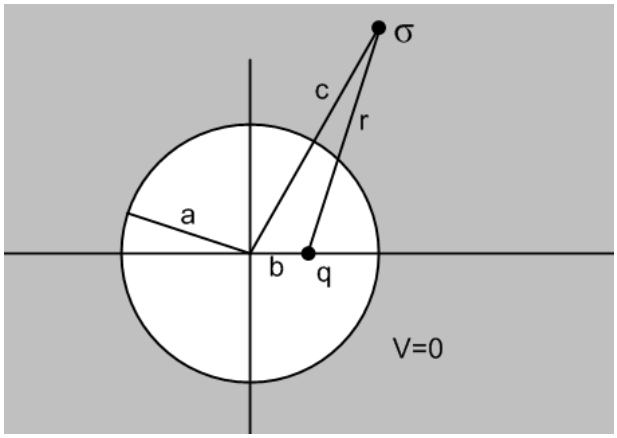
Status: So once again, I am blocked on a tributary feeder road into the "bowl problem" solution. Smythe gave me a path, but I cannot verify the first step of his path!

Note added 3.9.10. I eventually did the oblate spheroid problem [see " Exterior general Green's Function for an Oblate Spheroid.doc"] and the essence of the above discussion is correct. You have things like $Q_n^m(i\zeta)$ sitting in your Σ_{nm} expression for σ, and one sees no way to remove such factors to produce the elementary function Smythe shows below as the solution.

D. Problem 3. A point charge outside a grounded metal disk. (3D)

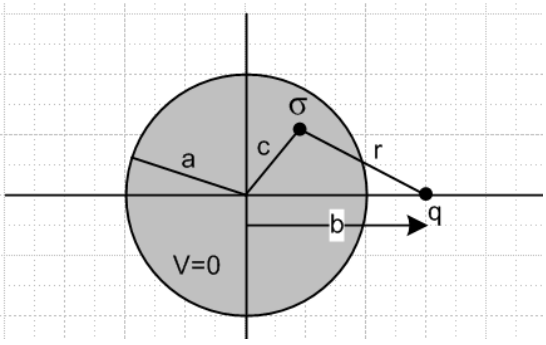
Smythe's Problem, its Claimed Result, and my Picture to go with it, is this

38. A charge q is at a distance b from the center of a circular hole of radius a in an infinite earthed flat conducting sheet and lies in the same plane. Show that the charge density induced on the sheet at a distance r from the charge and c from the center of the hole is $-q(a^2 - b^2)^{\frac{1}{2}}/[2\pi^2 r^2(c^2 - a^2)^{\frac{1}{2}}]$.



You would somehow assume that maybe the solution is best found directly in terms of the parameters shown in this picture.

I think somehow the following problem is the same by "some kind of inversion" [Note Added 7.9.10: this early conjecture was correct, see "doing the disk by inversion.." section (6). The disk and iris problems are related by analytic continuation, not inversion.]



Maybe this one is easier to think about. Disk is grounded with a wire, bring in your charge q, some charge distribution is induced on the disk, what is it? I keep wanting to break this down into a superposition of rings which is why I started above with a ring problem for Problem #1. Does this work or not?

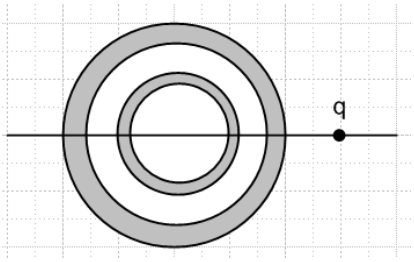
Problem 1: grounded ring at a_1 induced charge $\sigma(a_1,\theta)$

Problem 2: grounded ring at a_2 induced charge $\sigma(a_2,\theta)$

Problem 3: have both rings, both are grounded, both have above induced charge.

I guess it works! [wrong, Batman]

Say another way: consider this problem:



Assume both these thin rings are grounded. There *will be* an induced charge on each ring such that both rings are at zero potential. If I delete one of the rings, will the charge on the other ring change? Yes! This can be treated as a superposition of the type of Example 2, where the two objects are the two rings, and where we add our point charge q to Problem 1, say. When you get to Problem 3, you will find that neither ring is showing a constant potential. Another view: removing one ring removes some charge from the picture, and the other ring is going to adjust in response to this change. So, the ring method does not work!

E. Smythe Problem 38 attempted verification is nearly perfect (added 5.2.10)

In the doc above, we made this claim:

"So in order to show that Smythe's charge density really works, we have to show this:

$$\int_{\mathbf{d}}^{\infty} dh\, (h/\sqrt{h^2-d^2})\ \int_{\mathbf{0}}^{\mathbf{2\pi}} d\theta\ *\ 1/[\ (1 + h^2 - 2h\cos\theta)\ \sqrt{h^2 + \rho'^2\ - 2\rho' h\cos(\theta-\theta')}\ ]$$

$$=\ (\ 2\pi /\ \sqrt{d^2 - 1}\ )\ *\ 1/\ \sqrt{1 + \rho'^2 - 2\rho'\cos\theta'}\ ]\ "\qquad d = (a/b) = \text{a constant fraction} > 1$$

where a and b are fixed numbers in the Smythe problem.

I maybe be able to do the θ integral now that I have learned a little. Write it first this way

$$I \equiv \int_0^{2\pi} d\theta \; * \; 1/\left[\; (1 + h^2 - 2h\cos\theta) \; \sqrt{h^2 + \rho'^2 \; - 2\rho' h \cos(\theta-\theta')} \; \right]$$

Set

$a = h^2 + \rho'^2$

$b = 2\rho' h$

Go to the doc "cos(nx)...." where I do integrals like the above. There we show that

$$1/\sqrt{a - b\cos(x)} \; = \; (1/\pi)\sqrt{2/b} \; \Sigma_{n=0}^{\infty} \varepsilon_n \; Q_{n-1/2}(a/b) \cos(nx)$$

so we can rewrite the above integral as

$$I = (1/\pi)\sqrt{2/b} \; \Sigma_{n=0}^{\infty} \varepsilon_n \, Q_{n-1/2}(a/b) \; \int_0^{2\pi} d\theta \; \cos[n(\theta-\theta')] \, / \, (1 + h^2 - 2h\cos\theta)$$

Now we write

$$\cos[n(\theta-\theta')] \; = \cos(n\theta)\cos(n\theta') + \sin(n\theta)\sin(n\theta')$$

and we throw out the second term since it is odd in θ to get

$$I = (1/\pi)\sqrt{2/b} \; \Sigma_{n=0}^{\infty} \varepsilon_n \, Q_{n-1/2}(a/b) \cos(n\theta') \; \int_0^{2\pi} d\theta \; \cos(n\theta) \, / \, (1 + h^2 - 2h\cos\theta)$$

Now define

$$J = \int_0^{2\pi} d\theta \; \cos(n\theta) \, / \, (1 + h^2 - 2h\cos\theta) \; = 2 \; \int_0^{\pi} d\theta \; \cos(n\theta) \, / \, (1 + h^2 - 2h\cos\theta)$$

and we go find in GR7 page 391 the following fact

$$2.^6 \qquad \int_0^{\pi} \frac{\cos nx \, dx}{1 - 2a\cos x + a^2} = \frac{\pi a^n}{1 - a^2} \qquad \left[a^2 < 1, \quad n \geq 0\right]$$

$$= \frac{\pi}{(a^2 - 1)\, a^n} \qquad \left[a^2 > 1, \quad n \geq 0\right]$$

BI (65)(3)

where I am starting to "feel lucky". Set a = h and this is it. My h integral has $h \geq d$ and above I already said that d = (a/b) = a constant fraction > 1, so we have $h \geq a$ which means we use the second form above, so we get

$$J = 2 * \pi \, h^{-n} / (h^2 - 1)$$

We have overloading of variables a and b here, so let's keep "d" as d as long as we can, maybe we can clear the arena of the other a and b. So we have shown that

$$I = (1/\pi)\sqrt{2/b}\ \Sigma_{n=0}^{\infty}\ \varepsilon_n\ Q_{n-1/2}(a/b)\cos(n\theta')\ *\ 2\pi\ h^{-n}/(h^2-1) \qquad a = h^2+\rho'^2 \qquad b = 2\rho' h$$

$$= 2\sqrt{1/h\rho'}\ \ h^{-n}/(h^2-1)\ \ \Sigma_{n=0}^{\infty}\ \varepsilon_n\cos(n\theta')\ Q_{n-1/2}[(h^2+\rho'^2)/(2\rho' h)]$$

where yes, we have now gotten rid of this overloaded a and b. Our Smythe verification problem then requires us to show that

$$\int_{\mathbf{d}}^{\infty} dh\ (h/\sqrt{h^2-d^2}\ )2\sqrt{1/h\rho'}\ \ h^{-n}/(h^2-1)\ \ \Sigma_{n=0}^{\infty}\ \varepsilon_n\cos(n\theta')\ Q_{n-1/2}[(h^2+\rho'^2)/(2\rho' h)]$$

$$= \ (\ 2\pi\ /\ \sqrt{d^2-1}\ )\ *\ 1/\ \sqrt{1+\rho'^2-2\rho'\cos\theta'}\ \ ] \qquad d = (a/b) = \text{a constant fraction} > 1$$

so at least I have done *one* of the two integrals. Rewrite as

$$LHS = \Sigma_{n=0}^{\infty}\ \varepsilon_n\cos(n\theta')\ \ \int_{\mathbf{d}}^{\infty} dh\ (h/\sqrt{h^2-d^2}\ )2\sqrt{1/h\rho'}\ \ (h^{-n}/(h^2-1))\ Q_{n-1/2}[(h^2+\rho'^2)/(2\rho' h)]$$

$$= \ 2\ (2/\sqrt{\rho'}\ )\Sigma_{n=0}^{\infty}\ \varepsilon_n\cos(n\theta')\ \ \int_{\mathbf{d}}^{\infty} dh\ h^{-n+1-1/2}/\sqrt{h^2-d^2}\ )\ (1/(h^2-1))\ Q_{n-1/2}[(h^2+\rho'^2)/(2\rho' h)]$$

$$= \ 2\ (2/\sqrt{\rho'}\ )\Sigma_{n=0}^{\infty}\ \varepsilon_n\cos(n\theta')\ \ \int_{\mathbf{d}}^{\infty} dh\ h^{-n+1/2}\ [(h^2-1)\sqrt{h^2-d^2}\ ]^{-1}\ \ Q_{n-1/2}[(h^2+\rho'^2)/(2\rho' h)]$$

Now let's expand $1/\ \sqrt{1+\rho'^2-2\rho'\cos\theta'}$ in a similar series, and we find

$$1/\ \sqrt{1+\rho'^2-2\rho'\cos\theta'}\ \ =\ (1/\pi)\ (1/\sqrt{\rho'}\ )\ \Sigma_{n=0}^{\infty}\ \varepsilon_n\ \ Q_{n-1/2}[(1+\rho'^2)/(2\rho')]\cos(n\theta')$$

So comparing both sides, this is what we have to show

$$2\ (2/\sqrt{\rho'}\ )\Sigma_{n=0}^{\infty}\ \varepsilon_n\cos(n\theta')\ \ \int_{\mathbf{d}}^{\infty} dh\ h^{-n+1/2}\ [(h^2-1)\sqrt{h^2-d^2}\ ]^{-1}\ \ Q_{n-1/2}[(h^2+\rho'^2)/(2\rho' h)]$$

$$= \ (\ 2\pi\ /\ \sqrt{d^2-1}\ )\ (1/\pi)\ (1/\sqrt{\rho'}\ )\ \Sigma_{n=0}^{\infty}\ \varepsilon_n\ \ Q_{n-1/2}[(1+\rho'^2)/(2\rho')]\cos(n\theta')$$

or

$$(2)\ \ \int_{\mathbf{d}}^{\infty} dh\ h^{-n+1/2}\ [(h^2-1)\sqrt{h^2-d^2}\ ]^{-1}\ \ Q_{n-1/2}[(h^2+\rho'^2)/(2\rho' h)]$$

$$= \ (\ 1\ /\ \sqrt{d^2-1}\ )\ \ Q_{n-1/2}[(1+\rho'^2)/(2\rho')]$$

So we want to show now that

$$\int_{\mathbf{d}}^{\infty} dh\, h^{-n+1/2}\, [(h^2-1)\sqrt{h^2-d^2}\,]^{-1} Q_{n-1/2}[(h^2+\rho'^2)/(2\rho' h)] = (1/2)\,(d^2-1)^{-1/2} Q_{n-1/2}[(1+\rho'^2)/(2\rho')]$$

which looks like a pretty tall order. But let's give it a try. First let y = h/d so dy = dh/d and we get

$$\text{LHS} = \int_{\mathbf{1}}^{\infty} d\, dy\, (dy)^{-n+1/2}\, [\,(d^2y^2-1))\sqrt{d^2y^2-d^2}\,]^{-1}\ \ Q_{n-1/2}[(d^2y^2+\rho'^2)/(2\rho' dy)]$$

$$= d\ \ d^{-n+1/2}\ (d)^{-1}\ \int_{\mathbf{1}}^{\infty} dy\ \ y^{-n+1/2}\, [\,(d^2y^2-1))\sqrt{y^2-1}\,]^{-1}\ \ Q_{n-1/2}[(d^2y^2+\rho'^2)/(2\rho' dy)]$$

$$= d^{-n+1/2}\ \int_{\mathbf{1}}^{\infty} dy\ \ y^{-n+1/2}\, [\,(d^2y^2-1))\sqrt{y^2-1}\,]^{-1}\ \ Q_{n-1/2}[(d^2y^2+\rho'^2)/(2\rho' dy)]$$

We can express the Q argument this way

$(d^2y^2+\rho'^2)/(2\rho' dy) = 1 + (dy-\rho')^2/(2\rho' dy)$ and we then have

$$= d^{-n+1/2}\ \int_{\mathbf{1}}^{\infty} dy\ \ y^{-n+1/2}\,(d^2y^2-1)^{-1}\,(y^2-1)^{-1/2}\, Q_{n-1/2}[1 + (dy-\rho')^2/(2\rho' dy)]$$

I was hoping to get this into a form I see on GR7 p 774 which says this

7.137

1. $$\int_1^\infty x^{-\frac{1}{2}\mu-\frac{1}{2}}(x-1)^{-\mu-\frac{1}{2}}(1+ax)^{\frac{1}{2}\mu}\, Q_\nu^\mu(1+2ax)\,dx$$
$$= \pi^{-1/2}e^{-\mu\pi i}\,\Gamma\left(\tfrac{1}{2}-\mu\right)a^{\frac{1}{2}\mu}\left\{Q_\nu^\mu\left[(1+a)^{1/2}\right]\right\}^2$$
$$\left[|\arg a|<\pi,\quad \operatorname{Re}\mu<\tfrac{1}{2},\quad \operatorname{Re}(\mu+\nu)>-1\right]\quad \text{ET II 325(28)}$$

but I would have to have μ = 0, so it is all wrong. So here is where we are leaving things:

Must show that

$$\int_{\mathbf{1}}^{\infty} dy\ \ y^{-n+1/2}\, [\,(d^2y^2-1))\sqrt{y^2-1}\,]^{-1}\ \ Q_{n-1/2}[(d^2y^2+\rho'^2)/(2\rho' dy)]$$

$$= (d^{n-1/2}/2)\,(d^2-1)^{-1/2} Q_{n-1/2}[(1+\rho'^2)/(2\rho')]$$

One more time with vigor

$$d^{-2}\int_{\mathbf{1}}^{\infty} dy\ \ y^{-n+1/2}\, [\,((y+1/d)(y-1/d)\sqrt{y^2-1}\,]^{-1}\ \ Q_{n-1/2}[(d^2y^2+\rho'^2)/(2\rho' dy)]$$

$$= (d^{n-1/2}/2)\,(d^2-1)^{-1/2} Q_{n-1/2}[(1+\rho'^2)/(2\rho')]$$

Just out of curiosity, we see that the LHS integrand has a poles at y = ±1/d. Suppose we could somehow argue that this integral was just the residue of one or both of these poles. The pole at y = 1/d would yield this residue on the LHS

$$\text{LHS} = d^{-2}\,(1/d)^{-n+1/2}\,(2/d)^{-1}\;\;((1/d)^2-1)^{-1/2}\;\; Q_{n-1/2}[(1+\rho'^2)/(2\rho')] \,/\, (2\pi i)$$

$$\text{RHS} = (d^{n-1/2}/2)\,(d^2-1)^{-1/2} Q_{n-1/2}[(1+\rho'^2)/(2\rho')]$$

Now we can write $((1/d)^2-1)^{-1/2} = d\,(1-d^2)^{-1/2}$ so then to show equal we would have to show that

$$d^{-2}\,(1/d)^{-n+1/2}\,(2/d)^{-1}\;\; d\,(1-d^2)^{-1/2} / (2\pi i) = (d^{n-1/2}/2)\,(d^2-1)^{-1/2}$$

$$d^{-2}\,(1/d)^{-n+1/2}\,(d/2)\;\; d\,(1-d^2)^{-1/2} / (2\pi i) = (d^{n-1/2}/2)\,(d^2-1)^{-1/2}$$

$$(1/d)^{-n+1/2}\,(1/2)\,(1-d^2)^{-1/2} / (2\pi i) = (d^{n-1/2}/2)\,(d^2-1)^{-1/2}$$

$$(1/d)^{-n+1/2}\,(1/2)\,(1-d^2)^{-1/2} / (2\pi i) = (1/2)\,d^{n-1/2}\,(d^2-1)^{-1/2}$$

$$(1/d)^{-n+1/2}\,(1-d^2)^{-1/2} / (2\pi i) = d^{n-1/2}\,(d^2-1)^{-1/2}$$

$$(1-d^2)^{-1/2} / (2\pi i) = (d^2-1)^{-1/2}$$

$$(d^2-1)^{-1/2} / (2\pi) = (d^2-1)^{-1/2}$$

$$1/(2\pi) = 1$$

This would be pretty close. I could imagine making a 2π error somewhere. Due to the various cuts in the y plane, I have no justification for this pole residue evaluation of the integral, but it does suggest that the Smythe verification *could* somehow be done.

Further ideas on this: we have an integral on (1,∞). But maybe this could be related to the (-∞,-1) integral since the integrand is mostly even and we have

$$Q_\nu^\mu(-z) = -\,Q_\nu^\mu(z)\, e^{\pm i\pi\nu}$$

We have this integral

$$K = \int_1^\infty dy\;\; y^{-n+1/2}\,[\,(d^2y^2-1))\,\sqrt{y^2-1}\,]^{-1}\;\; Q_{n-1/2}[(d^2y^2+\rho'^2)/(2\rho' dy)]$$

Suppose we define y' = -y so this becomes

$$= \int_{-\infty}^{-1} dy'\;\; (-y')^{-n+1/2}\,[\,(d^2y'^2-1))\,\sqrt{y'^2-1}\,]^{-1}\;\; Q_{n-1/2}[(d^2y'^2+\rho'^2)/(2\rho' dy')] * (-e^{\pm i\pi(n-1/2)})$$

we combine some factors

$$(-y')^{-n+1/2}\,(-e^{\pm i\pi(n-1/2)}) = -(-1)^{-n+1/2}\,(y')^{-n+1/2}\,(-1)^{n-1/2} = -\,(y')^{-n+1/2}$$

so then we have

$$K = -\int_{-\infty}^{-1} dy'\;(y')^{-n+1/2}\,[\,(d^2y'^2-1))\sqrt{y'^2-1}\,]^{-1}\;Q_{n-1/2}[(d^2y'^2+\rho'^2)/(2\rho'dy')]$$

$$= -\int_{-\infty}^{-1} dy\;y^{-n+1/2}\,[\,(d^2y^2-1))\sqrt{y^2-1}\,]^{-1}\;Q_{n-1/2}[(d^2y^2+\rho'^2)/(2\rho'dy)]$$

This says that the $\int_1^{\infty}$ is the same as $\int_{-1}^{-\infty}$ which says we can replace our integral with

$$= (1/2)\,(\leftarrow + \rightarrow)$$

which means we then have a nice contour which we can maybe close up or down and end up with an integral (-1,1) which might then pick up a half residue from each pole with somehow the residual integral giving nothing, and maybe the pole residues are the same, but they are only half residues so we still have this (1/2) sitting there. I THINK this is really the way it will work out!

I am going to stop now because it is probably true, but I need a better way to show Smythe's result, this is just a super obscure indirect method. But I came so close in so many ways, I think it surely must be true.