**Iris Green's by partial eigenfunctions (cylindrical)** PhL 4.30.10

**Summary of this doc:**

In **Section 1** I use Stakgold's "partial eigenfunction method" to derive (full detail) an expression for the Green's function of a point charge with *no boundary conditions* in cylindrical coordinates. This leads to an expression for 1/R which is different from the one quoted by Jackson in his (3.149) page 86. Here is the comparison [ Green's point charge is at (ρ',φ'=0,z'), observation point is (ρ,φ,z) ]

$$1/R = (2/\pi)\,\Sigma_{m=0}^{\infty}\,\varepsilon_m\,\cos(m\varphi)\int_0^{\infty} dk\,\cos[k(z-d)]\,I_m(k\rho_<)\,K_m(k\rho_>) \qquad \text{// Jackson}$$

$$1/R = \Sigma_{m=0}^{\infty}\,\varepsilon_m\,\cos(m\varphi)\int_0^{\infty} dk\;\; J_m(k\rho)\,J_m(k\rho')\,e^{-k|z-z'|} \qquad \text{// me}$$

Both expressions are correct and I verified "mine" against various external sources. It is a matter of which coordinate you take to be non-oscillatory. For Jackson it is ρ, for me it is z.

In **Section** 2 I try to use the same method for the *iris* Green's function, without success. It seems that the boundary conditions entangle both the ρ and z coordinates and this makes the method not work.

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I don't expect to get far, but let's at least give this a shot. The method is discussed in Stak p 154, but he only does the 2D examples of a rectangle and a unit circle where he knocks out one variable using an expansion. In 3D you have to knock out two variables with such expansions, so how might this go?

My Wilcox guy below refers to Stak' "method of partial eigenfunctions" as "the reduced Green's function technique".

Rather than try the iris, I am going to first to the free space Green's function, which Stak always calls the fundamental solution. I use Jackson 4π conventions.

# 1. The Free Space Green's Function in Cylindricals with z as the non-osc.

The first order of business is to actually write down the Green's ODE, something I rarely do, and I want to do it in cylindricals. Here I imply that $\nabla^2$ is wrt unprimed variables, and primed variables are parameters marking the Green's charge location.

$$\nabla^2 g(\rho,\varphi,z;\, \rho',\varphi',z') \;=\; -\,4\pi\, \delta(z-z')\, \delta(\varphi-\varphi')\, \delta(\rho-\rho')/\rho \qquad \text{from } d^3r = dz\, \rho d\rho d\varphi$$

where I am staring at Jackson page 69 for conventions. This is Poisson's equation Jackson p 13 in the units system where V = q/R for a point charge.

**(a) Processing the φ dimension.**

Let's first try

$$g(\rho,\varphi,z;\, \rho',\varphi',z') \;= \Sigma_{m=0}^{\infty}\, g_m(\rho,z;\, \rho',z')\, \cos[m(\varphi-\varphi')]$$

We have now to write out, using Jackson page 69 (agrees with my TK page of course)

$$\nabla^2 \;= \partial_\rho^2 + (1/\rho)\partial_\rho \;+ (1/\rho^2)\partial_\varphi^2 + \partial_z^2 \;= \nabla_\rho^2 + \nabla_z^2 + \nabla_\varphi^2 \;=\; \nabla_{\rho z}^2 + \nabla_\varphi^2$$

where I am just making up names for the various pieces of $\nabla^2$. Note that

$$\nabla_\rho^2 = \partial_\rho^2 + (1/\rho)\partial_\rho \;= (1/\rho)\partial_\rho(\rho\partial_\rho)$$

Meanwhile, from my new transforms doc I know that

$$\Sigma_{m=0}^{+\infty} \varepsilon_m \cos[m(\varphi-\varphi')] \;=\; 2\pi\, \delta(\varphi-\varphi') \qquad \text{// completeness}$$

Now inserting our expansion we get

$$2\pi\, \nabla^2 \{\, \Sigma_{m=0}^{\infty}\, g_m(\rho,z;\, \rho',z')\, \cos[m(\varphi-\varphi')]\} \;=\; -\,4\pi\, \delta(z-z')\, \{\, \Sigma_{m=1}^{+\infty} \varepsilon_m \cos[m(\varphi-\varphi')]\}\delta(\rho-\rho')/\rho$$

We now have to figure out

$$\nabla^2 \{\, g_m(\rho,z;\, \rho',z')\, \cos[m(\varphi-\varphi')]\,\} \;=\; (\nabla_{\rho z}^2 + \nabla_\varphi^2)\{\, g_m(\rho,z;\, \rho',z')\, \cos[m(\varphi-\varphi')]\,\}$$

$$= \cos[m(\varphi-\varphi')]\, \nabla_{\rho z}^2\, g_m(\rho,z;\, \rho',z') \;+\; g_m(\rho,z;\, \rho',z')\, \nabla_\varphi^2 \cos[m(\varphi-\varphi')]$$

But of course $\nabla_\varphi^2 \cos[m(\varphi-\varphi')] = -(m^2/\rho^2) \cos[m(\varphi-\varphi')]$ so we have

$$= \cos[m(\varphi-\varphi')]\, \nabla_{\rho z}^2\, g_m(\rho,z;\, \rho',z') \;-\; (m^2/\rho^2)\; g_m(\rho,z;\, \rho',z')\, \cos[m(\varphi-\varphi')]$$

$$= [\, \nabla_{\rho z}^2\, g_m(\rho,z;\, \rho',z') - (m^2/\rho^2)\; g_m(\rho,z;\, \rho',z')\,]\, \cos[m(\varphi-\varphi')]$$

So our equation above becomes:

$$2\pi \Sigma_{m=0}^{\infty}[\ \nabla_{\rho z}^2\ g_m(\rho,z;\rho',z') - (m^2/\rho^2)\ g_m(\rho,z;\rho',z')\ ]\cos[m(\varphi-\varphi')]$$

$$= -4\pi\,\delta(z-z')\ \{\ \Sigma_{m=0}^{+\infty}\varepsilon_m \cos[m(\varphi-\varphi')]\}\delta(\rho-\rho')/\rho$$

$$= 2\pi \Sigma_{m=0}^{+\infty}[\ -(2\varepsilon_m)\,\delta(z-z')\,\delta(\rho-\rho')/(\rho)\ ]\cos[m(\varphi-\varphi')] \qquad (*)$$

Aside: Now if someone tells you that

$$\Sigma_{m=0}^{\infty} a_m \cos[m(\varphi-\varphi')] = \Sigma_{m=0}^{\infty} b_m \cos[m(\varphi-\varphi')]$$

where you know that $\cos[m(\varphi-\varphi')]$ is a "complete set" for expansion

$$\Sigma_{m=0}^{+\infty}\varepsilon_m \cos[m(\varphi-\varphi')] = 2\pi\,\delta(\varphi-\varphi')$$

you must conclude that $a_m = b_m$. We could of course prove this by setting $\varphi' = 0$ which would say

$$\Sigma_{m=0}^{\infty} a_m \cos[m(\varphi)] = \Sigma_{m=0}^{\infty} b_m \cos[m(\varphi)]$$

and then we could apply $\int_0^{\pi} d\varphi \cos(m'\varphi)$ to both sides and use orthogonality

$$\int_0^{\pi} d\varphi \cos(m\varphi)\cos(m'\varphi) = (\pi/\varepsilon_m)\,\delta_{mm'}$$

and we would conclude that $a_m = b_m$.

So, comparison of the two sides of (*) in terms of the complete set of functions shown tells us that

$$\nabla_{\rho z}^2\ g_m(\rho,z;\rho',z') - (m^2/\rho^2)\ g_m(\rho,z;\rho',z') = -(2\varepsilon_m)\,\delta(z-z')\,\delta(\rho-\rho')/(\rho)$$

$$[\nabla_{\rho z}^2 - (m^2/\rho^2)]\ g_m(\rho,z;\rho',z') = -(2\varepsilon_m)\ \delta(z-z')\,\delta(\rho-\rho')/\rho$$

and we have now successfully reduced our Green's problem by one variable, having replaced φ with m.

**(b) Processing the ρ dimension.**

I choose ρ because $J_m(\rho)$ is a complete set in the general atomic forms I want to use. We know that

$$\int_0^{\infty} d\rho\ \rho\ J_m(k\rho)\ J_m(k'\rho) = \delta(k-k')/k \qquad \text{// orthogonality}$$

$$\int_0^\infty dk\, k\, J_m(k\rho)\, J_m(k\rho') = \delta(\rho-\rho')/\rho \qquad \text{// completeness}$$

which is the Hankel transform (again, from transforms.doc). So let's try this:

$$g_m(\rho,z;\rho',z') = \int_0^\infty dk\; k\, g_{mk}(z;z')\, J_m(k\rho)\, J_m(k\rho')$$

Here I want to just expand on the $J_m(k\rho)$, but I know that a Green's function has to be symmetric. This is Theorem 2 of Stakgold section 6.6 (see meta meta Chap 6), and I think this is a very major step! The final function $g_{mk}(z;z')$ will also be symmetric in its arguments. I threw in a k above since I know it is the required weight function, just part of the definition then of $g_{mk}$.

We now need to compute

$$[\nabla_{\rho z}^2 - (m^2/\rho^2)]\, g_m(\rho,z;\rho',z') = [\nabla_z^2 + \nabla_\rho^2 - (m^2/\rho^2)]\, g_m(\rho,z;\rho',z')$$

$$= [\nabla_\rho^2 + \nabla_z^2 - (m^2/\rho^2)] \int_0^\infty dk\; k\, g_{mk}(z;z')\, J_m(k\rho)\, J_m(k\rho')$$

$$\int_0^\infty dk\; k\, J_m(k\rho)\, J_m(k\rho')\, [\nabla_z^2 - (m^2/\rho^2)] g_{mk}(z;z') + \int_0^\infty dk\; k\, g_{mk}(z;z')\, J_m(k\rho')\, [\nabla_\rho^2\, J_m(k\rho)]$$

We now turn to our Jackson cylindricals doc from which we quote:

$$[\, x\partial_x(x\partial_x) + (a^2x^2 - n^2)]\, J_n(ax) = 0$$

and translate to

$$[\, \rho\partial_\rho(\rho\partial_\rho) + (k^2\rho^2 - m^2)]\, J_m(k\rho) = 0$$

$$[\, (1/\rho)\partial_\rho(\rho\partial_\rho) + (k^2 - m^2/\rho^2)]\, J_m(k\rho) = 0$$

$$[\nabla_\rho^2 + (k^2 - m^2/\rho^2)]\, J_m(k\rho) = 0$$

$$[\nabla_\rho^2\, J_m(k\rho)] = (m^2/\rho^2 - k^2)\, J_m(k\rho)$$

We can then install this to get

$$[\nabla_{\rho z}^2 - (m^2/\rho^2)]\, g_m(\rho,z;\rho',z') =$$

$$= \int_0^\infty dk\; k\, J_m(k\rho)\, J_m(k\rho')\, [\nabla_z^2 - (m^2/\rho^2)] g_{mk}(z;z') + \int_0^\infty dk\; k\, g_{mk}(z;z')\, J_m(k\rho')\, [\nabla_\rho^2\, J_m(k\rho)]$$

$$= \int_0^\infty dk\; k\, J_m(k\rho)\, J_m(k\rho')\, [\nabla_z^2 - (m^2/\rho^2)] g_{mk}(z;z') + \int_0^\infty dk\; k\, g_{mk}(z;z')\, J_m(k\rho')\, [(m^2/\rho^2 - k^2)\, J_m(k\rho)]$$

$$= \int_0^\infty dk\ k\, J_m(k\rho)\, J_m(k\rho')\, [\nabla_z^2 - (m^2/\rho^2)] g_{mk}(z; z') + \int_0^\infty dk\ k\, J_m(k\rho)\, J_m(k\rho')\, [(m^2/\rho^2 - k^2)]\, g_{mk}(z; z')$$

$$= \int_0^\infty dk\ k\, J_m(k\rho)\, J_m(k\rho')\, \{ \nabla_z^2 - (m^2/\rho^2) + (m^2/\rho^2 - k^2) \} g_{mk}(z; z')$$

$$= \int_0^\infty dk\ k\, J_m(k\rho)\, J_m(k\rho')\, \{ \nabla_z^2 - k^2 \}\, g_{mk}(z; z')$$

$= - (\varepsilon_m/2\pi)\, \delta(z-z')\, \delta(\rho-\rho')/\rho$ // the RHS of our 2D Green's equation found above

$= - (\varepsilon_m/2\pi)\, \delta(z-z') \int_0^\infty dk\ k\, J_m(k\rho)\, J_m(k\rho')$ // installing Bessel completeness

So we now have

$$\int_0^\infty dk\ k\, J_m(k\rho)\, J_m(k\rho')\, \{ \nabla_z^2 - k^2 \}\, g_{mk}(z; z')$$

$$= \int_0^\infty dk\ k\, J_m(k\rho)\, J_m(k\rho')\, \{ - (2\varepsilon_m)\ \delta(z-z') \} \qquad (*)$$

Aside: Now if someone tells you that

$$\int_0^\infty dk\ k\, J_m(k\rho)\, J_m(k\rho')\, a(k) = \int_0^\infty dk\ k\, J_m(k\rho)\, J_m(k\rho')\, b(k) \qquad (**)$$

where you know that $J_m(k\rho)\, J_m(k\rho')$ is a "complete set" for expansion

$$\int_0^\infty dk\ k\, J_m(k\rho)\, J_m(k\rho') = \delta(\rho-\rho')/\rho$$

you must conclude that a(k) = b(k). We could of course prove this by applying $\int_0^\infty d\rho\ \rho$ to both sides of (**) ,

$$\int_0^\infty dk\ k\, \{ \int_0^\infty d\rho\ \rho\, J_m(k\rho)\, J_m(k\rho') \} a(k) = \int_0^\infty dk\ k\, \{ \int_0^\infty d\rho\ \rho\, J_m(k\rho)\, J_m(k\rho') \} b(k)$$

and we then use orthogonality

$$\int_0^\infty d\rho\ \rho\, J_m(k\rho)\, J_m(k'\rho) = \delta(k-k')/k$$

to get

$$\int_0^\infty dk\, k\, \{\, \delta(k-k')/k \quad \} a(k) \;=\; \int_0^\infty dk\, k\, \{\, \delta(k-k')/k \quad \} b(k)$$

$$a(k') = b(k')$$

Therefore, looking at equation (*) above we may conclude that

$$\{\, \nabla_z^2 - k^2 \} \, g_{mk}(z; z') \;=\; -\,(2\varepsilon_m)\;\delta(z-z')$$

**(c) Solve the 1D Green's function problem in z**

We have this problem now

$$\{\, \nabla_z^2 - k^2 \} \, g_{mk}(z; z') \;=\; -\,(2\varepsilon_m)\;\delta(z-z')$$

which is a 1D Helmholtz problem. Up to this point, I have said nothing about boundary conditions! The Green's solution for our iris problem has to vanish on the iris which is z = 0 but only when $\rho \geq a$. But I have ignored this fact totally, and I think we are "too late". So let's just find the Green's function in the case that there are NO boundary conditions and see what we get.

We solve the above like any other Green's problem, our interval is now $(-\infty,\infty)$ for z. We start with the obvious homogeneous solutions

$$g = Ae^{-k(z-z')} \qquad z > z'$$
$$g = Be^{+k(z-z')} \qquad z < z'$$

Our "pillbox" condition is found by integrating the ODE from $z'-\varepsilon$ to $z'+\varepsilon$ and that gives

$$\partial_z g(z = z'+\varepsilon) - \partial_z g(z = z'-\varepsilon) \;=\; -\,(2\varepsilon_m)$$

which is the famous "jump condition" in the 1D world. So:

$$\partial_z\, g = A\,(-k)e^{-k(z-z')} \qquad z > z'$$
$$\partial_z\, g = B(+k)e^{+k(z-z')} \qquad z < z'$$

$$A\,(-k) \;-\; B(+k) \;=\; -\,(2\varepsilon_m)$$
$$A\,(k) \;+\; B(+k) \;=\; (2\varepsilon_m) \qquad \Rightarrow \qquad (A+B)k \;= (2\varepsilon_m) \Rightarrow \qquad A+B = (2\varepsilon_m/k)$$

But of course the solution also has to be continuous at z = z' which says A = B. So

$$2A = (2\varepsilon_m/k) \qquad \Rightarrow \qquad A = B = (\varepsilon_m/k)$$

and our solution is then

$$g_{mk}(z; z') = (\varepsilon_m/k)\, e^{-k|z-z'|}$$

which is noted to be symmetric in z.

**(d) Assemble the complete Green's function solution**

$$g(\rho,\varphi,z;\,\rho',\varphi',z') = \Sigma_{m=0}^{\infty}\, g_m(\rho,z;\,\rho',z')\cos[m(\varphi-\varphi')]$$

$$g_m(\rho,z;\,\rho',z') = \int_0^{\infty} dk\;\; k\, g_{mk}(z;\,z')\, J_m(k\rho)\, J_m(k\rho')$$

$$g_{mk}(z;\,z') = (\varepsilon_m/k)\, e^{-k|z-z'|}$$

Therefore

$$g(\rho,\varphi,z;\,\rho',\varphi',z') = \Sigma_{m=0}^{\infty}\, \varepsilon_m \int_0^{\infty} dk\;\; J_m(k\rho)\, J_m(k\rho')\, e^{-k|z-z'|}\cos[m(\varphi-\varphi')]$$

This can be interpreted as the potential of a unit positive point charge located at (ρ',φ',z'), because our starting equation was

$$-\nabla^2 g = 4\pi\delta(\mathbf{r}-\mathbf{r}')$$

which is just Poisson's equation Jackson p 13 (1.28) without the 4π. We have V = q/R for a point charge. This, we have also shown that

$$1/R = 1/|\mathbf{r}-\mathbf{r}'| = \Sigma_{m=0}^{\infty}\, \varepsilon_m \int_0^{\infty} dk\;\; J_m(k\rho)\, J_m(k\rho')\, e^{-k|z-z'|}\cos[m(\varphi-\varphi')]$$

**(e) Comparison to Jackson and others.**

**Jackson** green (same in red) on page 84 derives a different form for 1/R. He starts with the same Green's equation (3.138) on page 84 (we ignore the 4π) , but he assumes oscillatory in the z direction, not the ρ direction, so his result is (3.149) which involves the I and K functions instead of the J functions.

(**Wilcox**) Make SURE you look soon at this web page:

http://en.wikipedia.org/wiki/Green%27s_function_for_the_three-variable_Laplace_equation

But for the moment we continue our search. I have a nice downloaded book chapter 3 on doing cylindrical electrostatics which claims: (the sum is on m)

$$\frac{1}{|\vec{x}-\vec{x}'|} = \int_0^{\infty} dk \sum_{-\infty}^{\infty} e^{im(\phi-\phi')}\, J_m(k\rho)J_m(k\rho')\, e^{-k(z_> - z_<)}. \quad (3.131)$$

⇓non-oscill.

⇑oscillatory

First off, if we expand $e^{im(\varphi-\varphi')} = \cos[m(\varphi-\varphi')] + i \sin[m(\varphi-\varphi')]$, the sine term is odd in m for non-zero m, and is zero for m=0, so makes no contribution at all. If we then fold the m<0 terms over, we get a doubling for m > 0 but not for m = 0, and this makes our $\varepsilon_m$ factor. So, this is full confirmation! I have not made any errors in the work above.

The above comes from Chapter 3 (spherical and cylindrical electrostatics) of a web book written by Walter Wilcox of the Bayler Physics department,

http://bearspace.baylor.edu/Walter_Wilcox/open_text/em3.html

" **Baylor University** in Waco, Texas, is a **private Baptist university**, and a nationally ranked liberal arts institution. Chartered in 1845 by the Republic of Texas, Baylor is the oldest, continually operating university in the state. Though 80 percent of our students come from within Texas, we are home to students from all 50 states, and 80 countries."

I downloaded the 2 part contents of his Jackson-like book, and a the first few chapters.

**Smythe** has Jacksons I K form of 1/R but not my form.

**M&F** vol 2 on page 1262 develop my result in an unusual manner, and here is their result:

$$\frac{1}{R} = \sum_{m=0}^{\infty} \epsilon_m \cos[m(\phi - \phi_0)] \int_0^{\infty} J_m(kr) J_m(kr_0) e^{-k|z-z_0|}\, dk \quad (10.3.23)$$

and I note that M&F use Neumann's factor $\varepsilon_m$. It probably appears in their Fourier section. It is good to compare this result to Jackson p 69 (see also p 65)

$1/R = 4\pi \Sigma_{\ell m} (2\ell+1)^{-1} r_<^{\ell} r_>^{-\ell-1} Y^*Y$ $\qquad$ $Y = ((2\ell+1)/4\pi)^{1/2} f(\ell,-m)^{1/2} P_\ell^m(z) e^{im\varphi}$

when we insert this Y twice, the obvious factors cancel and we get these to compare:

$1/R = \Sigma_{\ell m}$ $\quad$ $e^{im(\varphi-\varphi')}$ $\quad$ $f(\ell,-m)$ $\quad$ $r_<^{\ell} r_>^{-\ell-1}$ $\quad$ $P_\ell^m(z)\, P_\ell^m(z')$

$1/R \quad \Sigma_{m=0}^{\infty} \int_0^{\infty} dk \cos[m(\varphi-\varphi')]$ $\quad$ $\varepsilon_m$ $\quad$ $e^{-k|z-z'|}$ $\quad$ $J_m(k\rho)\, J_m(k\rho')$

I suppose there is some general way to write this in any coordinate system. But you have to decide which is your non-oscillatory coordinate, then you have to do the separation of variables, and so on.

## 2. The Iris Green's Function in Cylindricals with z as the non-osc.

OK, so how does the above presentation change, and does it become impossible? Our starting point is now this ODE system ( I don't bother to write g = 0 at ∞)

$\nabla^2 g(\rho,\varphi,z;\, \rho',\varphi',z') = -4\pi\, \delta(z-z')\, \delta(\varphi-\varphi')\, \delta(\rho-\rho')/\rho$ $\qquad$ $g(z=0) = 0$ $\qquad$ for $\rho > a$

$$\partial_z g(z=0+) = \partial_z g(z=0-) \quad \text{for } \rho < a$$

The continuity condition just says there is no surface charge in the hole of the iris. Were we to select z' = 0 for the Green's charge location, we could replace the second condition with $\partial_z g(z=0) = 0$, but let's try to stay general.

Let's now imagine we can carry out the φ processing as was done above, so that we arrive at this first intermediate point:

$$g(\rho,\varphi,z;\, \rho',\varphi',z') = \Sigma_{m=0}^{\infty} g_m(\rho,z;\, \rho',z') \cos[m(\varphi-\varphi')]$$

$$[\nabla_{\rho z}^2 - (m^2/\rho^2)]\, g_m(\rho,z;\, \rho',z') = -(2\varepsilon_m)\ \delta(z-z')\ \delta(\rho-\rho')/\rho$$

where we now apply our two BC's to the function $g_m(\rho,z;\, \rho',z')$. So we have this new PDE system:

$$[\nabla_{\rho z}^2 - (m^2/\rho^2)]\, g_m(\rho,z;\, \rho',z') = -(2\varepsilon_m)\ \delta(z-z')\ \delta(\rho-\rho')/\rho$$

$$g_m(z=0) = 0 \qquad \text{for } \rho > a$$
$$\partial_z g_m(z=0+) = \partial_z g_m(z=0-) \quad \text{for } \rho < a$$
$$\lim_{z\to\pm\infty} g_m(z) = 0$$

**(a) Plan A: try expanding on $J_m(k\rho)$**

The next step is not very clear to me. Suppose we *try* the same expansion as above:

$$g_m(\rho,z;\, \rho',z') = \int_0^{\infty} dk\ k\, g_{mk}(z;\, z')\, J_m(k\rho)\, J_m(k\rho')$$

Justification might be that we know that $g_m$ is going to be at least a continuous function of ρ on (0,∞), so we ought to be able to expand it on the interval (0,∞) in this manner, since the $J_m(k\rho)$ form a complete orthogonal set on that interval. And I think we are still symmetric in the arguments as before. Barring order-interchange issues, we can trace this down to the next conclusion which is this:

$$\{\nabla_z^2 - k^2\}\, g_{mk}(z;\, z') = -(2\varepsilon_m)\ \delta(z-z')$$

But something must be wrong at this point because ρ no longer appears anywhere, so how are you going to impose your required boundary conditions? This then suggests to me that the "form" shown above is not correct. One fact is that if we try the above form, applying $\partial_z$ does not help us much. We get the following "system"

$$\{\nabla_z^2 - k^2\}\, g_{mk}(z;\, z') = -(2\varepsilon_m)\ \delta(z-z')$$

$$\int_0^{\infty} dk\ k\, g_{mk}(0;\, z')\, J_m(k\rho)\, J_m(k\rho') = 0 \qquad \rho > a$$

$$\int_0^{\infty} dk\ k\, [\partial_z g_{mk}(0+;\, z') - \partial_z g_{mk}(0+;\, z')]\, J_m(k\rho)\, J_m(k\rho') = 0 \qquad \rho < a$$

I don't think this is a well-defined system, but it might be. What happens now if we apply

$\int_0^\infty d\rho'\ \rho'\ J_m(k'\rho')$ to these two boundary conditions? In each case, we think of ρ as fixed somewhere in its appropriate range, and we then get

$$\int_0^\infty dk\ \ k\ g_{mk}(0; z')\ J_m(k\rho)\ \ \int_0^\infty d\rho'\ \rho'\ J_m(k'\rho')J_m(k\rho')\ = 0 \qquad \rho > a$$

$$\int_0^\infty dk\ \ k\ [\partial_z g_{mk}(0+; z') - \partial_z g_{mk}(0+; z')\ ]\ J_m(k\rho)\ \ \int_0^\infty d\rho'\ \rho'\ J_m(k'\rho')\ J_m(k\rho')\ = 0 \qquad \rho < a$$

or

$$\int_0^\infty dk\ \ k\ g_{mk}(0; z')\ J_m(k\rho)\ \delta(k-k')/k\ \ = 0 \qquad \rho > a$$

$$\int_0^\infty dk\ \ k\ [\partial_z g_{mk}(0+; z') - \partial_z g_{mk}(0+; z')\ ]\ J_m(k\rho)\ \delta(k-k')/k \qquad = 0 \qquad \rho < a$$

or

$$g_{mk'}(0; z')\ J_m(k'\rho)\ = 0 \qquad \rho > a$$

$$[\partial_z g_{mk'}(0+; z') - \partial_z g_{mk'}(0+; z')\ ]\ J_m(k'\rho) \qquad \rho < a$$

which seems to imply that

$$g_{mk}(0; z') = 0$$
$$[\partial_z g_{mk'}(0+; z') - \partial_z g_{mk'}(0+; z')\ ]\ \ = 0$$

so maybe we just have this system as our result:

$$\{\ \nabla_z^2 - k^2\}\ g_{mk}(z; z')\ =\ -\ (2\varepsilon_m)\ \ \delta(z-z')$$
$$g_{mk}(0; z') = 0$$
$$[\partial_z g_{mk'}(0+; z') - \partial_z g_{mk'}(0+; z')\ ]\ \ = 0$$
$$\lim_{z\to\pm\infty} g_{mk}(z; z')\ \ = 0$$

But this is an overdetermined system. Instead of the two homo endpoint conditions, we have two BC's at z = 0, one at z = +∞ and one at z = - ∞. There is no solution to this system, so I guess the form must be no good, but I don't really know what is going wrong here, I cannot point to the step that is bad.

**(b) Plan B: try expanding on cos[k(z-z')].**

Here is where we were:

$$[\nabla_{\rho z}^2 - (m^2/\rho^2)]\ g_m(\rho,z;\ \rho',z') =\ -(2\varepsilon_m)\ \delta(z-z')\ \delta(\rho-\rho')/\rho$$

$$g_m(z=0) = 0 \qquad \text{for } \rho > a$$
$$\partial_z g_m(z=0+) = \partial_z g_m(z=0-) \qquad \text{for } \rho < a$$
$$\lim_{z\to\pm\infty} g_m(z) = 0$$

Suppose we sort of ignore the last condition and hope it works out and we then try this next step:

$$g_m(\rho,z;\,\rho',z') \;=\; \int_0^\infty dk\;\; g_{mk}(\rho;\rho')\cos[k(z-z')]$$

The justification here would be based on the completeness of the z functions,

$$\int_{-\infty}^{\infty} dz\,\cos[(k-k')z] \;=\; 2\pi\,\delta(k-k') \qquad \text{// orthogonality}$$

$$\int_{-\infty}^{\infty} dk\,\cos[k(z-z')] \;=\; 2\pi\,\delta(z-z') \qquad \text{// completeness}$$

The form is symmetric in $z \leftrightarrow z'$. Our first boundary conditions would then be

$$g_m(\rho,0;\,\rho',z') \;=\; \int_0^\infty dk\;\; g_{mk}(\rho;\rho')\cos[kz'] \;=0 \qquad \rho > a$$

To get the other BC, we would say

$$\partial_z g_m(\rho,z;\,\rho',z') \;=-\int_0^\infty dk\;\; k\, g_{mk}(\rho;\rho')\sin[k(z-z')]$$

$$\partial_z g_m(\rho,0\pm;\,\rho',z') \;=-\int_0^\infty dk\;\; k\, g_{mk}(\rho;\rho')\sin[kz']$$

The second BC would be met no matter what since 0± gives the same result. So we then have only that first BE which says:

$$\int_0^\infty dk\;\; g_{mk}(\rho;\rho')\cos[kz'] \;=0 \qquad \rho > a \qquad \text{// and = unknown for } \rho < a$$

So OK, maybe we can live with this "integral boundary condition". Let's now try to continue forward.

$$g_m(\rho,z;\,\rho',z') \;=\; \int_0^\infty dk\;\; g_{mk}(\rho;\rho')\cos[k(z-z')]$$

We then need to compute

$$[\nabla_{\rho z}^2 - (m^2/\rho^2)]\; g_m(\rho,z;\,\rho',z') \;=\; [\nabla_z^2 + \nabla_\rho^2 - (m^2/\rho^2)]\;\; g_m(\rho,z;\,\rho',z')$$

$$= \int_0^\infty dk\;\; \{\, g_{mk}(\rho;\rho')\,\nabla_z^2\cos[k(z-z')] \;+\; \cos[k(z-z')]\,[\nabla_\rho^2 - (m^2/\rho^2)]\, g_{mk}(\rho;\rho')\}$$

$$= \int_0^\infty dk\;\; \{\, g_{mk}(\rho;\rho')\,(-k^2)\cos[k(z-z')] \;+\; \cos[k(z-z')]\,[\nabla_\rho^2 - (m^2/\rho^2)]\, g_{mk}(\rho;\rho')\}$$

$$= \int_0^\infty dk\,\cos[k(z-z')]\,[\nabla_\rho^2 - (m^2/\rho^2) - k^2]\, g_{mk}(\rho;\rho')$$

Our starting position was this:

$$[\nabla_{\rho z}^2 - (m^2/\rho^2)]\, g_m(\rho,z;\rho',z') = -(2\varepsilon_m)\,\delta(z-z')\,\delta(\rho-\rho')/\rho$$

so we replace

$$\delta(z-z') = (1/2\pi) \int_{-\infty}^{\infty} dk \cos[k(z-z')]$$

and we end up with

$$\int_0^{\infty} dk \cos[k(z-z')]\,[\nabla_\rho^2 - (m^2/\rho^2) - k^2]\, g_{mk}(\rho;\rho') = \int_{-\infty}^{\infty} dk \cos[k(z-z')]\,(1/2\pi)\,-(2\varepsilon_m)\,\delta(\rho-\rho')/\rho$$

We then invoke our completeness of the cos[k(z-z')] to conclude that

$$[\nabla_\rho^2 - (m^2/\rho^2) - k^2]\, g_{mk}(\rho;\rho') = -(\varepsilon_m/\pi)\,\delta(\rho-\rho')/\rho$$

We then end up with this system:

$$[\nabla_\rho^2 - (m^2/\rho^2) - k^2]\, g_{mk}(\rho;\rho') = -(\varepsilon_m/\pi)\,\delta(\rho-\rho')/\rho$$
$$\int_0^{\infty} dk\; g_{mk}(\rho;\rho') \cos[kz'] = 0 \qquad \rho > a \qquad \text{// and = unknown for } \rho < a$$

But this is another of our "ill-defined looking" systems. The solution of the homo ODE is this:

$$A_{mk}\, I(k\rho) + B_{mk}\, K(k\rho)$$

We will get some solution that looks like this:

$$g_{mk}(\rho;\rho') = c_{mk}\, I_m(k\rho_<)\, K_m(k\rho_>)$$

where $c_{mk}$ is fully determined. As usual, our pillbox integration tells us that

$$\partial_\rho g_{mk}(\rho;\rho')^+ - \partial_\rho g_{mk}(\rho;\rho')^- = -(\varepsilon_m/\pi)/\rho'$$

$$\partial_\rho g_{mk}(\rho;\rho')^+ = \partial_\rho \{ c_{mk}\, I_m(k\rho_<)\, K_m(k\rho_>)\} = \partial_\rho \{ c_{mk}\, I_m(k\rho')\, K_m(k\rho)\} = c_{mk}\, I_m(k\rho')\, k\, K_m'(k\rho')$$

$$\partial_\rho g_{mk}(\rho;\rho')^- = \partial_\rho \{ c_{mk}\, I_m(k\rho_<)\, K_m(k\rho_>)\} = \partial_\rho \{ c_{mk}\, I_m(k\rho)\, K_m(k\rho')\} = c_{mk}\, K_m(k\rho')\, k\, I_m'(k\rho')$$

$$\partial_\rho g_{mk}(\rho;\rho')^+ - \partial_\rho g_{mk}(\rho;\rho')^- = k\, c_{mk}\, W[\, I_m(k\rho'), K_m(k\rho') = k\, c_{mk}\, (-1/[k\rho']) = -\, c_{mk}/\rho'$$

Therefore we find that

$c_{mk} = (\varepsilon_m/\pi)$

and

$g_{mk}(\rho;\rho') = (\varepsilon_m/\pi)\, I_m(k\rho_<)\, K_m(k\rho_>)$

Now what do I have to say about our "boundary condition"

$$\int_0^\infty dk\ g_{mk}(\rho;\rho')\cos[kz'] = 0$$

$$\int_0^\infty dk\ I_m(k\rho_<)\, K_m(k\rho_>)\cos[kz'] = 0 \qquad \rho > a \qquad \text{// and = unknown for } \rho < a$$

But this is an integral I ran into doing the Iris Attempt #3,

$$\int_0^\infty dk \cos[kd]\, I_m(kc)\, K_m(k\rho) = (1/2)\, 1/\sqrt{c\rho}\ \ Q_{m-1/2}\left[(\rho^2 + c^2 + d^2)/(2c\rho)\right]$$

$$\mathrm{Re}(\rho) > |\mathrm{Re}(c)| \qquad d>0 \qquad \mathrm{Re}(m) > -1/2$$

which I translate to be

$$\int_0^\infty dk \cos[kz']\, I_m(k\rho')\, K_m(k\rho) = (1/2)\, 1/\sqrt{c\rho}\ \ Q_{m-1/2}\left[(\rho^2 + \rho'^2 + z'^2)/(2\rho'\rho)\right]$$

$$\mathrm{Re}(\rho) > |\mathrm{Re}(\rho')| \qquad z'>0 \qquad \mathrm{Re}(m) > -1/2$$

So this result seems valid for $\rho > \rho'$ . So I can certainly find a value of $\rho$ which is larger than both a and $\rho'$. This integral is NOT zero, so our BC is NOT going to be met.

So again, things just "don't work". Some earlier assumption is wrong.

**(c) Conclusions:** The problem might have to do with Stakgold type limits where you have order interchange, as we saw in his boundary layers section. We do have some $\partial_z$ action going on here, after all. But just in general, this "method" seems to have trouble when you have boundary conditions that involve two of the three variables. In this problem the BC's involve both $\rho$ and z.

$$\nabla^2 g(\rho,\varphi,z;\rho',\varphi',z') = -4\pi\,\delta(z-z')\,\delta(\varphi-\varphi')\,\delta(\rho-\rho')/\rho$$

$$g(z=0) = 0 \qquad \text{for } \rho > a$$

$$\partial_z g(z=0+) = \partial_z g(z=0-) \qquad \text{for } \rho < a$$

I guess I can say that I have done "due diligence" in that I have at least tried to find the iris Green's function by the "partial eigenfunction expansion method" of Stakgold.