Meta review of iris attempts in this folder PhL 4.28.10

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Overview (2 pages, written 12.7.10)

The subject of these docs is finding the **in-hole Green's Function for an iris**.

In the **Introduction** I state a list of questions regarding iris and disk. This is April 2010 so I knew nothing about inversion methods. I review what Jackson does in green and red, where he uses cylindricals and pulls dual integral equation solutions out of his hat. I had rolled my own little Dirichlet Method of finding a Green's Function and I was successful on some simple problems 1,2,3,4 and the iris was problem 5. But the hole was causing me problems, so I started new docs on exactly this subject, and these are the three Attempt docs you see listed above.

In **Attempt #1** I started off with my little Dirichlet method using a simple Jackson-like cylindrical Smythian J_m form which I think even now is probably correct. It set it to -q/R on the iris metal as per the Dirichlet method, and I set $\partial_z\Phi = 0$ in the hole, and this leads at once to dual integral equations after you do a Fourier series in cosθ,

(1) $\int_0^\infty dk\, A_m(k)\, J_m(k\rho) = (-q/2)\, \varepsilon_m a_m(\alpha,\beta)$ $\rho \geq a$ $\alpha = b^2+\rho^2$ $\beta = 2c\rho$

(2) $\int_0^\infty dk\, k\, A_m(k)\, J_m(k\rho) = 0$ $0 < \rho < a$ $b^2 = c^2+ d^2$

Here $a_m(\alpha,\beta)$ is the Fourier coefficient of the 1/R in my -q/R , so $1/R = 1/\sqrt{\rho^2 + b^2 - 2b\rho\cos\theta}$. When I first did this work, $a_m(\alpha,\beta)$ was an impossible F(a,b,c,z) sum. I later learned this coefficient is "not so bad"

$$a_m(\alpha,\beta) = (2/\pi)(1/\sqrt{c\rho}\,)\, Q_{m-1/2}\,[(\rho^2+ b^2)/(2c\rho)]$$

Having now read Sneddon, and knowing the above for $a_m(\alpha,\beta)$, I could now probably solve this dual integral equation pair for $A_m(k)$, but I have never tried doing that. That was in my future only. So all I did here in Attempt #1 was "fiddle around". I put the known Smythe Problem 38 solution for σ on the RHS of (2) above for ρ < a. I then used the regular Hankel transform to invert (2) and got a messy integral for $A_m(k)$. Better than a poke in the eye with a sharp stick, but not much better. I took my $A_m(k)$ result and showed that it really does satisfy (2) above. I then of course wanted to see if my $A_m(k)$ solved (1). In doing that, I had to discover the following rather interesting integral (Appendix A)

$$\int_0^\infty dk\ J_m(\rho k)\, J_m(axk) = (1/\pi)\,(\rho ax)^{-1/2} Q_{m-1/2}[\,(ax/2\rho) + (\rho/2ax)]$$

But then in order to verify (1) I had to show this horrible fact

$$\int_1^\infty dx\ x^{1/2-m}\ [1/\sqrt{x^2-1}\ ]\ [(x^2+(b/a)^2)/(x^2-(b/a)^2)]\ Q_{m-1/2}[(\rho^2 + (ax)^2)/(2(ax)\rho]$$
$$= -\,2\,(a/b)^{m+1/2}\ Q_{m-1/2}\,[(\rho^2+ b^2)/(2b\rho)]/\sqrt{a^2-b^2}$$

and I simply stopped at that point. I know now that it is always VERY hard to show that something solves a dual equation because there is such a tangle of transforms involved in the theory used. It is never easy.

In **Attempt #2** I went on a wild goose chase with things like 4-region Smythian forms. I thought I had done something wrong in Attempt #1 regarding z symmetry, but later found out I had done it right after all. This attempt gets no more words here. The Green's charge bobbled in and out of the hole a few times.

In **Attempt #3** I start off gain exactly as in Attempt #1 and get the dual equations. I thought red Jackson's dual solution might work, but no go (that was Plan A). I then tried a different Smythian 2-region form for Plan B and this seems to have led to utter nonsense. In Plan C I try my I-K 1/R Bessel expansion for -q/R. All this did was make me realize the following has to be true,

$$\int_0^\infty dk\ \cos[kd]\ I_m(kc)\ K_m(k\rho) = (1/2)\ 1/\sqrt{c\rho}\ Q_{m-1/2}\,[(\rho^2+ c^2+ d^2)/(2c\rho)]$$

which is quite similar to the JJ integral shown in Attempt #1 above. I comment that it seems strange that a toroidal Q function is showing up in our cylindrical problem. Finally in Plan D repeat the above for the other cylinder expansion for 1/R, and I guess that just gives my JJ integral above.

To summarize, in all these attempts I just flailed at some dual integral equations which I could not solve, and which arose from my Dirichlet Green's function method applied to the iris. No solution was found. I was completely stymied in my efforts to derive or even verify Smythe's Problem 38 solution for iris σ.

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0. Introduction

This is another one of those complicated "maze" situations like the one I encountered doing the oblate hyperboloid Green's Function. Dozens of side canyons on the lake and you paddle around endlessly. Many "dead end" signs on the maze passage endings.

The motivations here are pretty straightforward:

(1) What is the Green's Function for an iris with arbitrary point charge location?
(2) What is it when the point charge is in the hole?
(3) Show that the above results imply the simple charge distribution that Smythe gives in his Problem 38.
(4) What are the answers for a disk instead of an iris?

I actually *have* answers to questions (1) and (2) where I treated the iris as a flat hyperboloid and I did the problem in oblate spheroidal coordinates. But I could not obtain from that the simple (3) result, because σ is a double sum of P and Q functions of unpleasant arguments,

$$\sigma(\zeta,\varphi) \sim \Sigma_{m=0}^{\infty} \varepsilon_m \Sigma_{n=-m+odd} F(n,m) P_n^m(\xi_0) q_n^m(\zeta) \cos(m\varphi)$$

$(\zeta_0, \xi_0, \varphi_0) = (0,\xi_0,0)$ = location of the Green's point charge in the hole
$(\zeta, \xi, \varphi) = (\zeta, 0, \varphi)$ = location on the iris where we are looking at σ

There must be some trick to simplifying this expression, but I don't know what it is. One has the feeling that one has done the problem in the "wrong coordinate system" which causes the messy result.

Jackson green treats the charged disk, and Jackson red does the iris in a distant parallel E field. In both cases, he uses cylindricals and quotes some dual integral equation result out of the blue. Perhaps Jackson stayed away from the iris and disk Green's Functions because they really are messy, who knows. I guess I could ask him at some point, but I have not really paddled around enough to do that.

Stakgold provides a list of methods of finding Green's functions. One method that is on his list but not really emphasized is what I call the Dirichlet (or Dirichlet/Neumann) method. You write a Smythian form for the potential due just to the induced charge on the iris, and impose Dirichlet and Neumann conditions on that potential. (As opposed to writing a Smythian form for the *total* potential.)

I did this successfully for an infinite plane and for a sphere in Sections 3 and 4 of the following doc,

math/Stakgold/Finding Green's Functions by the Dirichlet Method.doc

For the plane I used Cartesian coordinates and the development was amazingly complicated in terms of the fancy integrals required, though they were all found in Bateman ET (Erdelyi Transforms) and everything came out right.

I thought I would try to apply this Dirichlet method to the iris and have that be Section 5 of the above doc. But the iris (and the spherical bowl problem) are cases where the surface of interest contains a "hole" and this forces you to have mixed boundary conditions, unlike the simple sphere and plane problems. You have V = -q/R on the metal, and σ = 0 in the hole region. This always leads to a "dual integral equation" of some sort. So my efforts in these various attempts were mostly to obtain the dual integral equation and try to somehow "look up" a solution to it.

1. Iris Green's by Dirichlet method Attempt #1.doc

First, here is the contents of Attempt #1 doc:

PROBLEM 5: POINT CHARGE AND AN IRIS ..... **ERROR! BOOKMARK NOT DEFINED.**

Smythian Form and the Dual Integral Equations with coefficients $A_m(k)$Error! Bookmark not defined.

The Trantor Dual Integral Equation Model..... Error! Bookmark not defined.

Expression for the charge density on the iris sticky math surface in terms of $A_m(k)$..... Error! Bookmark not defined.

Trying to work it backwards from the Smythe Form for σ. ..... Error! Bookmark not defined.

Now go to the special case in which d = 0 (point charge in the hole).**Error! Bookmark not defined.**

Back to the Charge Distribution..... **Error! Bookmark not defined.**

Making use of the Hankel Transform to get a Smythe-σ based result for $A_m(k)$**Error! Bookmark not defined.**

Try to show that the Smythe-implied $A_m(k)$ satisfies the dual integral equations. Error! Bookmark not defined.

Appendix A: Compute $-\int_0^\infty$-dk $J_m(\rho k)\, J_m(axk)$ ..... Error! Bookmark not defined.

In this doc I took the following (assumed) Smythian form for the potential of the induced charge,

$$V(z,\rho,\theta) = \Sigma_{m=0}^{\infty} \cos(m\theta) \int_0^\infty dk\, e^{-|k|z} A_m(k)\, J_m(k\rho)$$

I discuss reasons why I think this is a reasonable form, and I explain why it is symmetric in z, even though the Green's charge is at an arbitrary location in space, and I note that it is the same form used by Jackson for his charged disk and hole in plate problems (both of which only involve m = 0). The location of the Green's charge is (z,ρ,θ) = (d,c,0) and $b^2 = c^2 + d^2$ is the squared distance of this charge from the origin, and the iris hole radius is a. The Dirichlet and Neumann boundary conditions then become

$$(1) \qquad \Sigma_{m=0}^{\infty} \cos(m\theta) \int_0^\infty dk\, A_m(k)\, J_m(k\rho) = -\, q/\sqrt{b^2 + \rho^2 - 2c\rho\cos\theta} \qquad \rho \geq a$$

$$(2) \qquad \Sigma_{m=0}^{\infty} \cos(m\theta) \int_0^\infty dk\, k\, A_m(k)\, J_m(k\rho) = 0 \qquad 0 < \rho < a$$

where the RHS of the first equation is the -q/R mentioned above. In order to remove θ from the problem, I had to expand the 1/R in cos(mθ) "partial waves" and then the above became the **dual integral equation**:

$$(1) \qquad \int_0^\infty dk\, A_m(k)\, J_m(k\rho) = (-q/2)\, \varepsilon_m a_m(\alpha,\beta) \qquad \rho \geq a \qquad \alpha = b^2+\rho^2 \quad \beta = 2c\rho$$

$$(2) \qquad \int_0^\infty dk\, k\, A_m(k)\, J_m(k\rho) = 0 \qquad 0 < \rho < a \qquad b^2 = c^2+ d^2$$

where $a_m(\alpha,\beta)$ is a messy finite sum of hypergeometric functions, [found to be a Q function, below!]

$$a_m(\alpha,\beta) = (2/\pi\sqrt{\alpha - \beta}\,)\, \Sigma_{s=0}^{m} (-1)^s\, (2m, 2m-2s)\, B[\, s+1/2,\, m-s+1/2\,]\, F\,[s+1/2,\, 1/2;\, m+1;\, -2\beta/(\alpha-\beta)\,]$$

I was happy to at least have arrived at *some* kind of dual integral equation. I scaled it to dimensionless coordinates and was then unhappy to find that it did not match the form given in Bateman, a special case of which green Jackson used for his charged disk. I was unaware at this time that red Jackson had another

form of the dual equations that DOES match (well not exactly). So, not knowing that, I looked into the very general Tranter form (1951) and at least got what I would call "a prescription" for solving for the coefficients $A_m(k)$.

I gave up at that point on trying to actually *find* the solution, and instead I wrote an expression for the charge density on the iris this way, the k coming from $\partial_z V$,

$$\sigma(\theta,\rho) = (1/2\pi)\, \Sigma_{m=0}^{\infty} \cos(m\theta) \int_0^{\infty} dk\, k\, A_m(k)\, J_m(k\rho) \qquad \rho > a$$

Knowing Smythe's claimed simple form for σ (he assumes Green's in the hole, so d = 0 and c = b), I then wrote the above as (using Smythe's σ),

$$(1/2\pi)\, \Sigma_{m=0}^{\infty} \cos(m\theta) \int_0^{\infty} dk\, k\, A_m(k)\, J_m(k\rho) = -\,(q/2\pi)\sqrt{a^2-b^2} \,/\, [\,\sqrt{\rho^2-a^2}\,(\,\rho^2 + b^2 - 2b\rho\cos\theta)\,]$$

So this is the thing I wanted to show was true for ρ > a. By fiddling around and using the Hankel Transform, I was able to show that the above equation is solved by [where $\alpha = -2b\rho/(\rho^2 + b^2)$]

$$A_m(k) = (q\,\varepsilon_m/2a)\sqrt{a^2-b^2} \int_a^{\infty} d\rho\, \rho\, J_m(k\rho)\, [1/\sqrt{\rho^2-a^2}\,]\,(1/\sqrt{1-\alpha^2})\,[\,(\sqrt{1-\alpha^2}-1)/\alpha\,]^m$$

$$= (q\,\varepsilon_m/2)\,(b/a)^m \sqrt{a^2-b^2} \int_1^{\infty} dx\, x^{1-m} J_m(kax)\,[1/\sqrt{x^2-1}\,]\,[(x^2+\alpha^2)/(x^2-\alpha^2)] \qquad \alpha = b/a < 1$$

So if I could somehow solve the dual integral equations to get the result above for $A_m(k)$, I would be verifying Smythe's simple result for σ. But I have no solution whatsoever for $A_m(k)$.

In a final section (which I added today 4.28.10), I decided to at least try to see if the above $A_m(k)$ expression *satisfied* the dual integral equations with point charge in the hole, c = b. I *was* able show that it satisfies the Neumann part $\int_0^{\infty} dk\, k\, A_m(k)\, J_m(k\rho) = 0$ for $\rho < a$. I then wanted to verify the Dirichlet part which reads (copying from above)

$$(1) \qquad \Sigma_{m=0}^{\infty} \cos(m\theta) \int_0^{\infty} dk\, A_m(k)\, J_m(k\rho) = -\,q/\sqrt{b^2 + \rho^2 - 2c\rho\cos\theta} \qquad \rho \geq a$$

Upon inserting the $A_m(k)$ expression, I required this integral (done in Attempt #1 Appendix A and verified against ET II)

$$\int_0^{\infty} dk\; J_m(\rho k)\, J_m(axk) = (1/\pi)\,(\rho a x)^{-1/2} Q_{m-1/2}[\,(ax/2\rho) + (\rho/2ax)]$$

in which the **toroidal** Q function appears (first time in anything I have ever done). Inserting the $A_m(k)$ then this integral and doing the same partial wave thing in θ, I found that I could complete the verification if I could show that

$$(1/\pi)\sqrt{a^2-b^2}\ (\rho a)^{-1/2}\ (b/a)^m \int_1^\infty dx\ x^{1/2-m}\ [1/\sqrt{x^2-1}\ ]\ [(x^2+(b/a)^2)/(x^2-(b/a)^2)]\ Q_{m-1/2}[(ax/2\rho)+(\rho/2ax)] = -\ a_m(\alpha,\beta)$$

$$= -\ (2/\pi(\rho-b))\ \Sigma_{s=0}^{m}\ (-1)^s\ (2m, 2m-2s)\ B[s+1/2, m-s+1/2]\ F[s+1/2, 1/2; m+1; -4b\rho/(\rho-b)^2]$$

I did not pursue this difficult integral any further. Even just for m = 0 it is not clear how you would even look it up. It has a Q strange argument, and a power, and a square root, and a quadratic rational function. Even if I could show the above was true, it is a very non-satisfying way to verify Smythe's result, and I want "a better way".

Note added 4.30.10. Below I seem to have discovered that this must be true, for general location of the point charge,

$$a_m(\alpha,\beta) = (2/\pi)(1/\sqrt{c\rho}\ )\ Q_{m-1/2}[(\rho^2+b^2)/(2c\rho)]$$

and for c = b this becomes

$$a_m(\alpha,\beta) = (2/\pi)(1/\sqrt{b\rho}\ )\ Q_{m-1/2}[(\rho^2+b^2)/(2b\rho)]$$

If so, then the above verification would require showing that

$$(1/\pi)\sqrt{a^2-b^2}\ (\rho a)^{-1/2}\ (b/a)^m \int_1^\infty dx\ x^{1/2-m}\ [1/\sqrt{x^2-1}\ ]\ [(x^2+(b/a)^2)/(x^2-(b/a)^2)]\ Q_{m-1/2}[(ax/2\rho)+(\rho/2ax)] =$$

$$= -\ (2/\pi)(1/\sqrt{b\rho}\ )\ Q_{m-1/2}[(\rho^2+b^2)/(2b\rho)]$$

which simplifies a bit to

$$\int_1^\infty dx\ x^{1/2-m}\ [1/\sqrt{x^2-1}\ ]\ [(x^2+(b/a)^2)/(x^2-(b/a)^2)]\ Q_{m-1/2}[(\rho^2+(ax)^2)/(2(ax)\rho] =$$

$$= -\ 2\ (a/b)^{m+1/2}\ Q_{m-1/2}[(\rho^2+b^2)/(2b\rho)]/\sqrt{a^2-b^2}$$

which is a very mysterious claim about the toroidal $Q_{m-1/2}$. [looks like a contour pole thing...]

2. Iris Green's by Dirichlet method Attempt #2.doc

First, here is the contents of this doc:

1. Introduction..... Error! Bookmark not defined.

2. Trying a two-region Smythian form, above and below z = 0. ..... Error! Bookmark not defined.

3. Trying a four-region Smythian form, above and below z = 0. ..... Error! Bookmark not defined.

(a) The matching condition on V at ρ = a between our two cylindrical regions:**Error! Bookmark not defined.**
(b) The matching condition on $\partial_\rho V$ at ρ = a between our two cylindrical regions: OUCH! .....**Error! Bookmark not defined.**

4. **The following sections are just idle spinning of wheels: .....** Error! Bookmark not defined.
(a) The matching condition on V at z = 0 regions for both cylinder regions:**Error! Bookmark not defined.**
(b) Recap to this point. ..... **Error! Bookmark not defined.**
(c) Idea #1: ..... **Error! Bookmark not defined.**
(d) Idea #2: ..... **Error! Bookmark not defined.**
(e) Now rewrite just the ones that are independent conditions: ..... **Error! Bookmark not defined.**
5. **Realization that V really is symmetric in z and why this is so. .....** Error! Bookmark not defined.

When I wrote the previous Attempt #1 doc, as you can see, I used a form for $V_{\texttt{Dirichlet}}$ that was *symmetric* in z. I then decided (erroneously) this was the wrong thing to do, so I started Attempt #2, thinking that $V_{\texttt{Dirichlet}}$ had to be *non*-symmetric in z if the Green's point charge was not in the z = 0 plane. So in Section 2 I tried a non-symmetric **2-region** Smythe form, but it led to a contradiction: you assume $A_{\mathtt{m}}(k) \neq B_{\mathtt{m}}(k)$, but then the z=0 match forces $A_{\mathtt{m}}(k) = B_{\mathtt{m}}(k)$ since each form valid for all ρ and you can use Hankel orthogonality. In Section 3 I then tried a **4-region** Smythe form to get around this contradiction. But then when I tried to match both V and $\partial_{\boldsymbol{\rho}} V$ at ρ = a, I got another contradiction, all coefficients are 0, where it says OUCH in the contents above. In actual fact, I did not discover this until section 4 (e), but I copied that section up to be 3 (b). This allows the very long Section 4 to be completely ignored. In Section 4 I was just trying to write down all potential and electric field matching conditions I could think of. I also was wrongly thinking that the tangent E field was 0 on the iris, but of course this is not true for the Dirichlet potential. So basically all of Section 4 is a complete wasted effort, but it is there for the record. Finally, in Section 5 I understood why in fact $V_{\texttt{Dirichlet}}$ is symmetric in z even if the Green's is not in the z=0 plane, and I wrote this up in detail. This of course was then consistent with Attempt #1 !

3. Iris Green's by Dirichlet method Attempt #3.doc

Again, here is the full contents of this doc,

PROBLEM 5: POINT CHARGE AND AN IRIS ..... **ERROR! BOOKMARK NOT DEFINED.**

1. **Attempting a solution by my Dirichlet method .....** Error! Bookmark not defined.
(a) Kinematics ..... **Error! Bookmark not defined.**
(b) The Dirichlet Potential on the iris ..... **Error! Bookmark not defined.**
(c) Charge density and Neumann part of the problem: statement of the Mixed BC's.....**Error! Bookmark not defined.**
(d) Smythian Form and statement of Mixed BC's in terms of same: the Dual Integral Equations ..... **Error! Bookmark not defined.**
(e) Playing with the Dual Integral Equations ..... **Error! Bookmark not defined.**
(f) The Charge Distribution on the Iris ..... **Error! Bookmark not defined.**
(g) Review and Status of our Dirichlet/Neumann Problem ..... **Error! Bookmark not defined.**

2. Discussion and looking for "leads" on how to do the iris problem .. Error! Bookmark not defined.

3. Plan A: Jackson's red book dual integral equations..... Error! Bookmark not defined.

4. Plan B: Let the "form" *be* for the total solution..... Error! Bookmark not defined.

(a) Try a 2-region Smyth form with $A_m(k)$ and $B_m(k)$, fitting now the *total* V.**Error! Bookmark not defined.**

(b) The Pillbox Abutment Theorem..... **Error! Bookmark not defined.**

(c) Point charge back in the hole, try a pillbox condition. ..... **Error! Bookmark not defined.**

(d) Cast our dual integral equations in red Jackson form and solve for $A_m(k)$**Error! Bookmark not defined.**

(e) Try a different approach to our dual integral equations (goes nowhere)**Error! Bookmark not defined.**

5. Plan C: Try expressing the potential of the point charge in terms of Bessel functions. ..... Error! Bookmark not defined.

Toroidal Coordinates. ..... **Error! Bookmark not defined.**

Now back to where we were:..... **Error! Bookmark not defined.**

6. Plan D: Repeat Plan C with other 1/R form. ..... Error! Bookmark not defined.

Section 1 is just a cleaned-up version of Attempt #1 where things are broken out into little subsections. We end up with the same dual integral equations of this form (as shown above)

(1) $\int_0^\infty dk\, A_m(k)\, J_m(k\rho) = (-q/2)\, \varepsilon_m a_m(\alpha,\beta)$ $\rho \geq a$ $\alpha = b^2+\rho^2$ $\beta = 2c\rho$

(2) $\int_0^\infty dk\, k\, A_m(k)\, J_m(k\rho) = 0$ $0 < \rho < a$ $b^2 = c^2+ d^2$

and there is nothing new in this section. In **Section 2** I start pondering other approaches I might take to this problem, a little list. Then I proceed to try a few new methods.

In the very brief **Section 3** I try **"Plan A"** -- I had just discovered the red Jackson dual integral equation form and I was eager to try to apply it to the iris problem. But I quickly found that, although the red dual form differs from the green one, it was *still* not the right form -- the 0 is in the wrong place. I realized I might get the 0 in the right place by reinterpreting the potential as the "total" V instead of the "Dirichlet" V.

So **in Section 4 "Plan B"** I try a simple 2-region Smythe form for the total potential. But in this case, the potential cannot be symmetric in z, so

In (a) I try my usual $A_m(k)$and $B_m(k)$ forms for ± z. This unfortunately leads to both dual equation RHS's being 0! The suggested solution $B_m(k) = - A_m(k)$ was then nonsense since this would make the total potential antisymmetric which is obviously wrong.

But in doing Section 4, I realized in (b) the **Pillbox Abutment Theorem**. You can think of this also with the 1D string where the "point charge" is the delta force spike out on the string. It says that where the point charge is located, you can never have a viable point charge pillbox condition if the potential has the same form all around the point charge! In particular, if the Green's charge lies on a boundary between two abutting potential regions, the potential must have different forms on the two sides. This just says that you cannot have the Green's point charge be "in the middle of some uniform region" of atomic form. This is obvious, but I need to drill it in.

(c) Still in Plan B, I then "put the point **charge back in the hole**" so I could then use my usual symmetric form for V (at least it is *slightly* different in the two regions!) This led to the following interesting dual integral equations:

$$\Sigma_{m=0}^{\infty} \cos(m\theta) \int_0^{\infty} dk\, k\, A_m(k)\, J_m(k\rho) = -(q/2c)\, \delta(\theta)\, \delta(\rho - c) \qquad \rho < a \quad // \; c < a \text{ of course}$$

$$\Sigma_{m=0}^{\infty} \cos(m\theta) \int_0^{\infty} dk\, A_m(k)\, J_m(k\rho) = 0 \qquad \rho > a \quad // \; V = 0 \text{ on iris}$$

where now the RHS of the first is a distribution instead of a function, and represents the point charge at its location in the hole. This first equation is like many I have obtained before by the pillbox method, but now it is part of a pair of dual integral equations! Since the first equation is not valid for all ρ, I cannot use the usual orthogonality method to solve it for $A_m(k)$.

In (d) I then went on to try to mechanically expand $\delta(\rho - c)$ in powers of ρ so I could use the red Jackson form. This led to a very strange singular expression for $A_m(k)$ containing derivatives of a delta function $\delta^{(j)}(c)$ centered at the point charge location,

$$A_m(k) = -(\varepsilon_m q / 4\pi c)\, (2ak)^{-1/2} \Sigma_{j=0}^{\infty} \delta^{(j)}(c) / \Gamma(j+3/2) * J_{j+3/2}(ak)$$

Perhaps one could make sense of this, but I moved on. The problem here is that the simple potential form has to represent the singular point charge and that makes $A_m(k)$ be very strange.

In (e) I tried just integrating the first equation of the pair above over ρ, but this led nowhere useful.

Now we come to **Section 5 "Plan C"** where things get more interesting. I wrote the point charge potential as a sum of Bessel functions as Jackson showed, so that now my residual Dirichlet Smythe form was once again my regular symmetric usual form I always use. The total potential is then the sum of these two things. But when working with the *sum*, you can say that V = 0 on the iris. So working with the sum, I arrived at this dual integral equation:

$$(1) \quad \int_0^{\infty} dk\, A_m(k)\, J_m(k\rho) = -(2q\varepsilon_m/\pi) \int_0^{\infty} dk \cos[kd]\, I_m(kc)\, K_m(k\rho) \qquad \rho > a$$

$$(2) \quad \int_0^{\infty} dk\; k\, A_m(k)\, J_m(k\rho) = 0 \qquad \rho < a$$

which has the same form as my earlier pair where I had $(-q/2)\, \varepsilon_m a_m(\alpha,\beta)$ on the RHS. So the **first** interesting idea is that maybe it is better to write q/R with R^{-1} expressed not as $1/\sqrt{b^2 + \rho^2 - 2c\rho\cos\theta}$, but expressed instead as $(2/\pi) \Sigma_{m=0}^{\infty} \varepsilon_m \cos(m\theta) \int_0^{\infty} dk \cos[k(z-d)]\, I_m(k\rho_<)\, K_m(k\rho_>)$ where $\rho_> = \max(c,\rho)$ and $\rho_< = \min(c,\rho)$. Notice that this is a 2-region model for a point charge. Inside a sphere of radius c with origin at the center, we have $I_m(k\rho)$ which is good at ρ = 0, while outside that disk we use the other function $K_m(k\rho)$. Having these two regions of course is consistent with our Abutment Theorem above.

In any event, the implication is that the following must be true:

$$(-q/2)\,\varepsilon_m a_m(\alpha,\beta) \;=\; -(2q\varepsilon_m/\pi)\int_0^\infty dk\,\cos[kd]\,I_m(kc)\,K_m(k\rho)$$

unless I made a mistake in my $1/\sqrt{b^2+\rho^2-2c\rho\cos\theta}$ expansion (to which I dedicated a whole doc). So this was the first result of interest, a new way to express a_m .

But then I was able to do the above integral two ways, both giving the same result (lookups in GR and Bateman ET), as shown Appendix A. I got

$$\int_0^\infty dk\,\cos[kd]\,I_m(kc)\,K_m(k\rho) \;=\; (1/2)\,1/\sqrt{c\rho}\;\;Q_{m-1/2}\left[(\rho^2+c^2+d^2)/(2c\rho)\right]$$

and the **second** interesting fact is that a "toroidal function" is suddenly appearing in my problem! For some reason we are getting a function which normally resides in the world of toroidal coordinates, not cylindrical or oblate coordinates. I cannot help thinking the appearance of such a function must have some significance.

The **third** implication of course is that this must be true,

$$a_m(\alpha,\beta) \;=\; (2/\pi)(1/\sqrt{c\rho}\,)\,Q_{m-1/2}\left[(\rho^2+c^2+d^2)/(2c\rho)\right]$$

It is hard for me to see right now how a finite sum of Legendre-like F functions could equal this one Legendre function. That remains a mystery. [but eventually I saw the light and this is correct]

I then took a quick look (for the first time ever) at **toroidal coordinates** which is just the 2D bipolar system rotated about the z axis.

So here then is the resulting dual integral equation:

$$(1)\quad \int_0^\infty dk\,A_m(k)\,J_m(k\rho) \;=\; -(q\varepsilon_m/\pi)\,1/\sqrt{c\rho}\;\;Q_{m-1/2}\left[(\rho^2+c^2+d^2)/(2c\rho)\right] \qquad \rho > a$$

$$(2)\quad \int_0^\infty dk\;\;k\,A_m(k)\,J_m(k\rho) = 0 \qquad \rho < a$$

I still don't know if my simple Smythe form is "viable" for this problem, nor do I have a simple dual integral equation lookup which has a full function in one of the equations (as Tranter did).

So new things have appeared on the horizon!

Section 5 "Plan D" obtains exactly the dual integral equations shown above, but uses a different cylindrical expansion for 1/R. So nothing new here.