Why spherical atoms don't work for origin on metal. PhL 1.21.10

Except for the last two sections, everything in this doc is with spherical coordinates. I don't solve the disk problem here, I just ponder how you might set it up in sphericals. You cannot! 27 pages! This document consists of a meta-meta review followed by a meta review followed by the raw document.

Meta Meta Review of this Doc [4.15.10] . **// read 12.7.10, is just fine.**

Meta meta review is 1 page, meta review is 4.5 pages, raw doc is 27 pages.

Using the examples of a metal disk and a cube and other, I learn in this document the fact that if you put a spherical-coordinates origin on a piece of metal some of which runs away from that origin along an r ray, then the near-zone potential cannot be expressed as a sum of spherical atomic terms, which is to say, the solution is not separable in spherical coordinates! I suspect the conclusion is true if you put the spherical-coordinates origin on any piece of metal such that there is no ball around the origin which is metal-free, but none of my examples dealt with this case. This result is referred to as the Spherical Atomic Form Theorem below in Section 9.

The converse fact is that as long as there is a metal-free ball around the selected origin, then regardless of the complexity of nearby metal shapes, you *can* in fact write the potential in that ball and I think on solid angles emanating from that ball (until they hit metal) as a sum of spherical atoms. The moral of the story is not to select a spherical origin on metal!

A second result -- called the Exterior Charged Metal Object Theorem in Section 6 below -- is that, regardless of the shape of a localized metal object, the potential outside that object *can* always be written as a sum of spherical atoms, even if the origin lies on the metal surface. In particular this is true outside any math sphere enclosing the metal object and I feel sure that the atomic solution can be extended on any inbound solid angle until it hits the metal object. In fact, this conclusion is correct for any localized collection of metal objects at various potentials and you can add local charges as well, and the spherical origin can be located in fact anywhere, as long as your enclosing math sphere centered on this origin is made large enough to enclose all the metal and charges. Again, this is an exterior claim, not a near field claim.

In contrast to the first theorem above, you *can* select a cylindrical-coordinates origin on a piece of metal like the disk even though some of that metal leaves the origin along a ρ ray. The reason is that ρ is an oscillatory dimension for the disk problem, whereas r is a non-oscillatory dimension, in the Sturm-Liouville sense. This means you can use orthogonality of the ρ functions to determine the atomic expansion coefficients, whereas there is no orthogonality for the r^n powers.

So the upshot is that for problems like the metal disk, you cannot get a separable solution in the near zone if you insist on using a spherical coordinate system with origin at disk center, no matter how hard you try! But you will be able to find a separable solution in cylindrical coordinates.

An important supporting idea (that required from me a separate document "Power series and f(x) constant in a finite region.doc" in math/misc) is that you cannot "model" or "represent" or "fit" a function of one variable having a slope discontinuity (I call this a "kink") with a power series. This seems very obvious perhaps, but it was not obvious to me in Jan 2010. You *can* model such a kink with a series of orthogonal functions. Similarly, if a function is constant in some finite range of its interval, again you cannot represent that function with a power series other than a series consisting of just the constant term,

but you can represent it with a series of orthogonal functions. Typically at the end of a constant region you will have a "kink", but I can imagine a very smooth kink. The relevance to our disk problem is that as you move away from the origin in r along the disk, the potential has a finite constant region which ends with some kind of mild kink at the edge of the disk, and by both forms of our claim just made, you cannot represent this potential as a power series in r, and this in turn invalidates the spherical atomic form.

Comments added 9.26.10 (1) Regarding power series fits, we know in general that powers are only a spanning set on any interval, so in general if you try to fit a function by an infinite series of powers, the coefficients "keep moving". Only if the function to be fit is C^∞ continuous can that series be stable, and then you have your power series. In contrast, a set of ortho polys on an interval is a complete basis, and in this case you can represent any piecewise continuous function by an infinite series of such polys. (2) When I say spherical coordinates above, I mean the usual r,θ,φ. You might be able to put the spherical origin on a piece of metal for a near solution if you use the alternate spherical atomic form which is in fact oscillatory in r,

$$(1/\sqrt{r})[\sin(\tau \ln r), \cos(\tau \ln r)]\ [\ P_{i\tau-1/2}^{m}(z),\ Q_{i\tau-1/2}^{m}(z)]\ [\ \sin(m\varphi), \cos(m\varphi)]$$

Meta Review of this Doc. [4.14.10] I reread *all* 27 pages of this doc today so now is a good time to summarize it so I won't have to read it again. This overview is really a "meta" document, but I will not put it in a separate file.

In **Section 1**, I make several "**attempts**" to model the potential of a charged metal disk using *spherical* coordinates with the origin at disk center. The attempts differ mainly in the way the space around the disk is partitioned into Smythian "regions".

Attempt #1 tries an above and below disk region with r^{-n-1} but of course this fails at the point r=0.

Attempt #2 tries regions inside and outside *sphere* r = a with r^n and r^{-n-1}. Here for the first time I realize the "power series problem" in r^n for the inner region. This led me later (see below) to write a separate math doc on the subject, the conclusion of which was basically this: you cannot model a function that has a "kink" with a power series; you can only model it with an orthogonal set.

Attempt #2A is same as #2 but we limit to hemispheres above the disk. Same conclusion as #2 case, namely, the Smythian form you keep writing keeps failing!

Attempt #3 takes the known problem solution and writes it in r,θ coordinates. I attempt to expand the θ dimension on $P_n(z)$ so that $V = \Sigma_n\ a_n(r)P_n(z)$, then I attempt to compute the $a_n(r)$. The interesting question here is whether I will find $a_n(r) = c_n r^n + d_n r^{-n-1}$. If yes, then that would say the solution can in fact be written in spherical Smythian form with origin at disk center. The known solution is a more or less uniform function (there are not several functions for different regions), so it seems that you would need $c_n = 0$ and also $d_n = 0$. I was unable to do the integrals to compute $a_n(r)$ so things were inconclusive. But my gut feeling was that $a_n(r)$ is in fact some complicated function which has the *limits* r^n and r^{-n-1} in the near and far zones, but that $a_n(r)$ is not simply a linear combination of these powers, and that therefore we do *not* match the spherical Smythian atomic forms.

Attempt #4 tries regions being inside and outside a *cylinder* enclosing the disk. But this has the same power series r^n problem as earlier attempts. I discuss some kind of "escape door" here, but it does not make much sense to me now. But in this discussion, I am realizing for the first time that the real problem

is putting the spherical origin "on a piece of metal". Before leaving this attempt, I ask whether maybe the outer cylinder region might be "OK" for spherical atomic forms, and it is only the inner region that has my power series problem. After all, the outer region does not have V = V_o for some range of the r coordinate. I then make an erroneous comparison to the bowl problem -- erroneous because that problem involves $P_n(z)$ which are an orthogonal set and not r^n which are not orthogonal. It was really at this point that I went off and wrote " Power series and f(x) constant in a finite region.doc" in math/misc.

In **Section 2** I then ask the big question which is this: when *can* you model a potential using spherical atoms? But I don't answer the question. Instead, I reconsider Attempt #3 and I conjecture that maybe somehow you really do get $a_n(r) = d_n r^{-n-1}$ for the *external* problem, albeit not $c_n r^n$ for the internal one. For the first time I get the idea that you can model the disk potential using spherical atoms, but only for the "exterior" solution. At this point I have not really understood that the problem is having the origin sitting on metal for the interior problem. The exterior problem does not include the metal and so does not have this origin problem.

In **Section 3** "getting more general" I ponder not just a disk, but an arbitrary origin-local piece of metal that is azimuthally symmetric, like a **coke bottle**. I make the argument that the *exterior* problem in this general case CAN be modeled as $\Sigma_n A_n r^{-n-1} P_n(z)$. The argument is this: there exists some answer to the total problem, and that solution potential has some value on a local math sphere enclosing the metal object. Since V (∞) = 0, the exterior problem is a well-defined exterior Dirichlet problem with the potential specified on a sphere and we know that the solution will have the form $\Sigma_n A_n r^{-n-1} P_n(z)$. We set r = a and we compute the A_n as integrals of the sphere-specified potential. I then wander off a bit with comments about cones that don't seem too useful now.

In **Section 4** I restate my "big question" this way: Can I find a potential problem whose known solution canNOT be written in spherical atoms? A counterexample. Spherical atoms really means "separation of variables", so I am looking for a case where you cannot get a solution in separated form. I find something on the web for 2D variables x,y which says separation only works if you have "homogeneous" boundary conditions. I think this statement is wrong, because a Dirichlet specification V(r=a,θ,φ) = f(θ,φ) is clearly *in*homogeneous (unless f = 0), yet we know that separation works. So this section peters out and has no conclusions at all.

In **Section 5** called "coke bottle" I restate what I said in the Section 3 summary above. Nothing new.

In **Section 6** I formally state my **Exterior Charged Metal Object Theorem** which says that, for an azisym piece of metal, you can model the exterior potential in spherical atoms. I claim this is in fact true for a local metal object of *any* shape, all just based on the surrounding math sphere idea.

I then **apply this theorem to the metal disk**: I take the known solution and compute V(r=a,θ) = (q/a) $\cot^{-1}$ ($\sqrt{|z|}$) and plot it. Then using this as a sphere boundary value of the ***exterior*** Dirichlet problem, I assume the exterior spherical atomic form $V(r,\theta) = \Sigma_n a_n (r/a)^{-n-1} P_n(z)$ and compute the coefficients $a_n = (2q/a) K_n$ where K_n is an elementary function of n.

Next, I consider the ***interior*** Dirichlet problem for the upper half sphere with this same V(a,θ) on the hemisphere and I think V = const = (q/a)(π/2) on the equatorial disk. Although this is a well defined Dirichlet problem with a closed boundary, the solution cannot be written in terms of spherical atoms in spherical coordinates with origin at the sphere center! This is proven by the power series in r argument.

This last interior problem provides the "counterexample" I was looking for. If you insist on putting your spherical coordinates origin at sphere center, meaning right on the metal "lid" of the bowl, the solution cannot be written in terms of spherical atoms $\Sigma_n a_n r^n P_n(z)$ and the problem is not separable! But of course the solution can be *written*, we know exactly what it is, but it is not separable. So here is a fascinating case where the exterior solution is separable but the interior solution is not separable.

I then consider the **2D analog of the disk problem**: a wire segment in the plane, enclosed by a math circle. The separated interior atomic form here would be $V(r,\theta) = \Sigma_n a_n r^n \cos(n\theta)$, but the interior problem again does not admit this form, due to our usual power series argument. The exterior problem on the other hand would have the form $\Sigma_n a_n r^{-n} \cos(n\theta)$ and you could compute the a_n for it.

At this point I **shift gears to Cartesian coordinates** and I list off the related atoms, and there are quite a few. This is in preparation for the next section.

In **Section 7** I consider a Dirichlet problem in a **box** a,b,c where only the far z-wall has $V \neq 0$. I assume a very simple Dirichlet prescription on this far wall and write the solution as follows, where m and n are eigenvalues which are quantized to be integers, where the Cartesian origin is at the obvious box corner:

$$\varphi_{nm}(x,y,c) = \sin(\pi n x/a)\sin(\pi m y/b)\sinh(h_{n,m}c) \qquad h_{n,m} = [\,(\pi n/a)^2 + (\pi m/b)^2\,]^{1/2}$$

I then simplify to a **cube** with $a = b = c = \pi$ and consider $\varphi_{11}(x,y,z) = \sin(x)\sin(y)\sinh(\sqrt{2}\,z)$. At this point, I have a Cartesian origin "on the metal" and things seem OK in these coordinates. I don't try here to express this in terms of a spherical coordinate system with origin at the corner because I know that will fail (I do this below). Instead, I rewrite the solution in terms of new (x,y,z) with origin at cube *center*, and I convert *this* to sphericals with that origin. My question then is this: can this cube solution be written in terms of spherical atoms with this cube-center origin? That question is the subject of the next section.

In **Section 8** I first write the cube-center Cartesian potential and then I expand it for small r in powers r^n and then I ask: will we find that $\varphi_{11}(r,\theta,\varphi) = \Sigma_n\ r^n f_n(\theta,\varphi)$ with $f_n(\theta,\varphi) = \Sigma_{m=-n}^{n} a_{nm} P_n(z) e^{im\varphi}$?? If so, then we have our spherical atom form with this origin and with spherical coordinates. I show explicitly that for r^1, r^2, r^3 and r^3 we get *exactly* the atomic forms expected! So the conclusion here is that it will work for all powers and that in fact the separable atomic form does in fact work if we choose our origin out in the middle of the cube so it is not on any metal surface.

In **Section 9** I state and prove the **Spherical Atomic Form Theorem**: regardless of how complex the situation, iff you pick as spherical origin a point which has a ball around it with no metal in it, you can write the potential in spherical atoms about that origin. Our hemisphere bowl with lid problem fails to have spherical atomic form because that lid-center origin violates this theorem. In Question #2 I claim that the cube-center spherical atomic form solution would give the right potential everywhere inside the cube, including the constant values on the cube faces (not totally sure of that, however).

In **Section 10** I wonder more about this last question of whether you would really duplicate V = 0 on the cube walls. So I examine a simpler problem, a "**waveguide**" with a square cross section. Still a 3D problem, but we know we can use 2D polar coordinate system atoms to model the solution in this case. I again take the polar origin at the center of the square, and I have V = 0 this time on all four walls so $V = \cos(x)\cos(z)$ where y now goes into the plane of paper. I do the small r expansion and add up the terms,

and sure enough, the potential is 0 on the 4 walls. I did a little Maple plot, but I don't know where the mws file went.

In **Section 11** (called comments) I note that, so far, I have not shown with cube example that the spherical atom form does not work if the origin is on metal, but that is coming soon. I then comment on the problem of a **cubic capacitor** where 5 sides are at V = 0 and one side is at V = 1. I know I can write the interior solution as a sum of Cartesian atoms cos(nx)cos(ny)sh(nz) and then express this in cube-center spherical coordinates, and the solution will be valid everywhere inside the cube. Outside the cube, this r^n expansion will exist, but won't give the exterior potential which we know is going to be an r^{-n-1} type affair. I have not DONE this exterior problem, it is not obvious to me how you would do it. But there is some solution, and if I knew it, I could find the potential on a math sphere outside the cube, and from that I could get a spherical atomic solution outside. This is just an example of the theorem of Section 6.

In **Section 12** I finally get around to trying my **cube** problem in sphericals with the **origin at a corner**. I want this to be a nice counterexample showing you cannot do it with spherical atoms. The potential in this case is $\varphi_{11}(r,\theta,\varphi) = \sin(r\sin\theta\cos\varphi)\,\sin(r\sin\theta\sin\varphi)\,\sinh(\sqrt{2}\,r\cos\theta)$ converting to corner-sphericals. Can this be written as $\varphi_{11}(r,\theta,\varphi) = \Sigma_{nm}\, a_{nm}\, r^n\, P_n^m(\cos\theta)\, e^{im\varphi}$ at least inside the box? In (a) I note that one way to see is try to compute the a_{nm} , but the integrals are too hard. In (b) I do a small-r expansion around the corner origin. The lowest term in the expansion is r^3 as you see from the above form, and for this term we CAN get the spherical atomic form. In (c) I realize that this r^3 term is really the product xyz. I show that it was just lucky that this polynomial in xyz was writable in this way. I show that xyz^2 does not have an atomic form, nor does xyz^3. But these polynomials appear in the expansion of φ_{11} above, so we then have our cube counterexample where we canNOT express the potential in spherical atoms with a corner origin. But then in (d) I show that if you consider the sum of such terms like xyz^3, the sum satisfies Cartesian Laplace. I conclude (I think) that I still have my counterexample.

In **Section 13** I state the conclusions for Section 12. I claim that with a cube-corner origin, you canNOT write the potential as a spherical atom sum, period, and this fits in with our Section 9 Theorem above. I go on to conjecture that if you do your expansion about any origin inside the cube which is a finite distance away from the wall, that solution applies in any solid angle beam until such beam hits the walls, and I have a little picture of this. I then restate various conclusions reached above, including the fact that the disc problem will not have a near-zone spherical atomic form for origin on the disk no matter how many "attempts" you make!

In **Section 14** for the first time I think about the charged disk in ***cylindrical*** coordinates, following Jackson's approach. Here I first write down the cylindrical "atoms" and my first idea is to keep the cylindrical origin off the disk. But then I gradually get the idea that $J_0(k\rho)$, which is the analog of r^n, with k the analog of n, might form an orthogonal set, so my kink power series argument would then not apply in cylindricals. It slowly sinks in that the cylindrical atoms are going to work for the disk problem's Smythian form even with the coordinate system origin on the disk. I then note that the big difference here is that r is the non-oscillatory SL dimension for the spherical case (for integer n), whereas ρ is an oscillatory SL dimension in the cylindrical case.

In **Section 15**, aka **Appendix A**, written much later than the original document, I firm up the distinction between spherical and cylindrical, and write the Hankel Transform in precise detail.

end of meta review

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1. Can you do a Smythe Spherical Form Fit (and maybe find sol) for the charged disk?

I now know the exact potential for the charged disk, namely,

$$V(x,y,z) = (q/a)\,\cot^{-1}(\xi) \;=\; (q/a)\,\sin^{-1}\left[2a \,/\, \left(\,\left|\,\sqrt{(\rho-a)^2+ z^2}\,\right| \;+\; \sqrt{(\rho+a)^2+ z^2}\,\right)\right]$$

where the first form is in oblate coordinates and the second in Cartesians where $\rho^2 = x^2+ y^2$. I just spent a lot of time understanding the nature of the first radical (the one with the absolute value on it). See the following for details: Ahlfors/ " A study of power branch points.doc". It is now completely clear and unambiguous.

Here is what I want to understand now: can you arrive at the above formula by doing Smythe style "form fitting?" If so, let's do it. If not, need to know why not. I will start in spherical coordinates here and then perhaps move on to other systems as needed.

Attempt #1. The charged disk assuming the Above/Below Form . (3D)

We always have the problem of selecting Smythian Regions to work with. Here we make a stupid choice, one form for above the disk and one for below the disk. We know m = 0 so we have

$$V_{\mathbf{above}}(r,\theta,\varphi) \;=\; \Sigma_n\, A_n r^{-n-1} P_n(z) \qquad 0 \le z \le 1$$

$$V_{\mathbf{below}}(r,\theta,\varphi) \;=\; \Sigma_n\, B_n r^{-n-1} P_n(z) \qquad -1 \le z \le 0$$

where we throw out the r^n terms since this has to be 0 at infinity. But then we have only r^{-n-1} terms and these all blow up at the center of the disk, where potential should be finite, so $A_n = B_n = 0$ and we are done with this "form". Move on.

Attempt #2. The charged disk assuming the Inner/Outer Spherical Form.

Suppose we start instead with this "form"

$$V_{\mathbf{outer}}(r,\theta,\varphi) \;=\; \Sigma_n\, A_n\; r^{-n-1} P_n(z) \qquad r \ge a$$
$$V_{\mathbf{inner}}(r,\theta,\varphi) \;=\; \Sigma_n\, B_n\; r^{n}\; P_n(z) \qquad r \le a$$

Since this form is supposed to be valid both above and below the disk, it cannot have n = odd terms in the Legendre sum, so we now have

$$V_{\mathbf{outer}}(r,\theta,\varphi) \;=\; \Sigma_{n,\mathbf{even}}\, A_n\; r^{-n-1} P_n(z) \qquad r \ge a$$
$$V_{\mathbf{inner}}(r,\theta,\varphi) \;=\; \Sigma_{n,\mathbf{even}}\, B_n\; r^{n}\; P_n(z) \qquad r \le a$$

Only the inner form is allowed to touch the boundary condition disk flat surface, and we get

$$V_{\mathbf{inner}}(r,\pi/2,\varphi) \;=\; \Sigma_{n,\mathbf{even}}\, B_n\; r^{n}\; P_n(0) \;= V_0 \qquad r \le a$$

We know that $P_n(0) \neq 0$ for n even. The only way this can be true for a continuum of r in $r \leq a$ is then to have $B_n = 0$ except for n=0. [Here we have made use of various ideas " Power series and f(x) constant in a finite region.doc" in math/misc: if a convergent power series is constant on all or even part of a region, then it is constant on all of that region. So here we have our series in r being a constant on a finite region say r = [0,a] or any portion of that region. The unique solution is $B_n P_n(0) = \delta_{n0} V_0$.] So again we are dead in the water: our Smythian fit collapses down to $V_{inner}(r,\theta,\varphi) = \Sigma_n B_n r^n P_n(z) = V_0$ at all points in the inner sphere.

See Appendix A below which I added 4.14.10.

Attempt #2A. The charged disk assuming another Inner/Outer Spherical Form.

Start with the same form above, and assume it is *only valid "above"* the disk. When we find a solution, we will mirror it below the disk. Then we don't get our restriction only to even terms, so we have

$$V_{outer}(r,\theta,\varphi) = \Sigma_n A_n r^{-n-1} P_n(z) \qquad r \geq a$$
$$V_{inner}(r,\theta,\varphi) = \Sigma_n B_n r^n P_n(z) \qquad r \leq a$$

Then we apply

$$V_{inner}(r,\pi/2,\varphi) = \Sigma_n B_n r^n P_n(0) = V_0 \qquad r \leq a$$

But we use the fact that $P_n(0) = 0$ for n odd, so the above line says

$$V_{inner}(r,\pi/2,\varphi) = \Sigma_{n,even} B_n r^n P_n(0) = V_0 \qquad r \leq a$$

and this is exactly the same as our previous attempt so our form fails.

Attempt #3: The charged disk with a global $P_n(z)$ series form.

Look at the known solution of the charged disk problem:

$$V(x,y,z) = (q/a) \cot^{-1}(\xi) = (q/a) \sin^{-1} [2a / (| \sqrt{(\rho-a)^2 + z^2} | + \sqrt{(\rho+a)^2 + z^2})]$$

We know we can write this in spherical coordinates:

$$\rho = r\sin\theta$$
$$z = r\cos\theta$$

$$V(r,\theta) = (q/a) \cot^{-1}(\xi) = (q/a) \sin^{-1} [2a / (| \sqrt{(r\sin\theta-a)^2 + (r\cos\theta)^2} | + \sqrt{(r\sin\theta+a)^2 + (r\cos\theta)^2})]$$

If we put dummy |...| around the second radical and take $\theta \rightarrow -\theta$, it is pretty clear that this potential is even in θ, so in any $P_n(z)$ expansion, it really will have only even terms. Since V is defined for θ in $(0,\pi)$, we can Legendre-Fourier analyze it (which would take a LOT of effort, I think), and we could get coefficients with terms of this form:

$a_n(r)P_n(z)$ n even $V(r,\theta) = \Sigma_n\, a_n(r)P_n(z)$

assuming that n = 0,1,2,3... is the correct valid range. It seems unlikely that we will find that $a_n(r)$ has the form $c_n r^n$ or $d_n r^{-n-1}$! Suppose $a_n(r) = c_n r^n + d_n r^{-n-1}$. Then our solution is invalid at large r unless c_n = 0, and it is invalid at r=0 unless d_n= 0, so we would need $c_n = d_n = 0$ which means $a_n = 0$. If is far *more* likely that $a_n(r)$ will come out being some fairly complicated function which behaves as $d_n r^{-n-1}$ for large r, and behaves as $c_n r^n$ for small r (maybe). This clinches the matter really, and says that you cannot write the solution to this problem using a single global form involving the atoms $r^n P_n(z)$ and $r^{-n-1}P_n(z)$.

Attempt #4: The charged disk assuming the Inner/Outer Cylindrical Boundary Form.

Suppose we take as our two regions the inside and outside of a cylinder which houses the disk, and also we will assume we are only "below" the disk. But we shall stay in spherical coordinates, just to be obstinate, because somehow it has to work (does it?). Then we have these forms

$V_{outside} = \Sigma_n\, A_n\, r^{-n-1}P_n(z)$ $\rho \geq a$

$V_{inside} = \Sigma_n\, A_n\, r^n\, P_n(z)$ $\rho \leq a$

The cylinder inside region is where we have the z = -1 axis, so we must have n = integer in the V_{inside} expression. But then we go onto the disk surface with z = 0 and <u>we have our same problem</u>:

$V_{inside} = \Sigma_n\, A_n\, r^n\, P_n(0) \;=\; \Sigma_{n,even}\, A_n\, r^n\, P_n(0)$ for continuum of r values

We have ruled out the r^{-n-1} terms because we have to be able to take our inside solution to the center of the plate which is r = 0 and it has to be finite there. So we are dead meat once again!

[Note that this inner/outer cylindrical region form in sphericals would produce a very ugly matching boundary on the walls of the cylinder, even if it were viable.]

<u>Here is a possible exit door. [no good]</u> We know that the answer to the problem is even in θ, and we want to say that we can therefore expand it on the even $P_n(z)$ functions and that means $r^n\, P_n(z)$ [if we assume the solution is a series of separated atomic terms] . But, our region here is only "above the disk" which means z is in the range (0,1), and the $P_n(z)$ are NOT a complete set on that range, as I recall from my flailing with the half sphere problem.

On the other hand, our known solution expression is valid on both side of the disk, so doesn't that mean we have the full range (-1,1) for z ?

But then the counter-argument to that is this: we don't have any "analytic continuous region" which has the full range of z. The plate cuts any such region in half, and it is a singular plate.

So maybe I can buy this argument that we cannot do a simple spherical form fit in a small region just above the plate because the $P_n(z)$ are not a complete set for such a region.

I shall return to this subject somewhere below.

OK, what about the region outside the cylinder? Our candidate solution there would be

$V_{outside} = \Sigma_n A_n \; r^{-n-1} P_n(z)$ $\rho > a$

But this does touch the edge of the plate when z = 0 so we would then need

$\Sigma_n A_n \; a^{-n-1} P_n(0) = V_0$

This at least seems possible to satisfy. In fact, if we look at the plane outside the disk, and if we know that a single form fits all (the known answer), we sort of assume that we can have only even terms in our sum in this "outside region form". Then our edge condition is this:

$\Sigma_{n,even} A_n \; a^{-n-1} P_n(0) = V_0$ $A_n = 0$ for n odd

and this is still viable, since these $P_n(0)$ do not vanish.

In general, then, our outside problem does not seem so impossible, we can probably match the disk edge potential with some A_n values. We decay at infinity. Yes, we have to match the inside solution at the cylinder wall, but we don't have a form for the inside solution!

One small extra concern. Since the outside region does not contain the z axis, it could in theory allow for non-integer values of n. But that would make matching impossible I think.

But here comes another nail in the coffin (but it's not a real nail) . Think of the metal bowl problem. Take a sphere just inside the bowl sphere. Origin at center, full range of z being (-1,1), must have integer n, so then you ought to be able to fit the potential inside this way

$V = \Sigma_n A_n \; r^n \; P_n(z)$

Now you do your Dirichlet condition against the bowl and you have

$\Sigma_n A_n \; a^n \; P_n(z) = V_0$ for continuum range of z being z in (z_0, 1)

So here we have that same "continuum problem", this time where z varies in some range. [but the $P_n(z)$ are an orthogonal set!]

This shows that I have a very basic misunderstanding about something, probably about expansions. Who says you can't have a Dirichlet BC with V = constant on some finite region?

Note added 1.23.10. It was at this point that I went off and wrote " Power series and f(x) constant in a finite region.doc", and learned that the above is viable since $P_n(z)$ is an orthogonal basis function. It is not viable when you have a power series where the z^n are then not orthogonal. See that doc "main conclusions" for more.

This is one of those amusing situations where you cannot rearrange the powers in the power series to make the $P_n(z)$ series be the same as a z^n series. The $P_n(z)$ series converges, but the z^n series won't converge at a "kink" which ends a constant region.

_____ adder _____

2. Question: *When can* you represent a potential in the form $V(r,z) = \Sigma_n a_n r^n P_n(z)$?
Under what conditions?

The charged disk once more. Let's ponder more on our Attempt #3 above, just as a case study. The known solution V(r,z) to the charged disk problem is this:

$$V(x,y,z) = (q/a)\cot^{-1}(\xi) = (q/a)\sin^{-1}\left[2a / \left(\left| \sqrt{(\rho-a)^2+ z^2} \right| + \sqrt{(\rho+a)^2+ z^2} \right)\right]$$
$$= (q/a)\sin^{-1}\left[2a / \left(\left| \sqrt{(r\sin\theta-a)^2+ (r\cos\theta)^2} \right| + \sqrt{(r\sin\theta+a)^2+ (r\cos\theta)^2} \right)\right]$$
$$= V(r,z)$$

Alternately, [reader knows that "z" is overloaded]

$$V(r,z) = (q/a)\cot^{-1}(\xi) \quad \text{where} \quad \xi^2 = (1/2a^2)\left\{ (r^2 - a^2) + \sqrt{|r^2-a^2|^2 + 4a^2r^2\cos\theta^2} \right\}$$

For a **fixed** r < a, we could expand V(r,z) onto the $P_n(z)$. That expansion would look like this:

$$V(r,z) = \Sigma_n\, a_n(r)\, P_n(z)$$

We could exactly compute the coefficients a_n in such an expansion. We would do this as follows:

$$a_n(r) \sim \int_{-1}^{1} dz\, V(r,\theta)P_n(z)$$

This is a very ugly integral, and it certainly seems *extremely unlikely* that we will find that

$$a_n(r) = A_n\, r^n$$

but it *could* come out that way! We know, however, for reasons discussed above, that the series given by $V(r,z) = \Sigma_n\, A_n\, r^n\, P_n(z)$ is not viable and if we force it to meet the disk BC we get $A_n = \delta_{n0}A_0$.

Getting back to that "unlikely" remark, for r > a, I will argue below that the coefficient $a_n(r)$ does come out having this form

$$a_n(r) = A_n\, r^{-n-1}$$

but only when r > a, not when r < a. This also seems "unlikely", but this is how it comes out!

So, based on our earlier arguments, the charged disk seems to be a case where you canNOT expand the solution to Laplace in the manner $\Sigma_n a_n r^n P_n(z)$, at least for the inner sphere $r \le a$. And certainly we have to rule out the form $\Sigma_n a_n r^{-n-1} P_n(z)$ as well to stay finite at r = 0. [I still think this is all true 1.25.10]

3. Getting more general.

We know that the form $\Sigma_n A_n r^n P_n(z)$ is a *possible* form for the solution of an azisym Legendre problem, but did we ever claim that ALL solutions must have this form? That is a bugaboo question that I never tried to answer before. Is it a fact that "bad boundary conditions" can rule out a separable solution of this form? In other words, is it possible that "bad boundary conditions" don't just make it hard to compute the coefficients A_n, but they make the whole form non-viable?

Consider the problem of a **charged metal object of some very complex arbitrary azisym shape**, perhaps an elaborately carved table leg or a coke bottle. We know a solution to Laplace exists. But does that mean we can write it in the form $V(r,\theta) = \Sigma_n A_n r^n P_n(z)$? Or at least inside some sphere?

Maybe think of the disk problem as really a cone problem. Then the n in the fit are not integers, they are values of ν such that $P_\nu(z=0) = 0$. Well, ν = odd integers are good solutions there. Ignore this cone comments for the moment, more later on Green's functions.

Well, I am being swayed. When we do the separation, we find a set of possible solution functions that work in the PDE, I like to call them separation atoms, little building blocks,

$r^n P_n(z)$ $r^{-n-1} P_n(z)$ etc etc

We know that any linear combination of these is also a solution. We know that if we take a linear combination of these and make that satisfy our BC's, then that is THE solution, because we know the solution is unique. But if we have unruly BC's, maybe we just cannot satisfy them with these basic atomic forms, and then we have a case where the solution is NOT of the atomic form.

The question would be whether the forms shown above form a "complete set" in some sense. We can think of the φ and z functions as complete sets, I have fiddled with this a lot before, even when n is not integer. But the r^n stuff does not look very complete set like!

The eigenfunctions of a sphere ($\varphi = 0$ on boundary) have a better chance at being complete I think. But they in r these are $j_n(kr)$ things, not r^n things.

Idea: maybe the form $r^n P_n(z)$ provides a solution only when you have r = constant BC's, because then the Y_{nm} are known to be complete on the sphere. Maybe for z = constant BC's, or φ = constant BC's, when you evaluate on the boundary, you are stuck with this r^n and r^{-n-1} powers which do not form a complete set, so your atoms then might not work. In other words, you have no way to invert those BC's because the r^n are not orthogonal, so you can't compute anything. [I think this is right. On your surface, you need to have two "oscillatory" coordinates, neither can be a "power/expo" type coordinate. Otherwise, you cannot match some BC on that surface in two dimensions, V = constant is just a small example.]

Here is an example of this. Consider a *charged metal cone* at potential V_0, maybe of some finite extent. Inside the cone we consider a solution of the form $\Sigma_n a_n r^n P_n(z)$. But then we evaluate on the cone boundary and we find that $\Sigma_n a_n r^n P_n(z_0) = V_0$. Since we are not doing a Green's function here, having n be the zeros of $P_n(z_0)$ does not do anything for us. We can assume whatever discrete n spectrum we want, when we get to $\Sigma_n a_n r^n P_n(z_0) = V_0$, we have our classic continuum problem in a power series in r, and we are forced to $a_n = \delta_{n0} a_0$. If we take the limit $z_0 = 0$, this becomes the charged disk problem.

[I suspect in oblate coordinates we have a similar situation. The $Q_n(i\zeta)$ replace the $r^{n,-n-1}$ and they too fail to be an orthogonal set of basis functions, so the form $\Sigma_n a_n Q_n(i\zeta)P_n(\xi)$ only works for BC's which have ζ = constant. This includes holes in plates and disks and of course "oblate spheroids.]

4. General Question: Are there potential problems whose solutions cannot be written as series of the separated atomic forms?

Let's take a little web scan on this:

There are at least two conditions that partial differential equation (PDE) of diffusion and its boundary conditions (BCs) must satisfy if the method of separation of variables (SOV) is to be used:

i) PDE must be linear and homogeneous
ii) BCs must be linear and homogenous (Initial conditions don't matter)

You might try the book "Symmetry and Separation of Variables" by W. Miller, Addison-Wesley 1977.

What is meant by a "homogeneous boundary condition" when you are talking PDE language?

For an ODE I know it means something like this:

$Au(a) + Bu'(a) = 0$ at an endpoint of the interval, say

or maybe both endpoints could be involved in the same BC.

Here is some text, he means G≠0 in the first line:

If $G = 0$, the PDE is non-homogeneous, examples of non-homogeneous PDEs are

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + B\frac{\partial u}{\partial t} + Au = 2$$

The same concepts of linearity and homogeneity apply to the initial and boundary conditions. For example

$u(x, 0) + u_x(x, 0) = 0$ linear, homogeneous

$u(x, 0) + u_x(x, 0) = 1$ linear, non-homogeneous

$u(x, 0) + u_x^2(x, 0) = 1$ nonlinear, non-homogeneous

I don't know how to apply this to sphericals and V(r,θ,φ). When we say V(r,θ,φ) = f(θ,φ) for Dirichlet on a sphere, that would seem to be an *inhomo*geneous BC, yet we still have a solution as a series of our separable atoms. So maybe I have to claim the first web claim above is false. [agreed 4.14.10: instead of

endpoints, in N dimensions we have a closed boundary surface σ of N-1 dimensions and V = f above would in fact have to be called "inhomogeneous" since f ≠ 0.]

So at least here are some examples of the animals in question, he is in 2D. Must y = 0?

The superposition principle is also applied to the boundary conditions. If u_1 and u_2 satisfy a linear homogeneous boundary condition then

$$u = C_1u_1 + C_2u_2$$

will also satisfy the boundary condition.

Maybe this is the idea:

$$A\,\varphi(\sigma) + B\,\partial_n\varphi(\sigma) = f(\sigma)$$

but this is not homogeneous. So my question remains unanswered. [My web scan was not successful.]

5. Coke Bottle Exterior Problem. Charged azisym metal coke bottle at the origin. Consider a fairly close surrounding sphere r = a. There is some V(a,θ) potential on this sphere, I just don't know what it is, call it f(θ). Consider a second sphere at r = ∞ on which V = 0. Try this fit

$$V(r,\theta) = \Sigma_n a_n r^{-n-1} P_n(z)$$

We must have integer n because the z = -1 ray is included. So write

$$V(a,\theta) = \Sigma_n a_n a^{-n-1} P_n(z) = f(\theta)$$

Use orthogonality to find the a_n all done. [This is really just Dirichlet inside a sphere in the 1/r space.]

6. Exterior Charged Metal Object Theorem: The potential outside any azisym charged metal object can be represented by the usual atomic form $r^{-n-1}P_n(z)$. This is true no matter how complex the shape of the azisym object. This applies in particular to the charged disk for the region outside the disk. The conclusion can be generalized in the obvious way to the potential outside a charged metal object of ANY shape. [The region of interest is just the infinite shell outside some sphere which surrounds your object. The position of the origin does not matter, because in effect this is a 1/r Dirichlet problem with origin located at infinity.]

Note added 4.17.10. Let's state this more clearly. Suppose you have a localized collection of metal objects perhaps held at different potentials. Throw in if you like some point (or sticky) charges as well, so we have a collection of charges and of metal objects carrying strange induced charge distributions. We know that there is some unique global solution to this problem. Given that solution, we know potential V on any sphere we choose which encloses all these objects. We then have a Dirichlet problem for the space between that sphere and the Great Sphere (V=0) on which the potential is prescribed. This space is "metal free". We know there is a unique Laplace solution in this space. If we try a fit with

$V = \Sigma_{nm}\ a_{nm}\ r^{-n-1}P_n^m(z)\ e^{im\varphi}$, with nm being "the usual values", we know we can use orthogonality in the z and φ dimensions to evaluate a_{nm}. Therefore, we have a solution which works and V = 0 at r=∞, so this must be THE solution. It seems pretty clear that, for any point in this space, we could find a sphere centered at that point which encloses all the metal and charges. Thus, there is no restriction on the location of the spherical origin.

Apply to charged disk exterior potential: We know the potential,

$$V(r,\theta) = (q/a)\ \sin^{-1}\left[2a \,/\, \left(\,\left|\ \sqrt{(r\sin\theta - a)^2 + (r\cos\theta)^2}\ \right|\ +\ \sqrt{(r\sin\theta + a)^2 + (r\cos\theta)^2}\ \right)\right]$$

So set r=a and we have

$$f(\theta) = V(a,\theta) = (q/a)\ \sin^{-1}\left[2a \,/\, \left(\,\left|\ \sqrt{(a\sin\theta - a)^2 + (a\cos\theta)^2}\ \right|\ +\ \sqrt{(a\sin\theta + a)^2 + (a\cos\theta)^2}\ \right)\right]$$

$$= (q/a)\ \sin^{-1}\left[2 \,/\, \left(\,\left|\ \sqrt{(\sin\theta - 1)^2 + (\cos\theta)^2}\ \right|\ +\ \sqrt{(\sin\theta + 1)^2 + (\cos\theta)^2}\ \right)\right]$$

$$= (q/a)\ \sin^{-1}\left[2 \,/\, \left(\,\left|\ \sqrt{2(1-\sin\theta)}\ \right|\ +\ \sqrt{2(1+\sin\theta)}\ \right)\right]$$

Well, maybe this is easier for setting r = a:

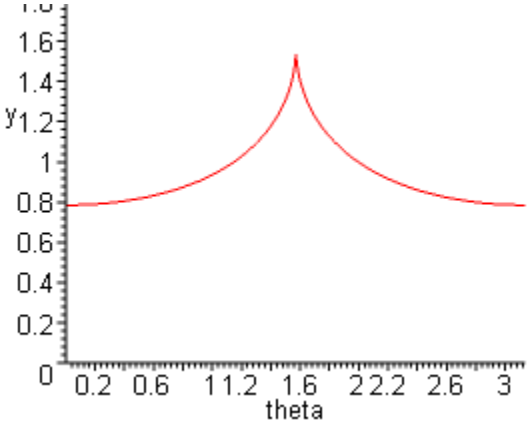
$$V(x,y,z) = (q/a)\ \cot^{-1}(\xi) \qquad \text{where} \quad \xi^2 = (1/2a^2)\ \{\ (r^2 - a^2) + \sqrt{|r^2 - a^2|^2 + 4a^2z^2}\ \}$$

So $\xi^2 = (1/2a^2)\ \{\ 0 + 2az\} = (z/a) = (r/a)|\cos\theta|$

$$V(a,\theta) = (q/a)\ \cot^{-1}(\sqrt{|\cos\theta|}\ ) = (q/a)\ \cot^{-1}\ (\sqrt{|z|}\ )$$

The potential at z = 0 is therefore $(q/a) * (\pi/2)$ [C = $2a/\pi$ for the disk]

Here is a plot of V(a,θ), you see how it peaks at the edge of the disk. It also has a nice doubly infinite discontinuity there, reflecting the infinite charge on both sides at the edge! I like it.



So now our outside coefficients will involve

$$\int_{-1}^{1} dz\, P_n(z) \cot^{-1}(\sqrt{|z|}\,)$$

Only the **even** coefficients survive, and we need

$$= 2 \int_{0}^{1} dz\, P_n(z) \cot^{-1}(\sqrt{z}\,)$$

Maple and Wolfram can't do it, but it certainly is well defined and non-singular and we could get our a_n.

Compute the integral to learn the external coefficients:

$$(2n+1)\, P_n(z) = P_{n+1}'(z) - P_{n-1}'(z) \qquad \text{parts} = P_{n+1}(z) \cot^{-1}(\sqrt{z}\,)\big|_0^1 \equiv p(n+1)$$

$$\int_{0}^{1} dz\, P_{n+1}'(z) \cot^{-1}(\sqrt{z}\,) = -\int_{0}^{1} dz\, P_{n+1}(z) [\cot^{-1}(\sqrt{z}\,)]' + P_{n+1}(z) \cot^{-1}(\sqrt{z}\,)\big|_0^1$$

$$-\int_{0}^{1} dz\, P_{n+1}(z) \{ -(1+z)\, \partial_z \sqrt{z}\ \} + \text{parts} \quad = (1/2) \int_{0}^{1} dz\, P_{n+1}(z)\, (1+z)/\sqrt{z} + \text{parts}$$

But then

$$\int_{0}^{1} dz\, P_{n+1}(z)\, (1+z)/\sqrt{z} \;=\; \int_{0}^{1} dz\, P_{n+1}(z)\, z^{-1/2} + \int_{0}^{1} dz\, P_{n+1}(z)\, z^{+1/2}$$

These last two integrals are in GR p 796 7.126.1 and Maple knows it, but gives a series instead of the simple GR result. so use GR.

7.126

1. $$\int_0^1 P_\nu(x) x^\sigma\, dx = \frac{\sqrt{\pi}\, 2^{-\sigma-1}\, \Gamma(1+\sigma)}{\Gamma\left(1+\frac{1}{2}\sigma-\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}\sigma+\frac{1}{2}\nu+\frac{3}{2}\right)} \qquad [\mathrm{Re}\,\sigma > -1]$$ EH I 171(23)

Let's call this integral $I(\nu,\sigma)$ so we get

$$\int_{0}^{1} dz\, P_{n+1}(z)\, (1+z)/\sqrt{z} \;=\; I(n+1,-1/2) + I(n+1,1/2)$$

Then here is the wrap-up

$$\int_{0}^{1} dz\, P_n(z) \cot^{-1}(\sqrt{z}\,) \;=\; (2n+1)^{-1} \left[\int_{0}^{1} dz\, P_{n+1}'(z) \cot^{-1}(\sqrt{z}\,) - \int_{0}^{1} dz\, P_{n-1}'(z) \cot^{-1}(\sqrt{z}\,) \right]$$

$$= (2n+1)^{-1} \left[(1/2) \int_{0}^{1} dz\, P_{n+1}(z)\, (1+z)/\sqrt{z} + p(n+1) - (n+1) \to (n-1) \right]$$

$$= (2n+1)^{-1}\,[(1/2)\,\{I(n+1,-1/2) + I(n+1,1/2)\} + p(n+1) \;-\; (n+1)\rightarrow(n-1)\;] \;\equiv\; (2n+1)^{-1}K_n$$

Meanwhile

$$p(n) = P_n(z)\,\cot^{-1}(\sqrt{z}\,)|^1_0 \;= P_n(1)\,\cot^{-1}(1) \;- P_n(0)\,\cot^{-1}(0) \;= P_n(1)\,\pi/4 \;- P_n(0)\,\pi/2$$

$$= \pi/4 \;- P_n(0)\,\pi/2$$

$$p(n+1) - p(n-1) \;=\; -\pi/2 * [P_{n+1}(0) - P_{n-1}(0)] \;= 0 - 0 = 0 \;\text{ since n is even}$$

$$2K_n \;= [\;\; I(n+1,-1/2) + I(n+1,1/2)\;\;] \;- [\;\; I(n-1,-1/2) + I(n-1,1/2)\;\;]$$

So fine, I have the result in closed form. This means we can write the external potential for the charged disk in this way

$$V(a,\theta) = \;(q/a)\,\cot^{-1}(\sqrt{|z|}\,) \;=\; \Sigma_n\, a_n\, P_n(z)$$

$$a_n \;= (2n+1)/2 * \;(q/a) * \;\int_{-1}^{1} dz\, P_n(z)\,\cot^{-1}(\sqrt{|z|}\,) \qquad\qquad \text{// Schaum p 147 25.28}$$

$$= (2n+1)\,(q/a)\,2\;\int_{0}^{1} dz\, P_n(z)\,\cot^{-1}(\sqrt{z}\,) \;=\; (2n+1)\,(q/a)\,2\,(2n+1)^{-1}K_n \;= (q/a)\,2K_n$$

$$= (q/a)\;\{\;[\;\; I(n+1,-1/2) + I(n+1,1/2)\;\;] \;- [\;\; I(n-1,-1/2) + I(n-1,1/2)\;\;]\;\}$$

Then we have

$$V(r,\theta) = \;\Sigma_n\, a_n\,(r/a)^{-n-1}P_n(z) \qquad\qquad a_n \;= (2q/a)\,K_n$$

This is explicitly the exterior potential of the charged disk as a spherical atom series. I entered this into Maple and got this result for K_n : (see Maple1.mws in this folder)

$$K_n = 2\sqrt{2}\;\{\,\sin((\pi/4)(3+2n))\,(1+2n)/((-1+2n)(3+2n)) \;- \sin((\pi/4)(5+2n))\,((-3+2n)(5+2n))\,\}$$

Summary: We can model the potential *exterior* to our disk as $V(r,z)\;\Sigma\, a_n r^{-n-1}P_n(z)$, and we know exactly how to compute the coefficients.

What about the inside sphere? We know the potential on its surface, $(q/a)\,\cot^{-1}(\sqrt{|z|}\,)$. If we were to glue that potential on this sphere and have nothing inside, that potential would be constant around the equator for sure. This would be a Dirichlet problem of some sort inside the sphere and there would be some solution everywhere inside. The solution inside would in fact be this for that Dirichlet problem:

$$V(r,\theta) = \Sigma_n a_n r^n P_n(z)$$

where the a_n would be the same as they were for our exterior sphere problem (unit sphere). For this solution, then, we could examine the radial dependence of V on the equatorial plane. We find:

$$V(r,\pi/2) = \Sigma_n a_n r^n P_n(0) = A_0 + A_2 r^2 + A_4 r^4 + ... \neq \text{constant} \quad // \text{ unless } A_n = \delta_{n0} A_0$$

This cannot then be the potential of the charged disk, which is constant on the plane. Nor is this the potential of a charged metal ring, since it would not likely have that exact same potential on the sphere! So this is some unknown and unrelated problem! [It is the problem of a Dirichlet hemispherical bowl with a constant potential "lid".]

Our problem of interest is a Dirichlet problem where the volume is a half sphere. The potential on the half dome is specified as above, and is constant on the equatorial disk, matching the ring values. But I don't know how to solve this Dirichlet problem! I have the potential on the entire σ surface, true, but then what do you do? Of course I know the answer, but I don't think it has the form $V(r,\theta) = \Sigma_n a_n r^n P_n(z)$! As noted earlier, if it did have this form, then we get $V(r,\pi/2) = \Sigma_n a_n r^n P_n(0) = V_0$ for all r in (0,a). But in our hard won power series work above, we know that if a power series f(r) is analytic on the disk of our disk and is constant for all r, the only a_n that work are $a_n = \delta_{n0} a_0$. But this does not fit the known potential there!

Conclusion: Inside the sphere, you cannot write the potential as $\Sigma_n a_n r^n P_n(z)$. It is not the superposition of any spherical "atomic forms" you can list. It is not a separated solution! So there ARE places where the solution to Laplace cannot be expressed as a superposition of atomic terms. [for a given coordinate system and a given origin.] In this case, you could do that atomic term form on the outside only.

Another example would be the potential inside a cubic box with some V(σ) specified on the walls of the box. There exists some solution inside, but I don't think it would be $\Sigma_n a_n r^n P_n(z)$! [Well, I work on this problem below.]

resume here

Trying polar atomic form on 2D charged "disk" (wire) problem. Consider a math disk enclosing a metal horizontal line segment on which $V = V_0$. This is a charged metal 2D wire segment (a problem I have done somewhere). The atoms for this problem are basically $r^n \cos(n\theta)$ but for n=0 we have the special case C ln(r) + D. So we attempt

$$V(r,\theta) = \Sigma_n a_n r^n \cos(n\theta) + C \ln(r) + D$$

We throw out ln(r) since it blows up, then just include D in the first term allowing n = 0 there:

$$V(r,\theta) = \Sigma_n a_n r^n \cos(n\theta)$$

One BC is at θ = 0 so we get $\cos(n\theta) = 1$ and

$$V(r,0) = \Sigma_n a_n r^n = V_0$$

The only solution valid for r in [0,1] (unit sphere say) is $a_n = \delta_{n0} a_0$ so $V(r,\theta) = V_0$ everywhere inside our unit disk. Conclusion: the separated atomic fails for this problem on the interior disk, very similar to our 3D charged disk problem.

Looking for a Counter Example where atomic form fails. I want to see a very simple Laplace problem where the solution in one coordinate system cannot, with 100% certainty, be expressed as a series of atoms of another system. I don't want any impenetrable integrals making the conclusion hazy.

So what are the atoms for Cartesian coordinates? Just do the separation right here:

$$\partial_x^2 XYZ + \partial_y^2 XYZ + \partial_z^2 XYZ = 0$$
$$YZ\partial_x^2 X + XZ\partial_y^2 Y + XY\partial_z^2 Z = 0$$
$$\partial_x^2 X/X + \partial_y^2 Y/Y + \partial_z^2 Z/Z = 0$$

$$\partial_x^2 X/X = k_x^2$$
$$\partial_y^2 Y/Y = k_y^2$$
$$\partial_z^2 Z/Z = k_z^2 = -k_x^2 - k_y^2$$

Two separation constants, I write k_x^2 as if plus, but it could be either sign, etc. So k_x and k_y are like the n and m of sphericals. A typical "term" in a form would then be

$$\sin(k_x x)\sin(k_y y)\exp(k_z z)$$

which would apply in a problem with a cube on which $\varphi = 0$ on all faces except the far-z face. So this would be for Dirichlet problem having 5 cube faces grounded and a 6th face at some variable potential. Note that in the 1D version of this problem, you have only k_x and $k_x = 0$ therefore, so Ax+B are the two atoms.

In general, the true atoms for the 3D problem are these, where we include the homo solution:

$$X(x) = A\exp(k_x x) + B\exp(-k_x x) + Cx + D$$

So the general term would be this: // for example, A' is really A'(k_y)

$$\varphi_k(x,y,z) = [A\exp(k_x x) + B\exp(-k_x x) + Cx + D] * \quad (*)$$
$$[A'\exp(k_y y) + B'\exp(-k_y y) + C'y + D'] *$$
$$[A''\exp(k_z z) + B''\exp(-k_z z) + C''z + D''] \quad \text{where } k_z^2 = -k_x^2 - k_y^2$$

So the set of "atoms" is fairly large:

$$[1, x, y, z, xy, yz, xz, xyz]$$

$$[1, y, z, yz]\exp(\pm k_x x) \quad \text{and cyclic}$$

$$[1, z]\exp(\pm k_x x)\exp(\pm k_y y) \quad \text{and cyclic}$$

$\exp(\pm k_x x)\ \exp(\pm k_y y)\ \exp(\pm k_z z)$

7. Cube Dirichlet problem solution with origin in two different locations.

In a problem where $\varphi = 0$ for x=0 and x=a (vanish on two walls of a box), if we look at (*) we know at once that C = D = 0 and we have to set A and B to get $\sin(k_x x)$, and we k_x gets quantized. So let's consider the Dirichlet problem of a box where $\varphi = f(x,y)$ on the far face only. We have the following candidate solution:

$$\varphi(x,y,z) = \sin(k_x x)\ \sin(k_y y)\ \sin(k_z z) \qquad k_z^2 = -k_x^2 - k_y^2$$

$$k_x = (\pi/a)n \qquad k_y = (\pi/b)m \qquad n,m = 0, \pm 1, \pm 2 \ldots$$

Then $k_z = i\,[\,(\pi n/a)^2 + (\pi m/b)^2\,]^{1/2} = i\,h_{n,m}$ $\qquad h_{n,m} = [\,(\pi n/a)^2 + (\pi m/b)^2\,]^{1/2}$

$$\varphi_{nm}(x,y,z) = \sin(k_x x)\ \sin(k_y y)\ \sinh(h_{n,m}\, z)$$

On the far face we have

$$\varphi_{nm}(x,y,c) = \sin(\pi n x/a)\ \sin(\pi m y/b)\ \sinh(h_{n,m} c) \qquad h_{n,m} = [\,(\pi n/a)^2 + (\pi m/b)^2\,]^{1/2}$$

Now let's take as our Dirichlet problem a simple case -- one where exactly what you see above is the Dirichlet prescribed potential on the far face. That is to say

$$f(x,y) = \sin(\pi n x/a)\ \sin(\pi m y/b)\ \sinh(h_{n,m} c) = g_{nm}(x,y)$$

In this case, the unique solution to our BC Laplace problem is $\varphi_{nm}(x,y,c)$ as shown above. The simplest case I can think of that is non-trivial is to pick n,m = 1,1

$$\varphi_{11}(x,y,z) = \sin(\pi x/a)\ \sin(\pi y/b)\ \sinh(h_{1,1}\, z) \qquad h_{1,1} = [\,(\pi/a)^2 + (\pi/b)^2\,]^{1/2}$$

Now make it even simpler by setting c = b = a so that $h_{1,1} = \sqrt{2}\,\pi/a$

$$\varphi_{11}(x,y,z) = \sin(\pi x/a)\ \sin(\pi y/a)\ \sinh(\sqrt{2}\,\pi z/a)$$

and now just set $a = \pi$ to make even simpler

$$\varphi_{11}(x,y,z) = \sin(x)\ \sin(y)\ \sinh(\sqrt{2}\, z)$$

So this is the simplest case I can think of in 3D Cartesians with a finite volume of interest.

Our origin is a cube corner, but I would rather have it be at cube center, so let's shift

$$\varphi_{11}(x,y,z) = \sin(x+\pi/2)\sin(y+\pi/2)\sinh[\sqrt{2}(z+\pi/2)]$$

$$= \cos(x)\cos(y)\{\sinh[\sqrt{2}z]\sinh[\sqrt{2}\pi/2] + \cosh[\sqrt{2}z]\cosh[\sqrt{2}\pi/2]\}$$

$$= \cos(x)\cos(y)\{A\sinh[\sqrt{2}z] + B\cosh[\sqrt{2}z]\}$$

where $A = \sinh[\sqrt{2}\pi/2]$ $B = \cosh[\sqrt{2}\pi/2]$

Now we can write this in spherical coordinates centered at the center of the cube:

$$\varphi_{11}(r,\theta,\varphi) = \cos(r\sin\theta\cos\varphi)\cos(r\sin\theta\sin\varphi)\{A\sinh(\sqrt{2}r\cos\theta) + B\cosh(\sqrt{2}r\cos\theta)\}$$

We could expand this for small x,y,z (meaning small r) about the origin to get

$$\varphi_{11}(r,\theta,\varphi) = \Sigma_n\ r^n f_n(\theta,\varphi)$$

The question is this: will you always find that

$$f_n(\theta,\varphi) = \Sigma_{m=-n}^{n} a_{nm}P_n(z)e^{im\varphi}$$

which in turn means that each term in a small r expansion is a spherical atom. I have done lots of cases below and this always seems to be true. I did n = 1,2,3,4.

---

8. Try spherical atomic form for origin at the center of the cube.

Now back to the x,y,z version

$$\varphi_{11}(x,y,z) = \cos(x)\cos(y)\{A\sinh[\sqrt{2}z] + B\cosh[\sqrt{2}z]\}$$

Let's expand this around the origin for a few terms

$$= [1 - x^2/2! + ...][1 - y^2/2! + ...]$$
$$\{A[(\sqrt{2}z) + (\sqrt{2}z)^3/3! + ...] + B[1 + (\sqrt{2}z)^2/2! + ...]\}$$

We see in this expansion lots of polynomial terms. We know that x,y,z are all proportional to r in sphericals, so let's find terms sorted in this manner:

1
z
x^2, y^2, z^2
x^2z, y^2z, z^3
$x^4, y^4, z^4, x^2y^2, x^2z^2, y^2z^2$

It is pretty obvious that 1 and z= rcosθ = $zP_1(\cos\theta)$ can be expressed as spherical atoms. The quadratic terms are these:

$$-1/2\ Bx^2 - 1/2\ By^2 + B\ z^2 \quad = (B/2)(-x^2 - y^2 + 2z^2) = (B/2)(-r^2 + 3z^2) = (B/2)r^2(-1 + 3\cos^2\theta)$$

Can we write the sum of these terms in the form $\Sigma_{nm}\ a_{nm}\ r^n\ P_n^m(\cos\theta)\ e^{im\varphi}$? If so, then we must have n = 2 and m = 0, so the sum would have to look like $a_{20}\ r^2\ P_2(\cos\theta)$. We would then need

$$a_{20}\ r^2\ P_2(\cos\theta) = (B/2)r^2(-1 + 3\cos^2\theta)$$

or

$$a_{20}\ P_2(z) = (B/2)(-1 + 3z^2)$$

or

$$a_{20}\ (1/2)(3z^2-1) = (B/2)(-1 + 3z^2)$$

So the r^2 situation gives the same result as the r^1 and r^0 : you get spherical atoms.
For the r^3 case we get

$$-x^2/2\ \sqrt{2}\ A\ z\ -y^2/2\ \sqrt{2}\ A\ z\ + A\ 2\sqrt{2}\ z^3/3!$$

$$-x^2/2\ z\ -y^2/2\ z\ +\ 2z^3/3!$$

$$-3x^2\ z\ - 3y^2\ z\ + 2z^3 \quad = -3z(x^2 + y^2 - (2/3)\ z^2) = -3z(x^2 + y^2\ + z^2 - (5/3)\ z^2)$$

$$= -3z(r^2 - (5/3)\ z^2) \quad = -3r\cos\theta(r^2 - (5/3)\ r^2\cos^2\theta) = -3r^3\ \cos\theta(1 - 5/3\ \cos^2\theta)$$

Can we write the sum of these terms in the form $\Sigma_{nm}\ a_{nm}\ r^n\ P_n^m(\cos\theta)\ e^{im\varphi}$? If so, then we must have n = 3 and m = 0, so the sum would have to look like $a_{30}\ r^3\ P_3(\cos\theta)$. We would then need

$$a_{30}\ r^3\ P_3(\cos\theta) = -3r^3\ \cos\theta(1 - 5/3\ \cos^2\theta)$$

$$a_{30}\ P_3(\cos\theta) = -3\cos\theta(1 - 5/3\ \cos^2\theta)$$

$$a_{30}\ P_3(z) = -3z(1 - 5/3\ z^2)$$

$$a_{30}\ (1/2)z(5z^2-3) = -3z(1 - 5/3\ z^2)$$

and again "it **works**". You suspect that probably this will happen no matter what r^n coefficient we look at. My "counterexample" keeps evaporating. Let's try "just one more compile".

For the r^4 case we get x^4, y^4, z^4, x^2y^2, x^2z^2, y^2z^2 terms like so:

$$+Bx^4/4! + By^4/4! + B\ 4z^4/4! + Bx^2y^2/4 - (1/2)Bx^2z^2 - (1/2)By^2z^2$$

$$+x^4/4! + y^4/4! + 4z^4/4! + x^2y^2/4 - (1/2)x^2z^2 - (1/2)y^2z^2$$

$$+x^4 + y^4 + 4z^4 + 6x^2y^2 - 12x^2z^2 - 12y^2z^2$$

$$+x^4 + y^4 + 4z^4 + 6x^2y^2 - 12z^2(x^2+y^2)$$

$$+x^4 + y^4 + 4z^4 + 6x^2y^2 - 12z^2(r^2 - z^2)$$

$$= r^4 [\ \sin^4\theta\cos^4\varphi + \sin^4\theta\sin^4\varphi + 4\cos^4\theta + 6\sin^4\theta\cos^2\varphi \sin^2\varphi - 12\cos^2\theta\sin^2\theta\ ]$$

well, I am sure now this can in fact be written as $\Sigma_m a_{4m} r^4 P_4^m(\cos\theta) e^{im\varphi}$. Keep going just to shove mud in your eye.

$$= r^4 [\ \sin^4\theta\{e^{i\varphi} + e^{-i\varphi}\}^4 (2)^{-4} + \sin^4\theta\{e^{i\varphi} - e^{-i\varphi}\}^4 (2i)^{-4}$$

$$+ 4\cos^4\theta + 6\sin^4\theta\{e^{i\varphi} + e^{-i\varphi}\}^2 (2)^{-2} e^{i\varphi} - e^{-i\varphi}\}^2 (2i)^{-2} - 12\cos^2\theta\sin^2\theta\ ]$$

$$= r^4[(\sin^4\theta/16 + \sin^4\theta/16)e^{4i\varphi} + \text{lower } \varphi \text{ terms }]$$

So we would need to have

$$a_{44} r^4 P_4^4(\cos\theta) e^{i4\varphi} = r^4[(\sin^4\theta/16 + \sin^4\theta/16)e^{4i\varphi}]$$

$$a_{44} P_4^4(\cos\theta) = (1/8)\sin^4\theta = (1/8)(1-\cos^2\theta)^2$$

$$a_{44} P_4^4(z) = (1/8)(1-z^2)^2 = (1/8)(z^4 - 2z^2 + 1)$$

and sure enough, this one **works** too. Good algebra ma boy.

I think I have now arrived at an obvious Theorem:

9. The Spherical Atomic Form Theorem: Imagine a hugely complicated Laplace problem, perhaps Dirichlet, perhaps having charged conductors all over the place. Imagine that the complete solution $\varphi(x,y,z)$ of the problem is known and is very ugly.

Now pick any point in space about which you can put a ball in empty space without hitting a piece of metal. Place a spherical coordinate system centered at the center of that ball. Then take the super-complex known solution φ and expand it in a power series in spherical variable r of that spherical coordinate system.

This theorem then says that the coefficients of that expansion will be spherical harmonics.

Proof: The potential in the ball must satisfy Laplace. If you write the potential as a power series in r ,then since the powers r^n form a spanning set, each term separately must satisfy Laplace. That then tells us that the term coefficient is a spherical harmonic.

I have seen this fall out in my cube example where I put a ball at the center of the cube.

Suppose one of the boundaries of the problem is a flat piece of metal at some potential V_0. Suppose you form a ball centered at a point on this boundary, so that the metal forms an equatorial plane of the ball. This is a violation of our theorem assumption above.

Question #1: in that half-sphere above the metal, is it even possible to expand the potential in a power series in r ?

Answer: No! you need a clear ball around your expansion point, see "power series and f(x) constant" doc once again. [I just did a digression and updated that doc.]

Question #2: **What can we say about the cube problem in regard to this Theorem?**

Even for the BC's above where 5 sides of the cube are at 0 potential and the far face is at the sort of default potential shown above, you can still write the potential inside the inscribed sphere in the form $\Sigma_{nm}\, a_{nm}\, r^n\, P_n^m(\cos\theta)\, e^{im\varphi}$. Our exact solution is this.

$$\varphi_{11}(r,\theta,\varphi) = \cos(r\sin\theta\cos\varphi)\cos(r\sin\theta\sin\varphi)\ \{\ A\sinh(\sqrt{2}\, r\cos\theta) + B\cosh(\sqrt{2}\, r\cos\theta)\ \}$$

$$A = \sinh[\sqrt{2}\,\pi/2] \qquad B = \cosh[\sqrt{2}\,\pi/2]$$

This is the solution of the interior Dirichlet problem. It is not the external solution if we require that $V(\infty) = 0$ as part of the external surface, mainly because the sinh/cosh stuff blows up. However, it would be the solution to the exterior Dirichlet problem if we specified the $r=\infty$ potential to just be the $r=\infty$ limit of what you see above. This is not a practical problem, just a thought problem.

We expand the above interior solution in a power series in r. This expansion converges over all space. We find the expansion has the form $\Sigma_{nm}\, a_{nm}\, r^n\, P_n^m(\cos\theta)\, e^{im\varphi}$. If we exscribe our cube with a sphere, the solution is correct everywhere inside the cube and on all the walls! So I guess in this problem, our solution really can be written in the form $\Sigma_{nm}\, a_{nm}\, r^n\, P_n^m(\cos\theta)\, e^{im\varphi}$. It's just that the solution is very clumsy to find working only with spherical coordinates and such an expansion.

I really do think that if you wrote down the series and added up the first 100 terms, you would see the constant walls appear out of nowhere.

10. An extruded 2D version of this problem?

We still use 3D physics, but we have a square extrusion cross section with V = 0 on three sides and the single term result on the forth side. Solution is probably this:

$$\varphi(r,\theta,\varphi) = \underset{x}{\cos(r\sin\theta\cos\varphi)}\ \{\ \underset{z}{A\sinh(r\cos\theta)} + \underset{z}{B\cosh(r\cos\theta)}\ \}$$

$$A = \sinh[\pi/2] \qquad B = \cosh[\pi/2]$$

So x-z are the cross sectional coordinates of our "waveguide" here. Potential is constant in y direction going down the extrusion.

Suppose we expand this as power series in r and let the ball hit the walls. Will we see the constant boundaries materialize? I hear Maple calling! We would have to do some kind of 3D contour plot?

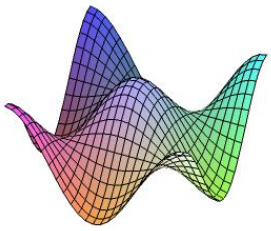
A simpler plotting case might be

$$f(r,\theta,\varphi) = \cos(r\sin\theta\cos\varphi)\cos(r\cos\theta) \quad = \cos(x)\cos(z)$$

We could pick $\varphi = 0$ and this is then across the wave guide cross section.

$$F(r,\theta) = \cos(r\sin\theta)\cos(r\cos\theta) \quad = \cos(x)\cos(z)$$

Let's expand terms for both cosines, and just see what Maple does. // I did this and sure, you see the zero potential boundaries:



11. Comments to this point.

1. My 3D and 2D *cube* Dirichlet studies have not produced an example of a Laplace solution which cannot be written in terms of spherical atoms. [Yes they have: origin at corner fails, see below!]

2. I could regard my cube walls as being metal set to V = 0. At infinity we have some V = ∞ form. There are no surface charges on the metal walls. The potential does not change slope going through them. They are just math walls set to $\varphi = 0$. So this is a rather special problem. [this makes no sense! In this case we would have V = 0 everywhere, I don't know what I was trying to say.]

3. <u>A different problem.</u> Imagine the cube problem with $\varphi = 1$ on the far face and $\varphi = 0$ on the others. This problem has some well defined solution in Cartesian coordinates, despite the discontinuity of potential at some corners. It will be an infinite sum of terms of the form cos(nx)cos(ny)sh(nz) instead of just one term. For each term we can do our r^n expansion, and regroup things globally to get a single power series expansion. The coefficients will be constants times spherical harmonics for any ball inside the cube.

Now imagine we have an external Dirichlet problem on the same cube with V=0 at ∞.

I think this is what happens:

(a) the r^n power series will be accurate for the entire inside of the cube, including on the walls. We get this situation when the cube is inscribed in the ball so that all the walls are included.

(b) outside the walls, this expansion exists, but does not describe the potential there of the external Dirichlet problem.

(c) there *is* surface charge on the metal walls. This is due to that V = 1 "face" which is making this thing some kind of capacitor. We know this because there will be a jump in the normal derivative across the

walls. This is like a capacitor of a square plate at V = 1 against infinity, where we have added some extra grounded square plates to the situation.

12. Try spherical atomic form for origin at a corner of the cube.

Now, write this in spherical coordinates:

$$\varphi_{11}(r,\theta,\varphi) = \sin(r\sin\theta\cos\varphi)\,\sin(r\sin\theta\sin\varphi)\,\sinh(\sqrt{2}\,r\cos\theta)$$

where we take one corner of the cube as our origin. [The corner we take is the place where three grounded plates all meet, should be very bad. [

Question: can this potential be expressed as a sum of spherical atoms?

Can you write this in the following manner:

$$\varphi_{11}(r,\theta,\varphi) = \Sigma_{nm}\, a_{nm}\, r^n\ P_n^m(\cos\theta)\, e^{im\varphi}$$

We cannot allow r^{-n-1} inside our box since blows up at the innocent origin. Well, as usual, it seems "unlikely", but we want to show for sure that this form does not work. [We are assuming that this form is somehow valid in the octant of space that falls inside the cube..]

(a) Sure fire method fails.

The only sure-fire proof is to do the double transform and see, but once again, I don't think this can be done. For example let a = rsinθ and consider

$$\int d\varphi\ \ e^{-im'\varphi}\sin(a\cos\varphi)\,\sin(a\sin\varphi)$$

These integrals are too hard to do. Same for the θ integral.

(b) Try small r leading term in the expansion

My first idea is to try small r and see if even there you cannot do it. We then get

$$\varphi_{11}(r,\theta,\varphi) \approx r\sin\theta\cos\varphi * r\sin\theta\sin\varphi * \sqrt{2}\,r\cos\theta \ = \sqrt{2}\ r^3\sin^2\theta\cos\theta\sin\varphi\cos\varphi$$

This would require that

$$\Sigma_{m=-3}^{3}\, a_{3m}\, r^3\ P_3^m(\cos\theta)\, e^{im\varphi} \ = \sqrt{2}\ r^3\sin^2\theta\,\cos\theta\,\sin\varphi\,\cos\varphi$$

$$\Sigma_{m=-3}^{3}\, a_{3m}\, P_3^m(\cos\theta)\, e^{im\varphi} \ = \sqrt{2}\ \sin^2\theta\,\cos\theta\,\sin\varphi\,\cos\varphi$$

Write

$\sin\varphi \cos\varphi = (e^{i\varphi} - e^{-i\varphi})/2i \ * \ (e^{i\varphi} + e^{-i\varphi})/2 = (1/4i)\,[\, e^{2i\varphi} - e^{-2i\varphi}]$

so we now have

$\Sigma_{m=-3}^{3}\, a_{3m}\, P_3^m(\cos\theta)\, e^{im\varphi} = \sqrt{2}\ \ (1/4i)\ \sin^2\theta\cos\theta\,[\, e^{2i\varphi} - e^{-2i\varphi}]$

Therefore, only m = ± 2 can contribute to this sum, and we find that

$a_{32}\, P_3^2(\cos\theta) = \sqrt{2}\ \ (1/4i)\ \sin^2\theta\cos\theta$

$a_{3,-2}\, P_3^{-2}(\cos\theta) = -\sqrt{2}\ \ (1/4i)\ \sin^2\theta\cos\theta$

Now we look up

$P_3^2(x) = 15*x*(1-x\text{^}2) = 15\cos\theta\sin^2\theta$

and I am of course defeated. There is probably some reason this works.

(c) Try the next terms in the small r expansion

Let's back up to

$\varphi_{11}(x,y,z) = \sin(x)\sin(y)\sinh(\sqrt{2}\,z)$

and our limit here was just this: [notice this is a Cartesian atom, solves Laplace on its own]

$\varphi_{11}(x,y,z) \approx \sqrt{2}\,xyz$

Can *any* polynomial in x,y,z be written as $\Sigma_{nm}\, a_{nm}\, r^n\ P_n^m(\cos\theta)\, e^{im\varphi}$? Yes or no? And if no, then which ones can?

Consider $xyz^2 = (xyz)\,r\cos\theta$

This will have the same φ dependence as (xyz), so our problem will be this for the r^4 balance:

$a_{42}\, P_4^2(\cos\theta) = \text{const} * \sin^2\theta\cos^2\theta \sim (1-z^2)z^2$

Maxima tells us that

```
assoc_legendre_p (4, 2, z)/15;
```

$$3\left(\frac{7(1-z)^2}{6}-\frac{7(1-z)}{3}+1\right)(1-z^2)$$

```
expand(%/((1-z^2)));
```

$$\frac{7z^2}{2}-\frac{1}{2}$$

which is to say

$P_4^2(z) = (15/2)\,(7z^2 - 1)(1-z^2)$

But this is not proportional to $z^2\ (1-z^2)$ so we conclude that xyz^2 cannot be written as Σ_{nm}. [If it could, then xyz^2 would solve Laplace, but we know it does not.]

Let's try another one:

Consider $xyz^3 = (xyz)r^2\cos^2\theta$

This will have the same φ dependence as (xyz), so our problem will be this for the r^5 balance:

$a_{52}\ P_5^2(\cos\theta) = \text{const} * \sin^2\theta\ \cos^3\theta\ \sim\ (1-z^2)z^3$

Maxima tells us that

```
assoc_legendre_p (5, 2, z)/105;
```

$$\left(-\frac{3(1-z)^3}{2}+\frac{9(1-z)^2}{2}-4(1-z)+1\right)(1-z^2)$$

```
expand(%/((1-z^2)));
```

$$\frac{3z^3}{2}-\frac{z}{2}$$

which is to say

$P_5^2(z) = (1/2)\ 105\ (3z^3 - z)(1-z^2)$

But this is not proportional to $z^3(1-z^2)$ so we conclude that xyz^2 cannot be written as Σ_{nm}.

So, let's go back to our potential

$\varphi_{11}(x,y,z) = \sin(x)\sin(y)\sinh(\sqrt{2}\,z)$

and expand around the origin to get

$\varphi_{11}(x,y,z) = [\,x - x^3/3! + ...]\,[\,y - y^3/3! + ...]\,[\,(\sqrt{2}\,z) + (\sqrt{2}\,z)^3/3! + ...]$

This has a term xyz^3 , and we have just shown that this term cannot be written as $\Sigma_{nm}\, a_{nm}\, r^n\ P_n^m(\cos\theta)\, e^{im\varphi}$. So I think I finally have my counterexample?

(d) But have I really proven anything?

More generally we have

$$\varphi_{11}(x,y,z) = \Sigma_{k,odd}\ a_k x^k\ \Sigma_{i,odd}\ a_i y^i\ \Sigma_{j,odd}\ (\sqrt{2}\,)^j a_j z^j$$

$$= \Sigma_{k,odd}\,\Sigma_{i,odd}\,\Sigma_{j,odd}\ (\sqrt{2}\,)^j a_k\, a_i a_j\, x^k y^i z^j$$

Notice that the higher individual terms here are not "atoms" of the Cartesian system either, x^2yz^5 for example. But the sum IS a Cartesian atom. I have shown that some of the terms are not Spherical atoms, and it is pretty clear that the sum is not a Spherical atom.

Well consider the term xyz^3shown above not to be a spherical atom. How do we know that ∇^2 acting on this term is not cancelled by ∇^2 acting on other terms in the series? If that were the case, then the fact that the xyz^3 was not a spheroidal atom would not matter.

$$\nabla^2(xyz^3) = 6xyz\ = \nabla^2(x^3yz)\ = \nabla^2(xy^3z)$$

so there are other terms that could cancel. I think these are the only ones. So I still don't really have a counterexample! So now I have to consider all three terms together and re-examine things. Here are the coefficients of these three terms.

xyz^3 term : $(\sqrt{2}\,)^3/3!$

x^3yz term : $-(\sqrt{2}\,)/3!$

xy^3z term : $-(\sqrt{2}\,)/3!$

So naturally,

$$\nabla^2 [\text{ three terms }] = [(\sqrt{2}\,)^3/3! + -(\sqrt{2}\,)/3! + -(\sqrt{2}\,)/3!\,]\ 6xyz = 0\ !$$

So yes, in fact they do cancel, so my proof fails? No. They had to cancel because this potential satisfies Laplace!

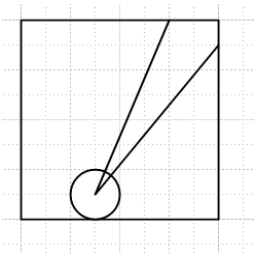
13. Summary of cube Dirichlet problem with spherical origin at a corner.

We have the known potential $\varphi_{11}(x,y,z) = \sin(x)\sin(y)\sinh(\sqrt{2}\,z)$ for our little Dirichlet here where the far face gets the default function form and the other 5 faces are at V = 0. We have put a spherical coordinate origin right where three metal plates meet! We expand the above potential in a power series in

r, and we find that the potential φ_{11} *cannot* be expressed as a spherical atoms series relative to this origin. Therefore, you cannot write φ_{11} in the form $\varphi_{11} = \Sigma_{nm}\, a_{nm}\, r^n\, P_n^m(\cos\theta)\, e^{im\varphi}$ for this origin. This series does not exist. You can write such a series of course, but you cannot find a_{nm} such that the series matches the known potential. So we say that such a series expansion for the known true potential does not exist.

On the other hand, if we consider the exact same problem and move the origin to the center of the sphere, we find that we suddenly *can* write the potential in the form $\Sigma_{nm}\, a_{nm}\, r^n\, P_n^m(\cos\theta)\, e^{im\varphi}$. And in fact this expansion is valid and gives the exactly correct results everywhere inside the cube.

This same conclusion is probably true if you pick the origin anywhere inside the sphere, not on the walls. I think what happens is this:



Although your ball hits the wall "early on", we have established the matching series potential in the finite ball, and we can sort of "continue it" correctly on any piece of solid angle until we hit the cube walls.

So the cases of origin at a corner and origin at the center seem to validate our Theorem quoted above regarding the need to have some finite ball about the point you use for the spherical origin.

Finally, regarding the charged disk problem, if we put our spherical origin at the center of the disk, we should expect that the known true potential cannot be represented as $\Sigma_n\, a_n\, r^n\, P_n(\cos\theta)$, and that is exactly what we have found in our various fiddlings a few days ago, see above. We could I think get such a series if we were willing to pick an origin off the surface, say on the symmetry axis. But if we tried to solve this charged disk problem a priori with such an origin, the BC of V= constant on the disk is pretty messy to even write down. It does not fall on a coordinate = constant surface.

As for the spherical bowl type problem, we do expect to get a $\Sigma_n\, a_n\, r^n\, P_n(\cos\theta)$ form using an origin at the center of the bowl sphere, at least for our interior solution.

The *exterior* problem for the charged disk can be written as $\Sigma_n\, a_n\, r^{-n-1}\, P_n(\cos\theta)$. You can think of this as an interior problem in 1/r space with nothing "inside", where infinity is the origin.

So, in this document we started out trying to set up for a solution of the charged disk by using "Smythian Forms", which is to say, spherical atoms. For an origin at the center of the disk, we find that we can write a form for the exterior problem in terms of spherical atoms, but we cannot do so for the interior problem, with origin at disk center, where we are talking about the half-sphere which subtends the disk. So all our attempts above were futile, though amusing.

14. What about doing the charged disk in Cylindrical Coordinates?

As Jackson shows, the atoms here are these

$$[\, J_m(k\rho),\, N_m(k\rho)]\;\; e^{\pm kz}\, e^{\pm im\varphi}$$

so that radial coordinate is no longer simple like $[\, r^n,\, r^{-n-1}]$. Here the m are integer, and the k are usually continuum values.

Based on the above discussions, here is what I imagine happens here.

(1) If you pick an origin away from metal surfaces so you can put a little "cylinder" around this origin, you will be able to expand the true potential in terms of the above atoms.

(2) For an azisym situation the form will be

$$[\, J_0(k\rho),\, N_0(k\rho)]\;\; e^{\pm kz}$$

So if we were to look for an interior solution to our charged disk problem, we would expect that the correct potential would be expandable in this way

$$\varphi(\rho,z) \;=\; \Sigma_k\, a_k\, J_0(k\rho)\, e^{-|k|z}$$

where the sum is really an integral. But maybe this would not be valid if the origin we pick lies on the disk! Similar to earlier fiddlings, if we did have such an expansion, we would conclude that

$$\varphi(\rho,z=0) \;=\; \Sigma_k\, a_k\, J_0(k\rho) \;= V_0$$

which we compare to $\Sigma_n\, a_n\, r^n\, P_n(0) \;= V_0$ we got many times above. The big question now is whether this thing $J_0(k\rho)$ is more like a power, or more like an orthogonal function and member of a complete set of such functions! Maybe for quantized values of k you can have an orthogonal set. I bet yes. $J_0(0) = 1$. See for example Jackson p 77 3.112 which is exactly this type of thing. And no quantization needed. He calls this the Hankel transform, but you have to do ρ integral 0,∞. [See Appendix A below!]

So maybe cylindricals will work better since we are getting a SL problem in our ρ coordinate system. Suppose we could get these quantized values. Then we have

$$V_0 = \Sigma_k\, a_k\, J_0(k\rho)$$

$$V_0 \int d\rho\, J_0(k'\rho) = \Sigma_k\, a_k \int d\rho\, J_0(k\rho)\, J_0(k'\rho) \;= \Sigma_k\, a_k\, \delta_{k,k'} \;\Rightarrow \qquad a_k = \; V_0 \int d\rho\, J_0(k\rho)$$

and then maybe we could get somewhere. Just a vague idea right now. [but the *right* idea!]

The other direction argument is this: Expand $J_0(k\rho)$ in powers of ρ and then conclude $a_k = 0$ and you cannot really have such an expansion with this choice of origin. But this is serious rearrangement and probably buys you nothing. [this is the whole issue of why t^n is not a basis and $P_n(t)$ is a basis!]

Jackson claims you really can write $\varphi(\rho,z) = \Sigma_k\, a_k\, J_0(k\rho)\, e^{-|k|z}$ for this "bad" choice of origin, see 3.170 on page 90, the integral goes $\rho = 0,\infty$. He even writes my $V_0 = \Sigma_k\, a_k\, J_0(k\rho)$ in 3.173. So the answer is definitely not " $J_0(k\rho)$ acts like a power". So we really can get a Smythian form in cylindrical coordinates, and it is a form that works over all space! And the a_k is given in 3.176 (called f(k) which in fact makes $V_0 = \Sigma_k\, a_k\, J_0(k\rho)$ for a continuum of ρ values!

Why can't we just do the Hankel thing? [In the following, I erroneously use (0,a) as endpoints. Alternatively, I erroneously assume the potential is 0 for $\rho > a$]

$\Sigma_k\, a_k\, J_0(k\rho) = V_0\ \rho < a, = 0\ \rho > a,]$. Apply $\int d\rho\ \rho\, J_0(k'\rho)$ to both sides and get

$$V_0 \int_0^a d\rho\ \rho\, J_0(k'\rho) = \int dk\ f(k) \int d\rho\ \rho\, J_0(k'\rho)\, J_0(k\rho) = \int dk\ f(k)\, \delta(k-k')/k = f(k')/k'$$

so we get

$$f(k) = kV_0 \int_0^a d\rho\ \rho\, J_0(k\rho)$$

$$\int x^{p+1}\, Z_p(x)\, dx = x^{p+1}\, Z_{p+1}(x)$$

GR

$$\int dx\ x\, J_0(x) = xJ_1(x)$$

So roughly we should have

$f(k) = kV_0$ (power of k) $J_1(k\rho)|_0^a = kV_0$ (power of k)$J_1(ka)$

But this is not the answer Jackson gives! Ay yes, we don't know that $\varphi = 0$ outside the disk, that is the problem. That is some other problem. Just as he says. [That problem would be a capacitor consisting of an inner metal disk at V_0 and the other conductor being an iris at V = 0.]

Comment: In the spherical case, we get r^n along the disk. This is not the Sturm-Liouville variable for sphericals, so there is no chance of having an expansion because r^n don't form an orthogonal set. In cylindricals, we have $\varphi_k(\rho) = J_0(k\rho)$ for the function along the disk, and this *is* a SL variable for this coordinate system. That is why we are allowed to put the origin right on the disk in this system. We cannot directly invert using Hankel, because we have the mixed BC and the dual equation situation. But

the point is we are much more "orthogonal function expansion" than we are "power series" in cylindricals. Wow, pretty fancy stuff.

Oblates are just a piece of cake for the charged disk. In sphericals, V = 1 on a sphere, you set equal to your atomic expansion and get $(r/a)^{-1}$, all done. For oblates, V = constant on spheroid, you set equal to your oblate atomic expansion and get $Q_0(i\zeta)/Q_o(i\zeta_0)$, all done. Then set $\zeta_0 = 0$ for disk.

15. Appendix A (added 4.14.10)

In Attempt #2 above, we try out the following Smythian form for doing the metal disk in sphericals:

$$V_{outer}(r,\theta,\varphi) = \Sigma_{n,even}\, A_n\, r^{-n-1} P_n(z) \qquad r \geq a$$
$$V_{inner}(r,\theta,\varphi) = \Sigma_{n,even}\, B_n\, r^n\, P_n(z) \qquad r \leq a$$

I shall now just quote text from above:

> "Only the inner form is allowed to touch the boundary condition disk flat surface, and we get
>
> $$V_{inner}(r,\pi/2,\varphi) = \Sigma_{n,even}\, B_n\, r^n\, P_n(0) = V_0 \qquad r \leq a$$
>
> We know that $P_n(0) \neq 0$ for n even. The only way this can be true for a continuum of r in $r \leq a$ is then to have $B_n = 0$ except for n=0. [Here we have made use of various ideas " Power series and f(x) constant in a finite region.doc" in math/misc: if a convergent power series is constant on all or even part of a region, then it is constant on all of that region. So here we have our series in r being a constant on a finite region say r = [0,a] or any portion of that region. The unique solution is $B_n P_n(0) = \delta_{n0} V_0$.] So again we are dead in the water: our Smythian fit collapses down to $V_{inner}(r,\theta,\varphi) = \Sigma_n B_n r^n P_n(z) = V_0$ at all points in the inner sphere. "

I just reread that above series doc and here is the main point: we end up with $B_n P_n(0) = \delta_{n0} V_0$ because the functions r^n are NOT an orthogonal set for r in (0,a).

Let's now compare all this to Jackson page 91 (3.173), disk discussion, which reads:

$$V(z=0, \rho) = \int_0^\infty dk\, f(k)\, J_0(k\rho) = V_0 \qquad \text{for } \rho \text{ in } (0,a)$$

We can think of the integral as a series of terms summed on k, and if the functions $J_0(k\rho)$ were NOT an orthogonal set, we would have to conclude that $f(k) = V_0\, \delta(k)/J_0(0)$ which is like saying only the n=0 term exists in the above sums we were talking about. In the Jackson problem, the interval for ρ is in fact $(0,\infty)$ and the integral above, as a function of ρ, is just an example where we have a kink at ρ = a. The integral is constant to the left of the kink, and has some non-constant behavior to the right. The potential is piecewise continuous. So on the interval $(0,\infty)$ we can evaluate the coefficients f(k) and therefore construct an explicit solution for the potential $V(z=0, \rho)$, because the $J_0(k\rho)$ are in fact orthogonal!

The orthogonality of the $J_0(k\rho)$ is embodied in the Hankel Transform which is valid not just for $\nu = 0$, but for any constant ν with $Re(\nu) > -1/2$ (see below) . I quote this transform from my Jackson cylindrical notes with $x \rightarrow \rho$ and $\mu \rightarrow k$:

$$F_\nu(k) = \int_0^\infty d\rho\ \rho\ J_\nu(k\rho)\ f(\rho) \qquad \text{projection}$$

$$f(\rho) = \int_0^\infty dk\ k\ J_\nu(k\rho)\ F_\nu(k) \qquad \text{expansion}$$

$$\delta(\rho-\rho')/\rho = \int_0^\infty dk\ k\ J_\nu(k\rho')\ J_\nu(k\rho) \qquad \text{completeness}$$

$$\delta(k-k')/k = \int_0^\infty d\rho\ \rho\ J_\nu(k\rho)\ J_\nu(k'\rho) \qquad \textbf{orthogonality}$$

Here is how this transform appears in Bateman Volume 5:

8.1. Общие формулы

	$f(x)$	$\int_0^\infty f(x)\ J_\nu(xy)\ (xy)^{1/2}\ dx = g(y;\ \nu),\quad y > 0$
(1)	$\int_0^\infty g(y)\ J_\nu(xy)\ (xy)^{1/2}\ dy,\quad \mathrm{Re}\ \nu > -1/2$	$g(y)$

See notes "Bateman ET II notes.doc" for more detail. The above says that if you insert the expansion shown on bottom left for f(x) into the projection shown top right, you get the expansion coefficient g(y). The main point of the above quote is to see the restriction which is $Re(\nu) > -1/2$, which is reminiscent of Legendre P functions which are symmetric about $\nu = -1/2$.

So the point is this: in the Jackson case, $F(\rho) = \int_0^\infty dk\ f(k)\ J_0(k\rho)$ with $F(\rho)$ having a *kink* at $\rho = a$, we are able to compute the f(k) because the $\varphi_k(\rho) = J_0(k\rho)$ form an orthogonal set. Jackson put a cylindrical coordinate system with origin at metal disk center, and we see that this does NOT cause the same problem we got above when we put a *spherical* coordinate system origin at the same location!

In spherical coordinates, the "radial" variable is r and our functions are r^n and it is the OTHER two dimensions θ and φ where we have orthogonal SL eigenfunctions. In cylindrical coordinates for our disk problem, we can think of the z direction as the "radial" coordinate, and we have orthogonal action in the OTHER two dimensions ρ and φ, and in ρ that orthogonality is expressed by the Hankel Transform.