Green's Function for a ring in 3D PhL 3.9.10

It was here that I realized I don't know the ring solution, I thought I did know it! I then flail a bit trying a few ideas to solve the ring Green's problem, but don't get very far.

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Overview. [1 page, written 4.18.10]

In **Section 1**, I basically say that the ring and sphere Green's problems are different problems, so the sphere solution simply does not apply to the ring case, and I give examples of places it would give an absurd result. It is just a bit tricky, nevertheless.

In **Section 2** I do what I think is a pretty good 3-region **Smythian form** for the ring problem using spherical atoms. I then consider the various matching conditions and try to evaluate the coefficients. I actually am able to compute one set of coefficients C_{nm}, but I need to compute the set D_{nm} from the equation $0 = \Sigma_{n=m}^{\infty} [D_{nm}\, a^{-n-1}]\ P_n^m(0)$ and I just don't know how to do it, this is a discrete integral equation. I am not sure this equation fully determines the D_{nm} . I allow the Green's charge to be at any location inside the sphere containing the ring.

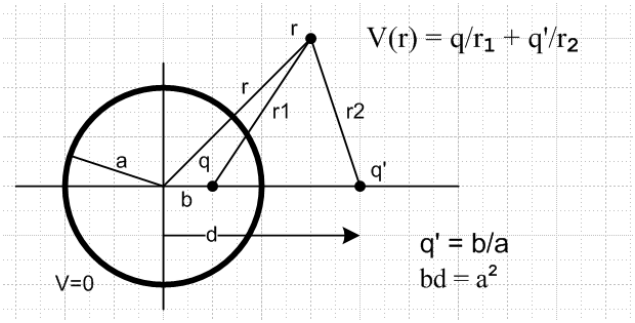
In **Section 3** I write the 1D **integral equation** for the ring induced charge $\sigma(\theta)$, and I proceed to expand things in **Legendre** polynomials since $1/\sqrt{1+\alpha^2-2\alpha\cos(\theta)} = \Sigma_{n=0}^{\infty} P_n(\cos\theta)\, \alpha^n$ seems a simple sum. I sort of wing it a bit and am able to end up with $\Sigma_{n=0}^{\infty} \sigma_n\, (n+m)!/(n-m)!\, B_n^{-m} = 0$ where $\sigma(\theta) = \Sigma_{n=0}^{\infty} \sigma_{an}\, P_n(\cos\theta)$, and thus I end up with another of those "digital integral equation" things I don't know how to solve. In my integral equation, the Green's charge is assumed to lie in the plane of the ring.

In **Section 4** I show that the ring integral equation is an **SO(2) convolution** equation, and I am able to diagonalize it and get σ_k where these are the projections of $\sigma(\theta)$ onto $e^{ik\theta}$. I find however that the kernel of the integration is singular and that a required projection integral $\int_{-\pi}^{\pi} d\theta\, \cos(n\theta)\, /|\sin(\theta/2)|$ diverges, and I connect this fact with the fact that the "surface" in this problem has two fewer dimensions that the space of the problem. I learn from **a pdf** that in fact there *is no solution* for the limit of an infinitely thin wire. The pdf does the exact problem in toroidal coordinates for a wire of finite cross section.

In **Section 5** I thought this problem appeared in Smythe, but that was not the case. I think for multiple concentric rings, the method of Section 4 could be brought to bear. [???]

1. Why the 3D sphere idea is not the solution to the ring problem

I thought I knew how to do this problem. The further you go, the more you realize that you don't know. So here was my supposed solution:



The solution implied by this picture is wrong, but it is not instantly obvious why. We have a Green's charge say inside the ring, and we imagine an image charge at the point q'. We know that the sum of the potentials of these two charges creates V = 0 on the math sphere on which our ring lies. So why isn't that the solution of this problem?

Well, this problem has a different 3D boundary condition, that's why. The BC here is the ring and the great sphere, and that differs from sphere + great sphere. The metal has a different shape.

For the sphere case, the q' charge "simulates" the effect of the surface charge on the inner surface of the sphere. This simulation is only valid inside the sphere. Outside the sphere we have V = 0 everywhere.

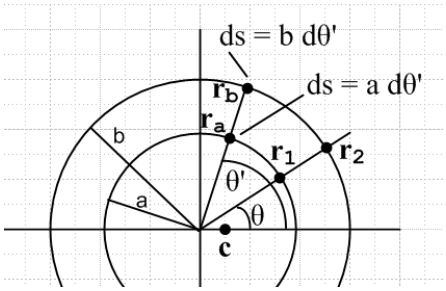
What happens if we "try" the sphere solution potential for the ring problem? Suppose we try it only inside the math sphere and forget the outside. What we are "trying" as our solution is the sum of the charges of the two point charges shown. If we go to a region near the math sphere surface, but on the inside, we would mysteriously find that V = 0 on the surface as we approach from the inside.

Now in our trial solution, we have V = 0 on the math sphere. A theorem then tells us that if there are no other objects outside the sphere, then we have V = 0 in all space outside the sphere. I guess the point is that you really have to consider the global picture of all space. Obviously V ≠ 0 everywhere outside, so this trial solution cannot be right.

Another small example of why it is wrong. It says that the out-of-paper electric field at the ring is zero. But we know that will not be the case, there will be surface charge on the ring, and there will be field lines coming off in all directions.

So OK, then what IS the solution to the ring problem? If we look at our two-ring document and set the outer ring's charge to zero, we end up with this integral equation:

$$0 = (q/c)/\sqrt{1+\alpha^2-2\alpha\cos(\theta)} \; + \int d\theta' \left[\; \sigma_a(\theta')/\sqrt{2[1-\cos(\theta-\theta')]} \right. \qquad \alpha = a/c$$



I show in the two-ring doc that if you expand on $P_n(\cos\theta)$, the integral equation diagonalizes in n (it did not in fact even do that!) , but then contains an infinite m-sum, and you have σ_{anm} as your ring charge coefficients and we don't have a solution, just a way to rewrite the integral equation.

What other approaches are there to solving this problem?

2. Attempting the Ring Green's function using the Smythian Form method in sphericals.

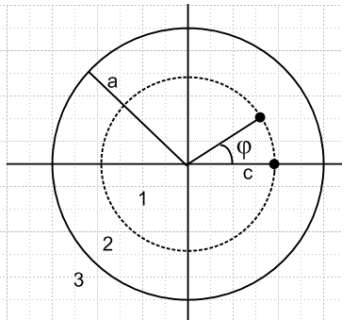
One method would be to make a Smythian form *in spherical coordinates*, with origin at the center of the ring. Put the z axis perp to the ring. Then as have z = ±1 in the problem, so we are forced to integer n. Let's make a different picture for this problem. We put our usual dotted line through the Green's charge, and we might as well assume the *Green's charge is at some arbitrary* location inside the math sphere of the ring. Then our Smythian form might be this

$V_1 = \Sigma_{nm} A_{nm} r^n P_n^m(\cos\theta) \cos(m\varphi)$ inner dotted sphere

$V_2 = \Sigma_{nm} [B_{nm} r^n + C_{nm} r^{-n-1}] P_n^m(\cos\theta) \cos(m\varphi)$ between dotted sphere and ring's sphere

$V_3 = \Sigma_{nm} [D_{nm} r^{-n-1}] P_n^m(\cos\theta) \cos(m\varphi)$ outside the ring sphere

We don't allow $Q_n^m(\cos\theta)$ because need to be finite at z = ±1. So is this too general? If so, I guess we will find out. The r dependence of the first and third lines is pretty solid I think. So let's start writing some conditions.



Condition 1: Potential vanishes on the ring.

V_2 (r=a-, θ=π/2,φ) = 0

V_3 (r=a+, θ=π/2,φ) = 0

or

$0 = \Sigma_{nm} [B_{nm} a^n + C_{nm} a^{-n-1}] P_n^m(0) \cos(m\varphi)$

$$0 = \Sigma_{nm} [D_{nm}\, a^{-n-1}]\ P_n^m(0) \cos(m\varphi)$$

Since cos(mφ) is a complete set for our problem, we must then have

$$0 = \Sigma_n [\, B_{nm}\, a^n + C_{nm}\, a^{-n-1}]\ P_n^m(0)$$
$$0 = \Sigma_n [D_{nm}\, a^{-n-1}]\ P_n^m(0)$$

where [for n,m the usual values , from leg prop doc]

$P_n^m(x=0) = (-1)^{(n+m)/2}\ (n+m-1)!!/\ (n-m)!!$ // n+m = even
$P_n^m(x=0) = 0$ // n+m = odd

The D_{nm} condition looks a bit strange, but I suppose it is conceivable that you could find some D_{nm} values that make that sum be zero. More on this below.

Condition 2. Another condition is this: (potential is continuous on a sphere containing the metal ring)

$$V_2\ (r=a-, \theta,\varphi) = V_3\ (r=a+, \theta,\varphi)$$

which says

$$\Sigma_{nm} [\, B_{nm}\, a^n + C_{nm}\, a^{-n-1}]\ P_n^m(\cos\theta) \cos(m\varphi) = \Sigma_{nm} [D_{nm}\, a^{-n-1}]\ P_n^m(\cos\theta) \cos(m\varphi)$$

If we invert the harmonics, we would get this condition:

$$B_{nm}\, a^n + C_{nm}\, a^{-n-1} = D_{nm}\, a^{-n-1}$$

or

$$B_{nm}\, a^n = (D_{nm} - C_{nm})\ a^{-n-1}$$

Condition 2A. Another condition is this: (potential is continuous on the dotted math sphere, at least away from the Green's point charge at which single point the potential is infinite)

$$V_2\ (r=a-, \theta,\varphi) = V_1\ (r=a+, \theta,\varphi)$$

which says

$$\Sigma_{nm} [\, B_{nm}\, a^n + C_{nm}\, a^{-n-1}]\ P_n^m(\cos\theta) \cos(m\varphi) = \Sigma_{nm} [A_{nm}\, a^n]\ P_n^m(\cos\theta) \cos(m\varphi)$$

If we invert the harmonics, we would get this condition:

$$B_{nm}\, a^n + C_{nm}\, a^{-n-1} = A_{nm}\, a^n$$

or

$$C_{nm}\, a^{-n-1} = (A_{nm} - B_{nm})\ a^n$$

Condition 3. The pillbox business for certain potential derivative. We have

$V_1 = \Sigma_{nm} A_{nm} r^n P_n^m(\cos\theta) \cos(m\varphi)$ inner dotted sphere

$V_2 = \Sigma_{nm} [B_{nm} r^n + C_{nm} r^{-n-1}] P_n^m(\cos\theta) \cos(m\varphi)$ between dotted sphere and ring's sphere

$\partial_r V_1 = \Sigma_{nm} A_{nm} n r^{n-1} P_n^m(\cos\theta) \cos(m\varphi)$

$\partial_r V_2 = \Sigma_{nm} [B_{nm} n r^{n-1} + (-n-1) C_{nm} r^{-n-2}] P_n^m(\cos\theta) \cos(m\varphi)$

$\partial_r V_1 - \partial_r V_2 = \Sigma_{nm} [n(A_{nm} - B_{nm}) r^{n-1} - (-n-1) C_{nm} r^{-n-2}] P_n^m(\cos\theta) \cos(m\varphi)$

For the oblate spheroid we had

$\partial_\zeta V_i - \partial_\zeta V_o = (q/\varepsilon) (h_2/h_3h_1) \delta(\varphi-\varphi_0) \delta(\xi - \xi_0)$

and here we have instead (using $V = q/(\varepsilon r)$ as point charge potential here)

$\partial_r V_i - \partial_r V_o = (q/\varepsilon) (h_r/h_\varphi h_z) \delta(\varphi-\varphi_0) \delta(z - z_0) = (q/\varepsilon) \delta(\varphi-\varphi_0) \delta(z - z_0)/c^2$

So we have our familiar pillbox condition

$(q/\varepsilon) \delta(\varphi-\varphi_0) \delta(z - z_0)/c^2 = \Sigma_{nm} [n(A_{nm} - B_{nm}) c^{n-1} - (-n-1) C_{nm} c^{-n-2}] P_n^m(z) \cos(m\varphi)$

and the Green's charge is at $\varphi_0 = 0$ and some z_0. We first use

$$\int_0^{2\pi} d\varphi \cos(m'\varphi) \cos(m\varphi) = \delta_{m,m'} 2\pi/(2-\delta_{m,0})$$

and get

$(q/\varepsilon) \cos(m\varphi_0) \delta(z - z_0)/c^2 = \Sigma_n 2\pi/(2-\delta_{m,0}) [n(A_{nm} - B_{nm}) c^{n-1} + (n+1) C_{nm} c^{-n-2}] P_n^m(z)$

Then we use

$$\int_{-1}^{1} dz P_n^m(z) P_k^m(z) = \delta_{n,k} (n+1/2)^{-1} f(n,m) \qquad n,k = m, m+1, m+2 \ldots\ldots \infty$$

and get

$(q/\varepsilon) \cos(m\varphi_0) P_n^m(z_0)/c^2 = 2\pi/(2-\delta_{m,0}) (n+1/2)^{-1} f(n,m) [n(A_{nm} - B_{nm}) c^{n-1} + (n+1) C_{nm} c^{-n-2}]$

and this gives us one condition on our three coefficients!

So here is what we have so far:

• potential vanishes on the metal ring:

$$0 = \Sigma_n [B_{nm} a^n + C_{nm} a^{-n-1}] \; P_n^m(0) \qquad 1$$
$$0 = \Sigma_n [D_{nm} a^{-n-1}] \; P_n^m(0) \qquad 2$$

• potential continuity between regions 2 and 3

$$B_{nm} a^n + C_{nm} a^{-n-1} = D_{nm} a^{-n-1} \qquad 3$$

• potential continuity between 2 and 1

$$C_{nm} a^{-n-1} = (A_{nm} - B_{nm}) \; a^n \qquad 4$$

• pillbox condition 5

$$(q/\varepsilon) \cos(m\varphi_0) P_n^m(z_0)/c^2 = 2\pi/(2-\delta_{m,0}) (n+1/2)^{-1} f(n,m) [n(A_{nm}-B_{nm})c^{n-1} + (n+1)C_{nm} c^{-n-2}]$$

I have been staring at this for an hour or two, I don't know where to go from here. I especially don't know how to use those first two conditions.

What can we do with these conditions? We seem to have four sets of coefficients A,B,C,D and five conditions. However, given condition 3, 1 and 2 are the same, so let's then throw out condition 1. We are left with this set of conditions:

$$0 = \Sigma_n [D_{nm} a^{-n-1}] \; P_n^m(0) \qquad 2$$
$$B_{nm} a^n + C_{nm} a^{-n-1} = D_{nm} a^{-n-1} \qquad 3$$
$$C_{nm} a^{-n-1} = (A_{nm} - B_{nm}) \; a^n \qquad 4$$
$$(q/\varepsilon) \cos(m\varphi_0) P_n^m(z_0)/c^2 = 2\pi/(2-\delta_{m,0}) (n+1/2)^{-1} f(n,m) [n(A_{nm}-B_{nm})c^{n-1} + (n+1)C_{nm} c^{-n-2}] \qquad 5$$

We can replace $(A_{nm} - B_{nm}) = C_{nm} a^{-2n-1}$ in condition 5 using condition 4, so condition 5 becomes

$$(q/\varepsilon) \cos(m\varphi_0) P_n^m(z_0)/c^2 = 2\pi/(2-\delta_{m,0}) (n+1/2)^{-1} f(n,m) [n C_{nm} a^{-2n-1}c^{n-1} + (n+1)C_{nm} c^{-n-2}]$$

$$= 2\pi/(2-\delta_{m,0}) (n+1/2)^{-1} f(n,m) C_{nm} [n a^{-2n-1}c^{n-1} + (n+1)c^{-n-2}]$$

This seems to give us a solution for the C_{nm} coefficients:

$$C_{nm} = (q/2\pi\varepsilon c^2) \cos(m\varphi_0) P_n^m(z_0) (2-\delta_{m,0}) (n+1/2) f(n,-m) [n a^{-2n-1}c^{n-1} + (n+1)c^{-n-2}]^{-1}$$

At least we have something! But there is nothing further I can so until I can solve condition 2 for the coefficients D_{nm} . Let's consider other possible conditions:

Condition 4. What happens if we "go far away" from the Green's charge plus ring? The ring will have some induced charge on it which will be less than the Green's, and we have $V = q_{net}/\varepsilon r$. But I don't think this puts any useful condition on our V_3 form, so we get nothing here. Of course if we had the answer, we could compute q_{net} in this way.

Condition 5. Can we apply some kind of pillbox thing around the metal ring? Let's first say this:

$V_2 = \Sigma_{nm} [B_{nm} r^n + C_{nm} r^{-n-1}] P_n^m(\cos\theta) \cos(m\varphi)$ between dotted sphere and ring's sphere

$V_3 = \Sigma_{nm} [D_{nm} r^{-n-1}] P_n^m(\cos\theta) \cos(m\varphi)$ outside the ring sphere

$\partial_r V_2 = \Sigma_{nm} [nB_{nm} a^{n-1} + (-n-1)C_{nm} a^{-n-2}] P_n^m(\cos\theta) \cos(m\varphi)$

$\partial_r V_3 = \Sigma_{nm} [(-n-1)D_{nm} a^{-n-2}] P_n^m(\cos\theta) \cos(m\varphi)$

Then

$$\partial_r V_2 - \partial_r V_3 = \Sigma_{nm} \{ nB_{nm} a^{n-1} + (-n-1)C_{nm} a^{-n-2} - (-n-1)D_{nm} a^{-n-2} \} P_n^m(\cos\theta) \cos(m\varphi)$$

Then we set this somehow equal to :

$(\sigma(\varphi)/\varepsilon) \delta(z-0)/a$

where I add a factor to get a dim match. So something like this:

$$(\sigma(\varphi)/\varepsilon) \delta(z-0)/a = \Sigma_{nm} \{ nB_{nm} a^{n-1} + (-n-1)C_{nm} a^{-n-2} - (-n-1)D_{nm} a^{-n-2} \} P_n^m(\cos\theta) \cos(m\varphi)$$

If we apply

$$\int_{-1}^{1} dz\, P_n^m(z) P_k^m(z) = \delta_{n,k} (n+1/2)^{-1} f(n,m) \qquad n,k = m, m+1, m+2 \ldots\ldots \infty$$

we get (using from above: $B_{nm} a^n = (D_{nm} - C_{nm}) a^{-n-1}$)

$$(\sigma(\varphi)/\varepsilon) P_n^m(0)/a = \Sigma_m (n+1/2)^{-1} f(n,m) \{ nB_{nm} a^{n-1} + (-n-1)C_{nm} a^{-n-2} - (-n-1)D_{nm} a^{-n-2} \} P_n^m \cos(m\varphi)$$

$$= \Sigma_m (n+1/2)^{-1} f(n,m) \{ nB_{nm} a^{n-1} - (n+1)C_{nm} a^{-n-2} + (n+1)D_{nm} a^{-n-2} \} P_n^m \cos(m\varphi)$$

$$= \Sigma_m (n+1/2)^{-1} f(n,m) \{ nB_{nm} a^{n-1} + (n+1)[D_{nm} - C_{nm}] a^{-n-2} \} P_n^m \cos(m\varphi)$$

$$= \Sigma_m (n+1/2)^{-1} f(n,m) \{ nB_{nm} a^{n-1} + (n+1) B_{nm} a^{n-1} \} P_n^m \cos(m\varphi)$$

$$= \Sigma_m (n+1/2)^{-1} f(n,m) (2n+1) B_{nm} a^{n-1} P_n^m \cos(m\varphi)$$

$$= 2 \Sigma_m f(n,m) B_{nm} a^{n-1} P_n^m \cos(m\varphi)$$

and I interpret this as saying that if we knew the solution coefficients, we would know $\sigma(\varphi)$.

Comments: Perhaps the most important condition we have is that V = 0 on the metal ring, but when I do the Smythian form method, I get a condition that just doesn't seem useful:

$$0 = \Sigma_{n=m}^{\infty} [D_{nm}\, a^{-n-1}]\; P_n^m(0)$$

I don't know what to do with this thing! Basically, this is an "integral equation" that has to somehow be solved. The "integration variable" is n which happens to be discrete. I suppose I could regard the sum as being from n = m to n = ∞. So the Smythe method has simply taken my integral equation and displayed it in another form, and as usual, we are no closer to a solution. But nice try.

Suppose someone tells you some constants a_{mn} and asks you to solves this for b_{nm}:

$$\Sigma_{n=0}^{\infty} a_{mn} b_{nm} = 0 \qquad\qquad \mathbf{a}_m \bullet \mathbf{b}_m^{T} = 0$$

Think of a_{mn} as being a vector $\mathbf{a}_m$ with components $(\mathbf{a}_m)_n = a_{mn}$. Then the problem is to find some other vector in this infinite dimensional space which is perpendicular to $\mathbf{a}_m$. It seems this is just a condition on the b_{nm} solution and does not determine the solution, ie, the vector b would not be unique. Parameter m is a bystander parameter here.

Look more now at our condition:

$$0 = \Sigma_{n=m}^{\infty} [D_{nm}\, a^{-n-1}]\; P_n^m(0)$$

$$P_n^m(x=0) = (-1)^{(n+m)/2}\, (n+m-1)!!/\,(n-m)!! \qquad \text{// n+m = even}$$

$$P_n^m(x=0) = 0 \qquad \text{// n+m = odd}$$

Let's define i = n-m so we can then write [n = m+i, n+m = 2m+i, n-m = i]

$$0 = \Sigma_{i=0}^{\infty} [D_{m+i,m}\, a^{-(m+i)-1}]\; P_{m+i}^m(0)$$

$$P_{m+i}^m(0) = (-1)^{(2m+i)/2}\, (2m+i-1)!!/\,(i)!! \qquad \text{// 2m+i = even, ie, i = even}$$

$$P_{m+i}^m(0) = 0 \qquad \text{// 2m+i = odd, ie, i = odd}$$

=>

$$0 = \Sigma_{i=even} [D_{m+i,m}\, a^{-m-1-i}]\; (-1)^{(m+i/2)}\, (2m+i-1)!!/\,(i)!!$$

Is there some kind of iterative solution where we perhaps think of 1/a as a smallness parameter? We might then write

$$0 = D_{m,m}\, a^{-m-1}\, (-1)^m\, (2m-1)!! + D_{m+2,m}\, a^{-m-3}]\; (-1)^{(m+1)}\, (2m+1)!!/\,2 + \ldots.$$

$$= a^{-m-1}\, (-1)^m \{ D_{m,m}(2m-1)!! - D_{m+2,m}\, a^{-2}\, (2m+1)!!/\,2 + D_{m+4,m}\, a^{-4}\, (2m+3)!!/\,8 - \ldots \}$$

I can imagine there is a solution here, but I don't know how to find it. As noted above, this is really an integral equation type of thing and the solution is not unique I suspect.

Somehow I don't like this general approach. I am thrashing blindly at this problem, and think soon I will be able to launch a more logical attack.

3. Attempt the ring Green's function via integral equation method with Leg polys.

Starting at the first-kind Fredholm integral equation (says potential is 0 on the ring) [This equation is derived in "Green's Function for two concentric rings in 3D.doc", a doc I started before starting this one, because I realized that I didn't know the 1-ring solution, so why continue on the 2-ring solution.]

$$0 = (q/c)/\sqrt{1+\alpha^2-2\alpha\cos(\theta)} + \int d\theta' \left[\sigma_a(\theta')/\sqrt{2[1-\cos(\theta-\theta')]} \right. \qquad \alpha = a/c$$

or

$$\int d\theta' \, \sigma_a(\theta') / |\sin[(\theta-\theta')/2]| = - (2q/c)/\sqrt{1+\alpha^2-2\alpha\cos(\theta)} \qquad \alpha = a/c$$

My comment is that "we", the general public, just aren't too familiar with "standard integral equations" the way we are with "standard differential equations". I have no clue as to how to solve this equation, nada! I might be able to "look it up" if I had a table of "transforms" for function csc(θ-θ'), but even Bateman volumes 4 and 5 don't have such transforms. Can we convert this to an ODE? After all, it is of the standard Fredholm type with kernel k(θ,θ') = csc(θ-θ'). I might explore this avenue soon. [Polyanin!]

We want to "expand" $\sigma_a(\theta')$ onto some basis functions, but none seem immediately useful. We know we can do this on the right:

$$1/\sqrt{1+\alpha^2-2\alpha\cos(\theta)} = \Sigma_{n=0}^{\infty} P_n(\cos\theta)\, \alpha^n$$

We are tempted to say

$$\sigma_a(\theta') = \Sigma_{n=0}^{\infty} \sigma_{an} P_n(\cos\theta')$$

Then we have

$$\Sigma_{n=0}^{\infty} \sigma_{an} \int d\theta' \, P_n(\cos\theta') / |\sin(\theta-\theta')/2| = - (2q/c) \Sigma_{n=0}^{\infty} P_n(\cos\theta)\, \alpha^n$$

$$\Sigma_{n=0}^{\infty} \sigma_{an} \int d\theta' \, P_n(\cos[\theta'+\theta]) / |\sin(\theta'/2)| = - (2q/c) \Sigma_{n=0}^{\infty} P_n(\cos\theta)\, \alpha^n \qquad (*)$$

Then conveniently we can expand $P_n(\cos[\theta'+\theta])$,

$$P_n(\cos\theta\cos\theta' - \sin\theta\sin\theta') = P_n(\cos\theta)\, P_n(\cos\theta') + 2 \Sigma_{m=1}^{\infty} P_n^{-m}(\cos\theta)\, P_n^m(\cos\theta') \quad \text{// more or less}$$

$$\text{LHS}(*) = \Sigma_{n=0}^{\infty} \sigma_{an} \int d\theta' \{ P_n(\cos\theta) P_n(\cos\theta') + 2 \Sigma_{m=1}^{\infty} P_n^{-m}(\cos\theta) P_n^{m}(\cos\theta') \} / |\sin(\theta'/2)|$$

$$= \Sigma_{n=0}^{\infty} \sigma_{an} \{ P_n(\cos\theta) \int d\theta' P_n(\cos\theta') / |\sin(\theta'/2)|$$

$$+ 2 \Sigma_{m=1}^{\infty} P_n^{-m}(\cos\theta) \int d\theta' P_n^{m}(\cos\theta') / |\sin(\theta'/2)| \}$$

I think the integrals shown converge except perhaps for n = 0. Assuming we could do the integrals, we get a form like this

$$\text{LHS}(*) = \Sigma_{n=0}^{\infty} \sigma_{an} \{ P_n(\cos\theta) A_n + 2 \Sigma_{m=1}^{\infty} P_n^{-m}(\cos\theta) B_n^{m} \}$$

$$\text{RHS}(*) = - (2q/c) \Sigma_{n=0}^{\infty} P_n(\cos\theta) \alpha^n \qquad \alpha = a/c$$

Putting all on the LHS we then get

$$\Sigma_{n=0}^{\infty} P_n(\cos\theta) [\sigma_{an} A_n + (2q/c) \alpha^n] + 2 \Sigma_{n=0}^{\infty} \sigma_{an} \Sigma_{m=1}^{\infty} P_n^{-m}(\cos\theta) B_n^{m} = 0 \qquad (**)$$

The problem now is that the set of functions $\{ P_n^m(z) \}$ with n fixed and m varying do not form a complete set (as far as I know), so we cannot use orthogonality to get a simpler equation. This would be a different SL problem than the one I am used to where m is fixed and n labels the eigenfunctions.

However, Bateman p 171 does give is this integral:

$$\int_{-1}^{1} dx P_n^m(x) P_n^k(x) / (1-x^2) = \delta_{mk} m^{-1}(n+m)!/(n-m)!$$

so again if we somehow avoid m = 0 we might get a result. So apply $\int_{-1}^{1} dx P_n^k(x) / (1-x^2)$ to both sides of (**) above and what do we get?

$$\Sigma_{n=0}^{\infty} \delta_{0k} 0^{-1}(n+0)!/(n-0)! [\sigma_{an} A_n + (2q/c) \alpha^n] +$$

$$2 \Sigma_{n=0}^{\infty} \sigma_{an} \Sigma_{m=1}^{\infty} \delta_{-m,k} m^{-1}(n-m)!/(n+m)! B_n^{m} = 0$$

or

$$\Sigma_{n=0}^{\infty} \delta_{0k} 0^{-1} [\sigma_{an} A_n + (2q/c) \alpha^n] + 2 \Sigma_{n=0}^{\infty} \sigma_{an} (n+k)!/(n-k)! B_n^{-k} = 0$$

Here k is probably any reasonable P label, so choose $k \neq 0$ and toss the first term and we get (I change k to m)

$$\Sigma_{n=0}^{\infty} \sigma_{an} (n+m)!/(n-m)! B_n^{-m} = 0$$

Surely this is not right, but if it were, we would have at least gotten rid of the m sum, but we still have the n sum, and this equation has the appearance of a "discrete integral equation". This is very similar to what happened in our Smythian sphericals approach above where we got $\Sigma_{n=m}^{\infty} [D_{nm}\, a^{-n-1}]\ P_n^m(0) = 0$. As before, I don't know what to do next.

4. Diagonalize the ring integral equation using group theory/ Fourier Series [4.18.10]

In "Diagonalization of Convolution Equations.doc" in math/group theory (which I just now wrote), I consider the notion of Fourier Transforms on groups, and I give the special case of the group SO(2) which is just the situation of the Fourier Series. I wrote this doc with tie ins to my thesis notation and with reference also to my Spectral Theory book. This is the first time in 30 years I have written on the general group subject. One result shown there is the diagonalization of a convolution equation on SO(2), which is just saying that a convolution equation is diagonalized by doing a Fourier Series. I quote:

We start with a convolution equation:

$$A(\theta) = \int d\theta_1/2\pi\ B(\theta_1)\ C(\theta-\theta_1) \qquad (*)$$

We read off the diagonalized equation (set $\sigma = n$ as traditional)

$$A^n = B^n\ C^n \qquad (**)$$

The various projections and expansions are:

$A^n = (1/2\pi) \int d\theta\ A(\theta)\ e^{+in\theta}$ // projection

$A(\theta) = \Sigma_n\ A^n\ e^{-in\theta}$ // expansion sum on all integers n

and similarly for B and C. If our unknown function is $B(\theta)$, then we have the following solution to our integral equation:

$$B(\theta) = \Sigma_n\ (A^n/C^n)\ e^{-in\theta}$$

Now, consider our integral equation from above, which was this:

$$-\ (2q/c)/\sqrt{1+\alpha^2-2\alpha\cos(\theta)} = \int d\theta'\ \sigma_a(\theta')\ /\ |\sin[(\theta-\theta')/2]|$$

Let's divide both sides by 2π to get this into our standard convolution form (*) shown above

$$-\ (q/\pi c)/\sqrt{1+\alpha^2-2\alpha\cos(\theta)} = \int d\theta'/2\pi\ \sigma_a(\theta')\ /\ |\sin[(\theta-\theta')/2]|$$

We then make these identifications:

$A(\theta) = -\ (q/\pi c)/\sqrt{1+\alpha^2-2\alpha\cos(\theta)}$

$B(\theta) = \sigma_a(\theta)$

$C(\theta) = 1/|\sin(\theta/2)|$

Therefore, since this integral equation exactly matches our convolution form, we can read off the solution:

$$\sigma_a(\theta) = \Sigma_n\ (A^n/C^n)\ e^{-in\theta}$$

where we know that

$$A^n = (1/2\pi) \int d\theta\ A(\theta)\ e^{+in\theta} = -\ (q/\pi c)\ (1/2\pi) \int d\theta\ e^{+in\theta}\ 1/\sqrt{1+\alpha^2-2\alpha\cos(\theta)}$$

$$= -\ (q/2\pi^2 c) \int d\theta\ \cos(n\theta)\ 1/\sqrt{1+\alpha^2-2\alpha\cos(\theta)} \qquad \text{// } \sin(n\theta) \text{ term} = 0$$

$$C^n = (1/2\pi) \int d\theta\ C(\theta)\ e^{+in\theta} = (1/2\pi) \int d\theta\ e^{+in\theta}/|\sin(\theta/2)|$$

The integral appearing in A^n above was studied in " cos(nx) over sqrt(a-bcosx).doc" math/integrals where we got this rather ugly result:

$$I = \int_0^{2\pi} dx\ \cos(nx)/\sqrt{a - b\cos(x)}$$

$$= (2/\sqrt{a - b}\ )\ \Sigma_{s=0}^{n}\ (-1)^s\ (2n, 2n-2s)\ B[\ s+1/2,\ n-s+1/2\ ]\ F\ [s+1/2,\ 1/2;\ n+1;\ -2b/(a-b)\ ]$$

If we set $a = 1+\alpha^2$ and $b = 2\alpha$ then this says: [$a-b = (\alpha-1)^2$]

$$I = \int_{-\pi}^{\pi} d\theta\ \cos(n\theta)/\sqrt{1+\alpha^2-2\alpha\cos(\theta)} = \int_{-\pi}^{\pi} d\theta\ e^{-in\theta}/\sqrt{1+\alpha^2-2\alpha\cos(\theta)}$$

$$= (2/|\alpha-1|))\ \Sigma_{s=0}^{n}\ (-1)^s\ (2n, 2n-2s)\ B[\ s+1/2,\ n-s+1/2\ ]\ F\ [s+1/2,\ 1/2;\ n+1;-4\alpha/(\alpha-1)^2)\ ]$$

$$\equiv f(\alpha, n) \qquad \text{// notice that } f(\alpha,-n) = f(\alpha,n)$$

Therefore, we have

$$A^n = -\ (q/2\pi^2 c)\ f(\alpha, n) \qquad \alpha = a/c$$

The C^n integral we need relates to this: [$C^n = (1/2\pi)\ J_n$]

$$J_n \equiv \int_{-\pi}^{\pi} d\theta\ e^{in\theta}/|\sin(\theta/2)| = \int_{-\pi}^{\pi} d\theta\ \cos(n\theta)/|\sin(\theta/2)| = 2\int_0^{\pi} d\theta\ \cos(n\theta)/\sin(\theta/2)$$

$= 4 \int_0^{\pi/2} dx \cos(2nx) / \sin(x)$ $\qquad\qquad\qquad\qquad$ $x = \theta/2$

I think this integral diverges! If we look at the lower end we have

$\int_0^{\varepsilon} dx\ 1/x = \ln(x) |^{\varepsilon}_0 = \ln(\varepsilon/0) = \infty$

So at this point our effort here comes to a grinding halt.

Comment: In Stakgold's "Surface Layers" section of Chap 6, we spent a lot of time studying what happens when a point approaches a surface σ. In N dimensional space, the surface was of N-1 dimensions. Stak shows that for a simple layer, nothing strange happens as you approach the surface. The potential in space just outside the surface smoothly approaches the potential at the surface, there are no extra terms that arise from delta function effects (as arise in the dipole surface case and with a Neumann boundary condition on the surface).

In our ring problem, however, our surface is of N-2 dimensions, something not addressed directly in the Stakgold discussion. In the N=3 case, when we let our potential-evaluation point x approach s on a surface, we had a situation with σdA /R with dA a small patch on the surface. If we thought of this patch as a sum of little annular rings of radius ρ, then we would have for such a ring σdA /R = σ 2πρdρ / ρ = 2πσdρ = finite. The contribution of any ring, no matter how small and close to the point of contact s, is finite, so this is why nothing strange happens. However, when the surface is our "wire", our σdA /R becomes σ dρ /ρ where now σ is a finite linear charge density on the wire. In this case, the contribution of a little piece of charge close to the contact point s is NOT finite and blows up, so we have a different situation. This is why we have the divergence noted above.

I am not sure how you would handle this. It may somehow involve the usual principle part 1/x distribution idea. It may even be that the result depends on the cross sectional shape of the wire! I do have a paper which does this problem with a finite toroidal ring wire using toroidal coordinates, and that would be one approach. Another would be to model the wire as a thin flat washer of some finite width and then take a limit on that result (as you would take with the toroid result). Yet another model would be to make the ring be a 3D washer with a rectangular cross section.

I have looked at the 2005 toroid paper and it basically does my Fourier Series diagonalization as above, but with a finite ring cross section. In the thin ring limit, the answer is this:

$$\sigma_\ell(\phi) = -\frac{q}{2\pi\sqrt{a r_s \sin\theta_s}} \sum_{m=0}^{\infty} \epsilon_m \frac{Q_{m-1/2}(\coth\xi_s)}{Q_{m-1/2}(1+2\alpha^2)} \cos m\phi \qquad (-\pi \le \phi \le \pi). \qquad (35)$$

where σ_ℓ means a linear charge density on the wire, and where the Green's point charge q is located at spherical coordinates r_s, θ_s, $\varphi_s = 0$ which are given in toroidal coordinates by

$$\coth\xi_s = \frac{r_s^2 + c^2}{2cr_s \sin\theta_s}, \qquad \cot\eta_s = \frac{r_s^2 - c^2}{2cr_s \cos\theta_s}. \qquad (27)$$

where a is the central line toroidal radius, b is the toroid cross section radius, and

$$c = \sqrt{a^2 - b^2} \qquad \text{and} \qquad \cosh \xi_0 = a/b. \tag{5}$$

and finally where α = b/2a, and they invent a symbol called the "Neumann number"

$$\epsilon_n = 2 - \delta_{0n}.$$

which is familiar to me though not with that name. Interestingly, an integral they have to deal with is this:

$$I(n,\alpha) = \int_{-\pi}^{\pi} \frac{\cos n\phi'}{\sqrt{\sin^2\left[\frac{1}{2}(\phi' - \phi)\right] + \alpha^2}}\, d\phi', \tag{60}$$

which is exactly my problem integral but regulated by α > 0 for a finite thickness wire.

Now what happens in our limit that α → 0? I think then $Q_{m-1/2}(1+2\alpha^2) \to \infty$ at a slow rate,

$$Q_{-1/2}(1 + 2\alpha^2) \xrightarrow[b/a \to 0]{} \ln \frac{2a}{b} \approx \xi_0 \tag{63}$$

so this particular one (which determines the total charge on the ring) → ln(1/α) → ∞ logarithmically, and the ability of the wire to carry charge just goes away! In other words, this problem does not have a "limit" in the sense I was expecting, though it does not surprise me after seeing the singular integrals involved. Oddly the paper authors don't comment directly on this fact.

This subject seems to arise in work with circular rings on a PC board, microstrip, etc. I just found a paper which purports to give the Green's function for a circular disk, and an annular ring, both results are pretty messy and involve Bessel functions. A little off my path right this moment but good to have.

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$= 8 \int_0^{\pi/2} dx \cos(nx) \sin(nx) / \sin(x)$

GR7 p 391 tells us this (this is for $n \geq 1$, but we note in passing that $s_{-n} = - s_n$)

4. $$\int_0^{\pi/2} \frac{\sin 2nx}{\sin x}\, dx = 2\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{k-1}}{2n-1}\right)$$ $= 2 s_n$

I think there is a typo and k-1 should be n-1 since that appears in a nearby expression. For example, if n = 3, the last term is 1/5 and the sign will be $(-1)^{n-1} = (-1)^2 = +1$, while if n = 2, last term is -1/3 etc. I just

sent off an errata email on this to Dan Zwillinger. Let's define s_n to be the parenthetical finite series shown, which seems to have a value between 0 and 1 for any n. If n = 1, series = 1. Then we have

$$C^n = (1/2\pi)\, J_n = (1/2\pi)\, 8\, s_n = (4/\pi)\, s_n$$

To summarize, we have done the two integrals we needed to solve our problem,

$$A^n = -\,(q/2\pi^2 c)\, f(\alpha, n) \qquad \alpha = a/c$$
$$C^n = (4/\pi)\, s_n$$

and our solution is then

$$\sigma_a(\theta) = \Sigma_n\, (A^n/C^n)\, e^{-in\theta} = \Sigma_n\, e^{-in\theta} \quad \{-\,(q/2\pi^2 c)\, f(\alpha, n)\}\ \{\pi/(4 s_n)\}$$

$$= -\,(q/2\pi^2 c)(\pi/4)\, \Sigma_n\, e^{-in\theta}\, f(\alpha,n)/s_n$$

$$= -\,(q/8\pi c)\, \Sigma_n\, e^{-in\theta}\, f(\alpha,n)/s_n$$

At this point, it is convenient to break out the n = 0 term and fold things in the usual way. We then have

$$-\,(q/8\pi c)^{-1}\sigma_a(\theta) = \Sigma_n\, e^{-in\theta}\, f(\alpha,n)/s_n = \Sigma_n \cos(n\theta)\, f(\alpha,n)/s_n \qquad \text{// since } \sigma = \text{real}$$

$$= f(\alpha,0)/s_0 + \Sigma_{n<0} \cos(n\theta)\, f(\alpha,n)/s_n + \Sigma_{n>0} \cos(n\theta)\, f(\alpha,n)/s_n$$

$$= f(\alpha,0) + \Sigma_{n>0} \cos(-n\theta)\, f(\alpha,-n)/s_{-n} + \Sigma_{n>0} \cos(n\theta)\, f(\alpha,n)/s_n$$

$$= f(\alpha,0) - \Sigma_{n>0} \cos(n\theta)\, f(\alpha,n)/s_n + \Sigma_{n>0} \cos(n\theta)\, f(\alpha,n)/s_n$$

$$= f(\alpha,0)$$

Our diagonalized equation was this

$$A^k = B^k\, C^k \qquad \text{with} \qquad A^k = -\,(q/2\pi^2 c)\, f(\alpha, k)$$

and our solution for the σ_k is then this:

$$\sigma^k = B^k = A^k/C^k = -\,(q/2\pi^2 c) f(\alpha, k)\, (\pi/4)\, (1/s_k) = -\,(q/8\pi c s_k)\, f(\alpha, k)$$

Then we recover $\sigma(\theta)$ from the expansion which goes with the above projection. Starting from above,

$$f(g) = \Sigma_\sigma\ d^\sigma\, \Sigma_{k,k'}\ f^{\sigma}{}_{k'k}\, D^{\sigma}{}_{k'k}(g)^*$$

we get

$$\sigma(\theta) = \Sigma_k\ \sigma^k\ e^{+ik\theta} = \Sigma_k\ \sigma^k \cos(k\theta) \qquad \sigma^k = -\,(q/8\pi c s_k)\, f(\alpha, k)$$

But of course f(α,k) is a huge mess, but at least we have our answer as a closed form series!

Recall that σ(θ) is the linear charge density on the ring, so the total ring charge is the integral of this around the ring: (induced on the ring, meaning pulled in from ∞)

$$q_{ind} = \int d\ell\, \sigma(\theta) = \int_0^{2\pi} (a\, d\theta)\, \sigma(\theta) = a \int_0^{2\pi} d\theta\, \sigma(\theta) = 2\pi a\, \sigma^0$$

So we can say that

$$q_{ind} = 2\pi a\sigma^0 = -2\pi a(q/8\pi c s_0)\, f(\alpha, 0) = -q\, \{\, \alpha/(4s_0)\, f(\alpha, 0)\, \} = -q\, \{\, (\alpha/4)\, f(\alpha, 0)\, \}$$

$$f(\alpha,0) = (2/(\alpha-1)) \Sigma_{s=0}^{k} (-1)^s\ (2n, 2k-2s)\ B[\, s+1/2, k-s+1/2\,]\ F\, [s+1/2, 1/2; k+1; -4\alpha/(\alpha-1)^2)\,]$$

$$= (2/(\alpha-1))\ (2n,0)\ B[1/2, 1/2\,]\ F\, [1/2, 1/2; 1; -4\alpha/(\alpha-1)^2)\,]$$

$$= (2/(\alpha-1))\ B[1/2, 1/2\,]\ F\, [1/2, 1/2; 1; -4\alpha/(\alpha-1)^2)\,]$$

$$= (2\pi/(\alpha-1))\ F\, [1/2, 1/2; 1; -4\alpha/(\alpha-1)^2)\,]$$

As noted in my doc on the integral,

$$K_{AS}(m) = (\pi/2)\ F(1/2,1/2;1;m) \qquad m = k^2 = -4\alpha/(\alpha-1)^2$$

so our answer seems to be $K = K_{AS}$

$$f(\alpha,0) = (2\pi/(\alpha-1))\ (2/\pi)\ K(k^2) \qquad k^2 = -4\alpha/(\alpha-1)^2 \qquad \alpha = a/c$$
$$= 4\, K(k^2)/(\alpha-1)$$

Then our induced charge is:

$$q_{ind} = -q\, \{\, (\alpha/4)\, f(\alpha, 0)\, \} = -q\, \{\, (\alpha/4)\, 4\, K(k^2)/(\alpha-1)\, \} = -q\, \{\, K(k^2)\ \alpha/(\alpha-1)\, \}$$

A quick web search did not reveal the correct answer. Just as a special case, if the Green's charge is at ring center, we have c = 0 and α = ∞ and $k^2 = 0$ and K(0) = π/2 (Schaum p 179 34.2) so we get

$$q_{ind} = -q\, \{\, K(k^2)\ \alpha/(\alpha-1)\} = -q\, \{\, \pi/2\ 1\} = -q\, \{\, \pi/2\, \}$$

which seems wrong, I expect {} to be something less than 1. I could redo this problem from scratch with the Green's charge at the center and probably get an answer fairly quickly, but it is not completely obvious how to do that looking at existing equations, so I put that off.

Another limit would be to take c → a so the Green's charge is smack against the ring. You would think in that case the induced charge would be -q. We find in that limit that c = a and α = 1 and k = ∞ so we need a formula on K. Somehow this limit looks wrong as well.

So here is a **statement of our solution** for the charge density on the ring in this ring Green's problem, where ring is radius a, and Green's charge is located c from the center on the +x axis:

$$\sigma(\theta) = (1/\pi) \Sigma_{n=1}^{\infty} \sigma_k \cos(k\theta)$$

where

$$\sigma_k = - (q/4\pi c) f(\alpha, k)/s(k) \qquad \alpha = a/c$$

$$f(\alpha, k) = (2/ (\alpha-1)) \Sigma_{s=0}^{k} (-1)^s \ (2n, 2k-2s) B[s+1/2, k-s+1/2] F [s+1/2, 1/2; k+1;-4\alpha/(\alpha-1)^2)]$$

$$s(k) = 1-1/3+1/5 ..... (-1)^{k-1}/(2k-1)$$

We know that f(α,k) is related to K and E elliptic functions.

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5. The Smythe Book Approach using a Finite Cylindrical Ring

The problem Smythe solves on page 189 is the Green's function for a point charge located inside the body of a 3D ring, and he uses cylindricals. That is to say, the point charge is inside a ring shaped enclosure of metal. But this is NOT the problem we are talking about here.