Green's Function for two concentric rings in 3D PhL 3.9.10

When I started this doc, I thought I *knew* the solution to the one ring problem and I wanted to see then what the two-ring solution might be in hopes of maybe assembling a flat disk from a set rings. I then realized that I didn't even know the one-ring solution, and so then I started my little one-ring doc and more or less abandoned this document.

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Overview (1/2 page, perhaps written 4.21.10)

In **Section 1** I "set up" the two-ring problem in terms of a pair of coupled 1D integral equations each of which says that the potential ON one of the two rings is 0. At section end, I simplify to one ring only so there is just one integral equation. (Only later in the one-ring doc did I learn that the intra-ring integrals in these integral equations in fact diverge.)

In **Section 2** I try doing formal expansions of the various 1/R factors in these integral equations in $P_n(\cos\theta)$ functions, mainly because the expansions are pretty simple. Since these are not representation functions of SO(2), the integral equations fail to formally diagonalize and I end up with a digital integral equation such that the integral over θ is now replaced by a sum over m, and I am nowhere man.

In **Section 3** I think about a "force" approach instead of a potential one, and quickly give up on that.

In **Section 4** I consider expanding all the 1/R factors in cos(nθ) functions, which actually is a more correct thing to do in this problem. But I cringe at the horrible F_n coefficients which arise, and at this time I am unaware that I have a simple O(2) diagonalization problem. So I quickly give up on this avenue. [These are the F_n things that I much later found were $Q_{n-1/2}$ functions.]

Then **Section 2A** is a fruitless rehash of Section 2 above with a few alterations. **Section 2B** is a similar effort.

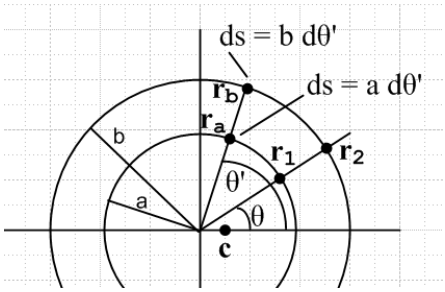
I now know that even the one-ring problem has "no solution" in the sense of infinitesimally thin wires, so it is a big waste of time to think about problems with more than one such wire! But of course I did not know that when I started this doc. I was somehow hoping to "build up" the disk from some wires, and that was my original vague motivation to think about a two-wire problem. I was sort of following Smythe's advice somewhere.

1. The setup. We start by picking an observation point on the inner ring at location $\mathbf{r_1} = (a,\theta)$. Point charge q is inside both rings at location $\mathbf{c} = c\hat{\mathbf{x}}$. Integration points are called $\mathbf{r_a} = (a,\theta')$ and $\mathbf{r_b} = (b,\theta')$. The inner ring has a linear charge density $\sigma_a(\theta')$ and ds = b dθ' as shown in the figure, and similarly for the other ring. The potential at our selected point $\mathbf{r_1}$ must be zero since it lies on a grounded metal ring, so we write

$$0 = V(\mathbf{r_1}) = \frac{q}{|\mathbf{r_1}-\mathbf{c}|} + \int d\theta' \left[\frac{a\sigma_a(\theta')}{|\mathbf{r_a}-\mathbf{r_1}|} + \frac{b\sigma_b(\theta')}{|\mathbf{r_b}-\mathbf{r_1}|} \right] \qquad \text{for all } \mathbf{r_1} \text{ on ring a}$$

Similarly for a point $\mathbf{r_2}$ on the outer ring we have

$$0 = V(\mathbf{r_2}) = \frac{q}{|\mathbf{r_2}-\mathbf{c}|} + \int d\theta' \left[\frac{a\sigma_a(\theta')}{|\mathbf{r_a}-\mathbf{r_2}|} + \frac{b\sigma_b(\theta')}{|\mathbf{r_b}-\mathbf{r_2}|} \right] \qquad \text{for all } \mathbf{r_2} \text{ on ring b}$$



So we seem to have two integral equations (in 1 dimension) with two unknowns. We can write distances in this way

$$|\, p - q|^2 = p^2 + q^2 - 2pq\cos(p \bullet q)$$

so then we have

$$|\, \mathbf{r_1}-\mathbf{c}\,|^2 = a^2 + c^2 - 2ac\cos(\theta)$$

$$|\, \mathbf{r_2}-\mathbf{c}\,|^2 = b^2 + c^2 - 2bc\cos(\theta) \qquad \text{// when we observe from here, also } \theta_1 \text{ say}$$

$$|\, \mathbf{r_a}-\mathbf{r_1}\,|^2 = a^2 + a^2 - 2a^2\cos(\theta-\theta') = 2a^2[1-\cos(\theta-\theta')]$$

$$|\, \mathbf{r_a}-\mathbf{r_2}\,|^2 = a^2 + b^2 - 2ab\cos(\theta-\theta')$$

$$|\, \mathbf{r_b}-\mathbf{r_1}\,|^2 = a^2 + b^2 - 2ab\cos(\theta-\theta')$$

$$|\, \mathbf{r_b}-\mathbf{r_2}\,|^2 = b^2 + b^2 - 2b^2\cos(\theta-\theta') = 2b^2[1-\cos(\theta-\theta')] = 4b^2\sin^2(\theta-\theta')$$

So here are our two integral equations:

$$0 = q/\sqrt{a^2+ c^2-2accos(\theta)} \;+ \int d\theta' \left[\; a\sigma_a(\theta')/\sqrt{2a^2[1-cos(\theta-\theta')]} \; + b\sigma_b(\theta')/\sqrt{a^2+b^2-2abcos(\theta-\theta')} \; \right]$$

$$0 = q/\sqrt{b^2+c^2-2bc\, cos(\theta)} \;+ \int d\theta' \left[\; a\sigma_a(\theta')/\sqrt{a^2+b^2-2abcos(\theta-\theta')} \; + b\sigma_b(\theta')/\sqrt{2b^2[1-cos(\theta-\theta')]} \; \right]$$

Let's process these just a bit to get a more dimensionless form:

$$0 = q/\sqrt{a^2+ c^2-2accos(\theta)} \;+ \int d\theta' \left[\; \sigma_a(\theta')/\sqrt{2[1-cos(\theta-\theta')]} \; + b\sigma_b(\theta')/\sqrt{a^2+b^2-2abcos(\theta-\theta')} \; \right]$$

$$0 = q/\sqrt{b^2+c^2-2bc\, cos(\theta)} \;+ \int d\theta' \left[\; a\sigma_a(\theta')/\sqrt{a^2+b^2-2abcos(\theta-\theta')} \; + \sigma_b(\theta')/\sqrt{2[1-cos(\theta-\theta')]} \; \right]$$

Write

$$a^2+ c^2-2accos(\theta) \;= c^2[\, 1+\alpha^2-2\alpha\, cos(\theta)] \qquad \alpha = a/c$$

$$a^2+ b^2-2abccos(\theta) \;= c^2[\alpha^2+\beta^2-2\alpha\beta\, cos(\theta)] \qquad \beta = b/c$$

$$b^2+c^2-2bc\, cos(\theta) \;= c^2[1+\beta^2-2\beta\, cos(\theta)]$$

Then we have these equations:

$$0 = (q/c)/\sqrt{1+\alpha^2-2\alpha\, cos(\theta)} \;+ \int d\theta' \left[\; \sigma_a(\theta')/\sqrt{2[1-cos(\theta-\theta')]} \; + \beta\sigma_b(\theta')/\sqrt{\alpha^2+\beta^2-2\alpha\beta\, cos(\theta-\theta')} \; \right]$$

$$0 = (q/c)/\sqrt{1+\beta^2-2\beta\, cos(\theta)} \;+ \int d\theta' \left[\; \alpha\sigma_a(\theta')/\sqrt{\alpha^2+\beta^2-2\alpha\beta\, cos(\theta-\theta')} \; + \sigma_b(\theta')/\sqrt{2[1-cos(\theta-\theta')]} \; \right]$$

where now all terms have dimensions charge/length in form q/c or σ . Let's simplify again using

$$\alpha^2+\beta^2-2\alpha\beta\, cos(\theta-\theta') \;= \alpha^2[1+(\beta/\alpha)^2- 2(\beta/\alpha)\, cos(\theta-\theta')]$$

to get

$$0 = (q/c)/\sqrt{1+\alpha^2-2\alpha\, cos(\theta)} \;+ \int d\theta' \left[\; \sigma_a(\theta')/\sqrt{2[1-cos(\theta-\theta')]} \; + \sigma_b(\theta')/\sqrt{1+(\alpha/\beta)^2- 2(\alpha/\beta)\, cos(\theta-\theta')} \; \right]$$

$$0 = (q/c)/\sqrt{1+\beta^2-2\beta\, cos(\theta)} \;+ \int d\theta' \left[\sigma_a(\theta')/\sqrt{1+(\beta/\alpha)^2- 2(\beta/\alpha)\, cos(\theta-\theta')} \; + \sigma_b(\theta')/\sqrt{2[1-cos(\theta-\theta')]} \; \right]$$

Then make the change $\theta'' = \theta-\theta'$ to get

$$0 = (q/c)/\sqrt{1+\alpha^2-2\alpha\cos(\theta)} + \int d\theta'' \left[\sigma_a(\theta-\theta'')/\sqrt{2[1-\cos(\theta'')]} + \sigma_b(\theta-\theta'')/\sqrt{1+(\alpha/\beta)^2-2(\alpha/\beta)\cos(\theta'')} \right]$$

$$0 = (q/c)/\sqrt{1+\beta^2-2\beta\cos(\theta)} + \int d\theta'' \left[\sigma_a(\theta-\theta'')/\sqrt{1+(\beta/\alpha)^2-2(\beta/\alpha)\cos(\theta'')} + \sigma_b(\theta-\theta'')/\sqrt{2[1-\cos(\theta'')]} \right]$$

and again changing back to single prime

$$0 = (q/c)/\sqrt{1+\alpha^2-2\alpha\cos(\theta)} + \int d\theta' \left[\sigma_a(\theta-\theta')/\sqrt{2[1-\cos(\theta')]} + \sigma_b(\theta-\theta')/\sqrt{1+(\alpha/\beta)^2-2(\alpha/\beta)\cos(\theta')} \right]$$

$$0 = (q/c)/\sqrt{1+\beta^2-2\beta\cos(\theta)} + \int d\theta' \left[\sigma_a(\theta-\theta')/\sqrt{1+(\beta/\alpha)^2-2(\beta/\alpha)\cos(\theta')} + \sigma_b(\theta-\theta')/\sqrt{2[1-\cos(\theta')]} \right]$$

I am not sure which pair will be best for solution.

If there is only one ring of radius a, we can set $\sigma_b = 0$ in the above and we don't have the second equation which says the potential is 0 on the outer ring. So we have just this integral equation in this case:

$$0 = (q/c)/\sqrt{1+\alpha^2-2\alpha\cos(\theta)} + \int d\theta' \left[\sigma_a(\theta')/\sqrt{2[1-\cos(\theta-\theta')]} \right]$$

or

$$\int d\theta' \, \sigma_a(\theta') / |\sin[(\theta-\theta')/2]| = -(2q/c)/\sqrt{1+\alpha^2-2\alpha\cos(\theta)} \qquad \alpha = a/c$$

2. How do you solve the integral equations?

I don't know! One obvious starting point is to expand $\sigma_a(\theta')$ in some carefully selected manner. I know now that you do not want to expand on cos(nθ) functions since things are a mess. Everything *seems* to point to Legendre polynomials. The famous generating function is shown page 146 Schaum

$$1/\sqrt{1+\beta^2-2\beta\cos(\theta)} = \Sigma_{n=0}^{\infty} P_n(\cos\theta)\,\beta^n$$

If we set $\beta = 1$ this becomes

$$1/\sqrt{2[1-\cos(\theta)]} = \Sigma_{n=0}^{\infty} P_n(\cos\theta)$$

where I have to wonder about convergence. But suppose we formally expand all 1/radicals in our two equations in this manner. Looking at our first pair of coupled equations above,

$$0 = (q/c)/\sqrt{1+\alpha^2-2\alpha\cos(\theta)} + \int d\theta' \left[\sigma_a(\theta')/\sqrt{2[1-\cos(\theta-\theta')]} + \beta\sigma_b(\theta')/\sqrt{\alpha^2+\beta^2-2\alpha\beta\cos(\theta-\theta')} \right]$$

$$0 = (q/c)/\sqrt{1+\beta^2-2\beta\cos(\theta)} + \int d\theta' \left[\alpha\sigma_a(\theta')/\sqrt{\alpha^2+\beta^2-2\alpha\beta\cos(\theta-\theta')} + \sigma_b(\theta')/\sqrt{2[1-\cos(\theta-\theta')]} \right]$$

we see that we will then find ourselves dealing with things like this:

$$\int d\theta' \, \sigma_a(\theta') \; P_n(\cos[\theta-\theta'])$$

We know that the Legendre function addition theorem will come into play here. First

$$\cos(\theta-\theta') = \cos\theta\cos\theta' + \sin\theta\sin\theta'$$

We can then use the form shown Bateman p 168-9 which is this: $\psi = 1$

$$P_n(\cos[\theta-\theta']) = P_n(\cos\theta\cos\theta' + \sin\theta\sin\theta') \;\; = P_n(\cos\theta)\, P_n(\cos\theta') \;\; + 2 \sum_{m=1}^{\infty} (-1)^m P_n^{-m}(\cos\theta)\, P_n^m(\cos\theta')$$

Then our integral above is this:

$$\int d\theta' \, \sigma_a(\theta')\, P_n(\cos[\theta-\theta'])$$

$$= \int d\theta' \, \sigma_a(\theta') \; \{ \, P_n(\cos\theta)\, P_n(\cos\theta') \;\; + 2 \sum_{m=1}^{\infty} (-1)^m P_n^{-m}(\cos\theta)\, P_n^m(\cos\theta') \, \}$$

$$= \; P_n(\cos\theta) \int d\theta' \, \sigma_a(\theta')\, P_n(\cos\theta') \;\; + \;\; 2 \sum_{m=1}^{\infty} (-1)^m P_n^{-m}(\cos\theta) \;\; \int d\theta' \, \sigma_a(\theta')\, P_n^m(\cos\theta')$$

So let's just define the following coefficients:

$$\sigma_{anm} \;\; \equiv \;\; \int d\theta' \, \sigma_a(\theta')\, P_n^m(\cos\theta')$$

so we then have

$$\int d\theta' \, \sigma_a(\theta')\, P_n(\cos[\theta-\theta']) \;\; = \;\; P_n(\cos\theta)\, \sigma_{an0} \;\; + \;\; 2 \sum_{m=1}^{\infty} (-1)^m P_n^{-m}(\cos\theta) \;\; \sigma_{anm} \quad (*)$$

Our integral equations are then going to have this sum over m sitting in them. We can balance the n sums term by term (maybe) and get rid of the n sum. All we then accomplish is replacement of the integral equations with sum over m equations.

Suppose we assume that $\sigma_b = 0$ and we ignore the second equation which said that the potential on ring b was zero. This reduces us to the **"one ring problem"** of radius a. We then have this integral equation to start with.

$$0 = (q/c)/\sqrt{1+\alpha^2-2\alpha\cos(\theta)} \;\; + \int d\theta' \, \sigma_a(\theta')/\sqrt{2[1-\cos(\theta-\theta')]}$$

Let's now actually insert our expansions quoted above, namely

$$1/\sqrt{2[1-\cos(\theta-\theta')]} \;\; = \sum_{n=0}^{\infty} \; P_n(\cos[\theta-\theta'])$$

$$1/\sqrt{1+\alpha^2-2\alpha\cos(\theta)} \;= \Sigma_{n=0}^{\infty}\; P_n(\cos\theta)\,\alpha^n \qquad\qquad \alpha = a/c$$

We then get

$$0 = (q/c)\,\Sigma_{n=0}^{\infty}\; P_n(\cos\theta)\,\alpha^n \;+ \Sigma_{n=0}^{\infty}\;\; \int d\theta'\,\sigma_a(\theta')\,P_n(\cos[\theta-\theta'])$$

We then insert our result (*) above and get:

$$0 = (q/c)\,\Sigma_{n=0}^{\infty}\; P_n(\cos\theta)\,\alpha^n \;+ \Sigma_{n=0}^{\infty}\;\; \{\, P_n(\cos\theta)\,\sigma_{an0} \;+\; 2\,\Sigma_{m=1}^{\infty}\,(-1)^m\,P_n^{-m}(\cos\theta)\;\sigma_{anm}\,\}$$

or

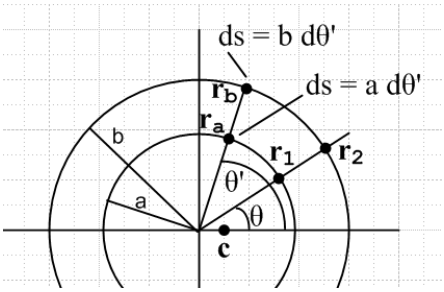
$$0 = \Sigma_{n=0}^{\infty}\,P_n(\cos\theta)\,[(q/c)\,\alpha^n \;+ \sigma_{an0}] \;+\; 2\,\Sigma_{n=0}^{\infty}\,\Sigma_{m=1}^{\infty}\,(-1)^m\,P_n^{-m}(\cos\theta)\;\sigma_{anm}$$

This result certainly is clumsy and suggests that I took a wrong term. The first term is a reasonable expansion on the P_n but the second term is a double expansion on P_n^m so we have not expanded on a complete set of basis functions, but on functions from different complete sets! Very ugly. You have to decide what complete set of functions you are aiming for, but here I just let the math pull me along and it dumped me in a ravine.

Below the **** line I show other attempts to deal with things that don't help much. So I conclude that even the "two-ring" problem is too hard for me to do! Ouch!

3. What about a "force" approach to finding the charge density?

Consider the point $\mathbf{r_1}$ in our picture:



Presumably the tangential force on a piece of charge at $\mathbf{r_1}$ is zero. Since the charge density even on the a ring is not azimuthally symmetric, we will have a force integral that is just as ugly if not uglier than that in the potential approach. We have $1/r^2$ so we get rid of the radicals, maybe that is a plus. But then we have the vector mess to deal with. I cannot imagine this is a simpler approach!

4. What about the horrible cos(nθ) expansion?

Here is one of our integral equations above

$$0 = (q/c)/\sqrt{1+\alpha^2-2\alpha\cos(\theta)} \;+\; \int d\theta' \left[\sigma_a(\theta')/\sqrt{2[1-\cos(\theta-\theta')]} \;+ \sigma_b(\theta')/\sqrt{1+(\alpha/\beta)^2-2(\alpha/\beta)\cos(\theta-\theta')} \right]$$

Suppose we go ahead and use this mess

Coefficients of $1/R = 1/|\mathbf{r}-\mathbf{r}'| = (1/r)\, 1/\sqrt{a - b\cos(\theta)}$ $\quad a=1+f^2 \quad b = 2f \quad f = r'/r$

$$= (1/\pi)\, \Sigma_n\, F_n(f) \cos(n\theta)$$

$$F_n(f) = (2/|1-f|)\, \Sigma_{s=0}^{n}\, (-1)^s\, (2n, 2n-2s)\, B[s+1/2, n-s+1/2]\, F[s+1/2, 1/2; n+1; -4f/(1-f)^2]$$

How would we apply this to one of our radicals?

$$|\mathbf{r_1}-\mathbf{c}|^2 = a^2 + c^2 - 2ac\cos(\theta) = c^2[1+\alpha^2-2\alpha\cos(\theta)] \qquad \alpha = a/c$$

$1/R = 1/|\mathbf{c}-\mathbf{r_1}| = (1/c)\, 1/\sqrt{a - b\cos(\theta)}$ $\quad a = 1+\alpha^2 \quad b = 2\alpha \quad \alpha = a/c \quad f = \alpha$

So using this, we can expand our outside term. But what about one of the integral terms:

$$|\mathbf{r_a}-\mathbf{r_1}|^2 = a^2 + a^2 - 2a^2\cos(\theta-\theta') = a^2[2-2\cos(\theta-\theta')]$$

$1/R = 1/|\mathbf{r_a}-\mathbf{r_1}| = (1/a)\, 1/\sqrt{a - b\cos(\theta)}$ $\quad a = 2 \quad b = 2 \quad f = 1$

In this case, our F formula diverges.

In any event, even if things were nice here, we still have $\Sigma_{s=0}^{n}$ sum left over so things do not diagonalize.

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These two sections are alternate versions of section 2 above, but neither was more successful!

2A. How do you solve the integral equations?

Here I tried a slightly different threading through the same math, where I added a sine factor(red) to make the required end-game integral look better. But no cigar.

I don't know! One obvious starting point is to expand $\sigma_a(\theta')$ in some carefully selected manner. I know now that you do not want to expand on cos(nθ) functions since things are a mess. Everything seems to point to Legendre polynomials. The famous generating function is shown page 146 Schaum

$$1/\sqrt{1+\beta^2-2\beta\cos(\theta)} = \Sigma_{n=0}^{\infty}\, P_n(\cos\theta)\, \beta^n$$

If we set $\beta = 1$ this becomes

$$1/\sqrt{2[1-\cos(\theta)]} = \Sigma_{n=0}^{\infty} P_n(\cos\theta)$$

where I have to wonder about convergence. But suppose we formally expand all 1/radicals in our two equations in this manner. We will then find ourselves dealing with things like this:

$$\int d\theta' \, \sigma_a(\theta') \, P_n(\cos[\theta-\theta'])$$

Suppose we simply expand the charge densities in the (almost) obvious manner:

$$[\sigma_a(\theta')/\sin\theta'] = \Sigma_{n=0}^{\infty} \sigma_{an} P_n(\cos\theta')$$

Then we will be stuck with this

$$\Sigma_{n=0}^{\infty} \sigma_{an} \int d\theta' \sin\theta' \, P_n(\cos\theta') \, P_n(\cos[\theta-\theta'])$$

We know that the Legendre function addition theorem will come into play here. First

$$\cos(\theta-\theta') = \cos\theta\cos\theta' + \sin\theta\sin\theta'$$

We can then use the form shown Bateman p 168-9 which is this: $\psi = 1$

$$P_n(\cos\theta\cos\theta' + \sin\theta\sin\theta') = P_n(\cos\theta) \, P_n(\cos\theta') + 2 \Sigma_{m=1}^{\infty} (-1)^m P_n^{-m}(\cos\theta) \, P_n^m(\cos\theta')$$

Then our integral above is this:

$$\int d\theta' \sin\theta' \, P_n(\cos\theta') \, P_n(\cos[\theta-\theta'])$$

$$= \int d(\cos\theta') \, P_n(\cos\theta') \, \{ P_n(\cos\theta) \, P_n(\cos\theta') + 2 \Sigma_{m=1}^{\infty} (-1)^m P_n^{-m}(\cos\theta) \, P_n^m(\cos\theta') \}$$

$$= \int d(\cos\theta') \, P_n(\cos\theta') \, P_n(\cos\theta')$$

$$+ \int d(\cos\theta') \, P_n(\cos\theta') \, 2 \Sigma_{m=1}^{\infty} (-1)^m P_n^{-m}(\cos\theta) \, P_n^m(\cos\theta')$$

$$= P_n(\cos\theta) \int d(\cos\theta') \, P_n(\cos\theta') \, P_n(\cos\theta')$$

$$+ 2 \Sigma_{m=1}^{\infty} (-1)^m P_n^{-m}(\cos\theta) \int d(\cos\theta') \, P_n(\cos\theta') \, P_n^m(\cos\theta')$$

$$= P_n(z) \int dz' P_n(z') P_n(z') + 2 \Sigma_{m=1}^{\infty} (-1)^m P_n^{-m}(z) \int dz' P_n(z') P_n^m(z')$$

If we just assume these were Bateman on-cut functions, we can use this rule

$$\int_{-1}^{1} dz P_n^m(z) P_k^m(z) = \delta_{n,k} (n+1/2)^{-1} [(n+m)! / (n-m)!] = \delta_{n,k} (n+1/2)^{-1} f(n,m)$$

but this does not tell us how to do the last integral! If I could cause an extra factor $(1-z'^2)^{-1}$ to be in the integrand, then the entire summation term vanishes, but the first term then diverges.

2B. How do you solve the integral equations?

Here I try yet another threading through the same math, but don't get anything very useful. I had in mind the integral p 171 (20) of Bateman.

I don't know! One obvious starting point is to expand $\sigma_a(\theta')$ in some carefully selected manner. I know now that you do not want to expand on cos(nθ) functions since things are a mess. Everything seems to point to Legendre polynomials. The famous generating function is shown page 146 Schaum

$$1/\sqrt{1+\beta^2-2\beta \cos(\theta)} = \Sigma_{n=0}^{\infty} P_n(\cos\theta) \beta^n$$

If we set β = 1 this becomes

$$1/\sqrt{2[1-\cos(\theta)]} = \Sigma_{n=0}^{\infty} P_n(\cos\theta)$$

where I have to wonder about convergence. But suppose we formally expand all 1/radicals in our two equations in this manner. Looking at our first pair of coupled equations above,

$$0 = (q/c)/\sqrt{1+\alpha^2-2\alpha \cos(\theta)} + \int d\theta' [\sigma_a(\theta')/\sqrt{2[1-\cos(\theta-\theta')]} + \beta\sigma_b(\theta')/ \sqrt{\alpha^2+\beta^2-2\alpha\beta \cos(\theta-\theta')}]$$

$$0 = (q/c)/\sqrt{1+\beta^2-2\beta \cos(\theta)} + \int d\theta' [\alpha\sigma_a(\theta')/ \sqrt{\alpha^2+\beta^2-2\alpha\beta \cos(\theta-\theta')} + \sigma_b(\theta')/ \sqrt{2[1-\cos(\theta-\theta')]}]$$

we see that we will then find ourselves dealing with things like this:

$$\int d\theta' \sigma_a(\theta') P_n(\cos[\theta-\theta'])$$

Suppose we simply expand the charge densities in the (almost) obvious manner:

$$\sigma_a(\theta') = \Sigma_{n=0}^{\infty} \sigma_{an} P_n(\cos\theta')$$

Then we will be stuck with this little item to deal with:

$$\Sigma_{n=0}^{\infty} \sigma_{an} \int d\theta' \, P_n(\cos\theta') \, P_n(\cos[\theta-\theta'])$$

We know that the Legendre function addition theorem will come into play here. First

$$\cos(\theta-\theta') = \cos\theta\cos\theta' + \sin\theta\sin\theta'$$

We can then use the form shown Bateman p 168-9 which is this: $\psi = 0$

$$P_n(\cos\theta\cos\theta' + \sin\theta\sin\theta') = P_n(\cos\theta) \, P_n(\cos\theta') + 2 \Sigma_{m=1}^{\infty} (-1)^m \, P_n^{-m}(\cos\theta) \, P_n^m(\cos\theta')$$

Then our integral above is this:

$$\int d\theta' \, P_n(\cos\theta') \, P_n(\cos[\theta-\theta'])$$

$$= \int d\theta' \, P_n(\cos\theta') \, \{ \, P_n(\cos\theta) \, P_n(\cos\theta') + 2 \Sigma_{m=1}^{\infty} (-1)^m \, P_n^{-m}(\cos\theta) \, P_n^m(\cos\theta') \, \}$$

$$= P_n(\cos\theta) \int d\theta' \, P_n(\cos\theta') \, P_n(\cos\theta')$$

$$+ \int d\theta' \, P_n(\cos\theta') \, 2 \Sigma_{m=1}^{\infty} (-1)^m \, P_n^{-m}(\cos\theta) \, P_n^m(\cos\theta')$$

$$= P_n(\cos\theta) \int d\theta' \, P_n(\cos\theta') \, P_n(\cos\theta')$$

$$+ 2 \Sigma_{m=1}^{\infty} (-1)^m \, P_n^{-m}(\cos\theta) \int d\theta' \, P_n(\cos\theta') \, P_n^m(\cos\theta')$$

$$= P_n(z) \int d\theta' \, P_n(z') \, P_n(z') + 2 \Sigma_{m=1}^{\infty} (-1)^m \, P_n^{-m}(z) \int d\theta' \, P_n(z') \, P_n^m(z')$$

Now write $z' = \cos\theta'$ so that $dz' = -\sin\theta' \, d\theta' = -\sqrt{1-z'^2} \, d\theta'$ and I think we then get

$$\int d\theta' = \int dz' / \sqrt{1-z'^2}$$

with the usual limits, so we continue the above as

$$= P_n(z) \int dz' \, P_n(z') \, P_n(z') / \sqrt{1-z'^2} + 2 \Sigma_{m=1}^{\infty} (-1)^m \, P_n^{-m}(z) \int dz' \, P_n(z') \, P_n^m(z') / \sqrt{1-z'^2}$$

These are assumed all to be Bateman on the cut functions, so we can now use Bateman p 171 (20) which says that this last integral vanishes except when m = 0, but m = 0 is not in the sum, so the entire second piece vanishes and we have only the first piece.