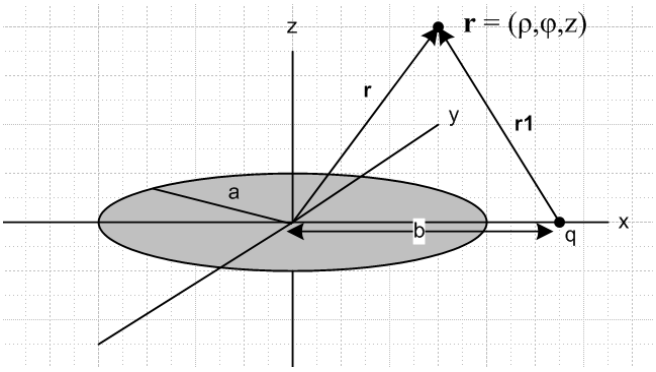
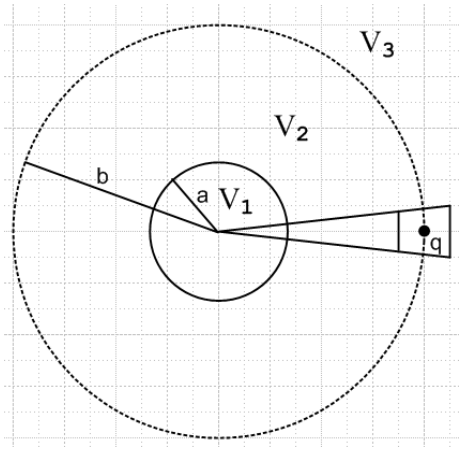
Disk Green Attempt #1 PhL 4.4.10

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1. Setting up the disk problem.

Our problem specifically is the Green's function where the point charge is in the plane of the disk or iris. Let's start with the disk and draw this picture showing where things are:





I have identified three cylindrically shaped regions that we need to worry about, as shown in the cross section on the right. For all three regions, we want $e^{-k|z|}$ and not $e^{-ik|z|}$ for the z dependence, since we know in the ± z direction the potential must go to 0, so we do not want any I or K functions. For our Bessel functions I shall choose

region 1	$J_m(k\rho)$	since good at $\rho = 0$
region 2	$J_m(k\rho)$, $H^{(1)}{}_m(k\rho)$ mixture	general mix
region 3	$H^{(1)}{}_m(k\rho)$	since good at $\rho = \infty$

So we might then write a form for our potential in each region

$$V_1(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk\, e^{-k|z|} A_m(k)\, J_m(k\rho)$$

$$V_2(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk\, e^{-k|z|} [\, B_m(k)\, J_m(k\rho) + C_m(k)\, H^{(1)}{}_m(k\rho)]$$

$$V_3(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk\, D_m(k)\, e^{-k|z|}\, H^{(1)}{}_m(k\rho)$$

I intend this to be the potential due to both the induced charge and the point charge, which is the way all other Smythian form problems have been done.

2. The potential matching condition between region 1 and region 2.

The potential must be continuous at every point on this math cylinder, so we write

$$V_1(a,\varphi,z) = V_2(a,\varphi,z)$$

$$\Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk\, e^{-k|z|} A_m(k)\, J_m(ka)$$

$$= \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk\, e^{-k|z|} [\, B_m(k)\, J_m(ka) + C_m(k)\, H^{(1)}{}_m(ka)]$$

Since the φ functions are complete, this tells us that

$$\int_0^{\infty} dk\, e^{-k|z|} A_m(k)\, J_m(ka) = \int_0^{\infty} dk\, e^{-k|z|} [\, B_m(k)\, J_m(ka) + C_m(k)\, H^{(1)}{}_m(ka)]$$

or

$$\int_0^{\infty} dk\, e^{-k|z|} \{\, B_m(k)\, J_m(ka) + C_m(k)\, H^{(1)}{}_m(ka) - A_m(k)\, J_m(ka) \,\} = 0$$

Although the $e^{-k|z|}$ are not formally a complete set, we presume that the correct way to fulfill this matching condition is to have

$$A_m(k)\, J_m(ka) = B_m(k)\, J_m(ka) + C_m(k)\, H^{(1)}{}_m(ka)$$

we here is one condition on our four sets of coefficients.

3. The Gaussian Pillbox condition between region 2 and region 3

In my first attempt here, I did things wrong, and this is worth a comment. In our spherical or oblate spheroid case, we computed things like this

$$\partial_\zeta V_i - \partial_\zeta V_o = (q/\varepsilon)\, (h_2/h_3 h_1)\, \delta(\varphi-\varphi_0)\, \delta(\xi - \xi_0)$$

where ζ was the "radial" coordinate. This was the direction of expo decay, and the other two directions φ and ξ were "oscillatory". The result was that we had complete functions sets in these other two directions, and this let us solve the "pillbox condition" for the coefficient. In the current case, we are tempted to think of ρ as the "radial" direction and compute $\partial_\rho V_2 - \partial_\rho V_1$. But in fact, in our current problem it is the z direction which has expo decay.

Using the M&M ordering for curvilinear coordinates we have

$q^1, q^2, q^3 = z, \rho, \varphi \qquad h_1 = 1 \qquad h_2 = 1 \qquad h_3 = \rho$

and from our oblate spheroid doc

$\partial_\zeta V_i - \partial_\zeta V_o = (q/\varepsilon)\,(h_2/h_3h_1)\,\delta(\varphi-\varphi_0)\,\delta(\xi - \xi_0) \qquad q_1, q_2, q_3 = \xi, \zeta, \varphi$

=>

$\partial_\rho V_i - \partial_\rho V_o = (q/\varepsilon)\,(h_2/h_3h_1)\,\delta(\varphi-0)\,\delta(z - 0) = (q/\varepsilon)(1/b)\,\delta(\varphi)\delta(z) = \sigma/\varepsilon$

and for our designated inner and outer regions this says

$\partial_\rho V_2 - \partial_\rho V_3 = (q/\varepsilon)(1/b)\,\delta(\varphi)\delta(z)$

I think I did this right, can come back later and fix it. Dimensions are correct Q/L^2 for σ.

Now we can compute our radial derivatives in the z = 0 plane. Start with the potentials at z=0:

$$V_2(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk\, e^{-k|z|} [\, B_m(k)\, J_m(k\rho) + C_m(k)\, H^{(1)}{}_m(k\rho)]$$

$$V_3(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk\, e^{-k|z|}\, D_m(k)\, H^{(1)}{}_m(k\rho)$$

Then we have, setting $\rho = b$ in the last line,

$$\partial_\rho V_2(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk\, k\, e^{-k|z|} [\, B_m(k)\, J_m'(k\rho) + C_m(k)\, H^{(1)}{}_m'(k\rho)]$$

$$\partial_\rho V_3(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk\, k\, e^{-k|z|} D_m(k)\, H^{(1)}{}'_m(k\rho)$$

$$\partial_\rho V_2 - \partial_\rho V_3 = \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk\, k\, e^{-k|z|} *$$

$$\{\, B_m(k)\, J_m'(kb) + C_m(k)\, H^{(1)}{}_m'(kb) - D_m(k)\, H^{(1)}{}'_m(kb)\,\}$$

Then, using our pillbox condition above, we get

$$(q/\varepsilon)(1/b)\,\delta(\varphi)\delta(z) =$$

$$\Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk\, k\, e^{-k|z|} \{\, B_m(k)\, J_m'(kb) + C_m(k)\, H^{(1)}{}_m'(kb) - D_m(k)\, H^{(1)}{}'_m(kb)\,\}$$

Now we first quote our result

$$\int_0^{2\pi} d\varphi \cos(m'\varphi) \cos(m\varphi) = \delta_{m,m'} 2\pi/(2-\delta_{m,0})$$

So apply $\int_0^{2\pi} d\varphi \cos(m'\varphi)$ to both sides of the above to get

$$(q/\varepsilon)(1/b)\,\delta(z) =$$

$$\Sigma_{m=0}^{\infty} \delta_{m,m'} 2\pi/(2-\delta_{m,0}) \int_0^{\infty} dk\, k\, e^{-k|z|} \{ B_m(k) J_m'(kb) + C_m(k) H^{(1)}{}_m'(kb) - D_m(k) H^{(1)}{}'_m(kb) \}$$

$$= 2\pi/(2-\delta_{m',0}) \int_0^{\infty} dk\, k\, e^{-k|z|} \{ B_{m'}(k) J_{m'}'(kb) + C_{m'}(k) H^{(1)}{}_{m'}'(kb) - D_{m'}(k) H^{(1)}{}'_{m'}(kb) \}$$

This must be true for every value of m'. Rewrite with m'→ m as usual

$$(q/\varepsilon)(1/b)\,\delta(z) =$$
$$2\pi/(2-\delta_{m,0}) \int_0^{\infty} dk\, k\, e^{-k|z|} \{ B_m(k) J_m'(kb) + C_m(k) H^{(1)}{}_m'(kb) - D_m(k) H^{(1)}{}'_m(kb) \}$$

I am not sure what to do right now, maybe integrate z from (-∞,∞) and use

$$\int_{-\infty}^{\infty} dk\, e^{-k|z|} = 2/k$$

Then we have

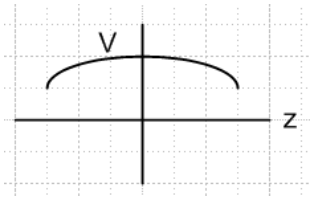
$$(q/\varepsilon)(1/b) = 4\pi/(2-\delta_{m,0}) \int_0^{\infty} dk \{ B_m(k) J_m'(kb) + C_m(k) H^{(1)}{}_m'(kb) - D_m(k) H^{(1)}{}'_m(kb) \}$$

This seems mighty strange to me. It is an integral condition on three of our coefficients.

********** **topic change** **********

I think this is all garbage to the end of the doc.

If we lower our point **r** onto the z = 0 plane outside the disk, we know that the potential will have this shape as we go up and down in z:



We don't know it is cupping down as I have drawn it, but we know that (1) V(z=0) = finite; (2) V(z) is symmetrical about z = 0. Therefore, we know that $\partial_z V = 0$ at z = 0, just as in Jackson's problem. But on the z = 0 plane, the quantity $\partial_z V$ is proportional to the charge density there which is of course zero (except near the Green's charge). So just as in Jackson's problem, we have these conditions:

$V(\rho,\varphi,0) = V_1$ $\quad \rho \le a$

$\partial_z V(\rho,\varphi,0) = 0$ $\quad \rho > a$ $\quad$ but away from the point charge

So we are going to think of this as a mixed Dirichlet/Neumann problem as Jackson did for the charged disk problem. We can restrict our interest to $z \ge 0$ I think, since the solution must be symmetric. Then we have out in region 3 where $V = V_3$,

$$\partial_z V(\rho,\varphi,z) = q\,\partial_z(1/r_1) - \Sigma_{m=0}^{\infty} \int dk\, A_m(k)\, k\, e^{-k|z|}\, H^{(1)}{}_m(k\rho) \cos(m\varphi) \qquad \rho > a \quad z \ge 0$$

It seems pretty obvious that $\partial_z(r_1) = 0$ when **r** is on the z = 0 plane. If you were to plot $r_1(z)$ you would get a plot similar to the one drawn above for V. It is finite at z = 0 and symmetrical about z = 0 so must have zero slope there. So we conclude that $\partial_z(1/r_1)|^{z\,=\,0} = 0$. Again, we stay away from the Green's charge.

So, we are thinking of the z = 0 plane as that on which the Dirichlet and Neumann BC's are established, so we have the statement of no charge density outside the disk on this plane, excepting of course our Green's charge

$$\partial_z V(\rho,\varphi,0) = -\Sigma_{m=0}^{\infty} \int dk\, A_m(k)\, k\, H^{(1)}{}_m(k\rho) \cos(m\varphi) = 0 \qquad \rho > a, \text{ away from Green's charge}$$

Our other condition is

$$V(\rho,\varphi,0) = q/r_1 + \Sigma_{m=0}^{\infty} \int dk\, B_m(k)\, J_m(k\rho) \cos(m\varphi) = V_1$$

where V_1 is the (as yet unknown) constant potential on the disk. So we end up with these dual integral equations:

$$q/r_1 + \Sigma_{m=0}^{\infty} \int dk\, B_m(k)\, J_m(k\rho) \cos(m\varphi) = V_1 \qquad \rho \le a$$

$$\Sigma_{m=0}^{\infty} \int dk\, A_m(k)\, k\, H^{(1)}{}_m(k\rho) \cos(m\varphi) = 0 \qquad \rho > a$$

A simple triangle picture connecting the origin, the Green's charge point and the **r** on the z=0 plane gives

$$r_1^2 = \rho^2 + b^2 - 2\rho b \cos(\varphi)$$

The above equations are of course considerably messier than Jackson's on page 91 (3.171), but we shall carry on and see where we go. Of course right now we have to coefficient sets which we now deal with.

We know that on the math cylinder of radius $\rho = a$, our two Smythian forms must agree, at least for $z \neq 0$. So let's consider:

$$V(a,\varphi,z) = q/r_1 + \Sigma_{m=0}^{\infty} \int dk\, A_m(k)\, e^{-k|z|}\, H^{(1)}{}_m(ka) \cos(m\varphi)$$

$$V(a,\varphi,z) = q/r_1 + \Sigma_{m=0}^{\infty} \int dk\, B_m(k)\, e^{-k|z|}\, J_m(ka) \cos(m\varphi)$$

If we set these equal, we get cancellation of the point charge so

$$\Sigma_{m=0}^{\infty} \int dk\, A_m(k)\, e^{-k|z|}\, H^{(1)}{}_m(ka) \cos(m\varphi) = \Sigma_{m=0}^{\infty} \int dk\, B_m(k)\, e^{-k|z|}\, J_m(ka) \cos(m\varphi)$$

Since we have a complete set of functions in $\cos(m\varphi)$ for this problem (which is symmetrical in φ), we can claim that the coefficients are equal, so

$$\int dk\, A_m(k)\, e^{-k|z|}\, H^{(1)}{}_m(ka) = \int dk\, B_m(k)\, e^{-k|z|}\, J_m(ka)$$

or

$$\int dk\, e^{-k|z|} \{ A_m(k)\, H^{(1)}{}_m(ka) - B_m(k)\, J_m(ka) \} = 0$$

Although $e^{-k|z|}$ is not officially a complete function set in z (maybe it is for this problem), we know that we can obtain a viable solution to our matching on the cylinder if we take

$$A_m(k)\, H^{(1)}{}_m(ka) = B_m(k)\, J_m(ka) \qquad B_m(k) = A_m(k)\, [H^{(1)}{}_m(ka)/\, J_m(ka)]$$

which somehow seems at least reasonable to me for this kind of problem. Let's look back at our potential,

$$V(\rho,\varphi,z) = q/r_1 + \Sigma_{m=0}^{\infty} \int dk\, A_m(k)\, e^{-k|z|}\, H^{(1)}{}_m(k\rho) \cos(m\varphi) \qquad \rho > a$$

$$V(\rho,\varphi,z) = q/r_1 + \Sigma_{m=0}^{\infty} \int dk\, B_m(k)\, e^{-k|z|}\, J_m(k\rho) \cos(m\varphi) \qquad \rho < a$$

and now we can replace to get

$$V(\rho,\varphi,z) = q/r_1 + \Sigma_{m=0}^{\infty} \int dk\, A_m(k)\, e^{-k|z|}\, H^{(1)}{}_m(k\rho) \cos(m\varphi) \qquad \rho > a$$

$$V(\rho,\varphi,z) = q/r_1 + \Sigma_{m=0}^{\infty} \int dk\, A_m(k)\, [H^{(1)}{}_m(ka)/\, J_m(ka)]\, e^{-k|z|}\, J_m(k\rho) \cos(m\varphi) \qquad \rho < a$$

Suppose we now define

$$A'_m(k) \equiv A_m(k) / J_m(ka)$$

Then we can write the above as

$$V(\rho,\varphi,z) = q/r_1 + \Sigma_{m=0}^{\infty} \int dk\, A'_m(k)\, J_m(ka)\quad e^{-k|z|}\, H^{(1)}_m(k\rho)\cos(m\varphi) \qquad \rho > a$$

$$V(\rho,\varphi,z) = q/r_1 + \Sigma_{m=0}^{\infty} \int dk\, A'_m(k)\, H^{(1)}_m(ka) e^{-k|z|}\, J_m(k\rho)\quad \cos(m\varphi) \qquad \rho < a$$

This is our familiar Smythian Form method, and we can then write the potential as

$$V(\rho,\varphi,z) = q/r_1 + \Sigma_{m=0}^{\infty} \int dk\, A'_m(k)\, e^{-k|z|}\, J_m(k\rho_<)\, H^{(1)}_m(k\rho_>)\cos(m\varphi) \qquad \text{for all } \rho$$

where as usual $\rho_> = \max(\rho,a)$ etc. Now back to our conditions

$V(\rho,\varphi,0) = V_1 \qquad \rho \le a \qquad\qquad\qquad\qquad \rho_< = \rho$

$\partial_z V(\rho,\varphi,0) = 0 \qquad \rho > a \qquad$ but away from the point charge

we have

$$q/r_1 + \Sigma_{m=0}^{\infty} \int dk\, A'_m(k)\, J_m(k\rho)\, H^{(1)}_m(ka)\cos(m\varphi) = V_1 \qquad \rho \le a$$

$$\Sigma_{m=0}^{\infty} \int dk\, k\, A'_m(k)\, J_m(ka)\, H^{(1)}_m(k\rho)\cos(m\varphi) = 0 \qquad \rho > a$$

where $r_1^2 = \rho^2 + b^2 - 2\rho b\cos(\varphi)$

Now for the second condition, we can use completeness of cos(mφ) to get

$$\int dk\, k\, A'_m(k)\, J_m(ka)\, H^{(1)}_m(k\rho) = 0 \qquad \rho > a$$

The first equation however requires an expansion of $1/r_1$ in cos(mφ). This is the painful work I did one day in " cos(nx) over sqrt(a-bcosx).doc". First, write

$$1/r_1 = 1/\sqrt{\rho^2 + b^2 - 2\rho b\cos(\varphi)} = (1/b)\left(1/\sqrt{1 + (\rho/b)^2 - 2(\rho/b)\cos(\varphi)}\right)$$

$$= (1/b)\left(1/\sqrt{1 + f^2 - 2f\cos(\varphi)}\right) \qquad \text{with } f = \rho/b$$

$$= (1/b)\left(1/\sqrt{\alpha - \beta\cos(\varphi)}\right) \qquad \text{with } \alpha = 1+f^2 > 1 \text{ and } \beta = 2f > 0$$

Then define

$$F_m(\alpha,\beta) = \int_0^{2\pi} d\varphi \cos(m\varphi)/\sqrt{\alpha - \beta\cos(\varphi)} = \int_0^{2\pi} d\varphi \cos(m\varphi)\,(b/r_1(\rho,\varphi))$$

$$= b \int_0^{2\pi} d\varphi \cos(m\varphi)/r_1$$

and I showed that F is given by this messy expression

$$F_m(\alpha,\beta) = (2/\sqrt{a-b}\,)\,\Sigma_{s=0}^{m}(-1)^s\ (2m, 2m-2s)\ B[\,s+1/2,\ m-s+1/2\,]\ F\,[s+1/2,\ 1/2;\ m+1;\ -2\beta/(\alpha-\beta)\,]$$

Here is the Fourier Series Schaum p 131 for even function of φ (we have $2L = 2\pi$ so $L = \pi$)

$$f(\varphi) = a_o/2 + \Sigma_{m=1}^{\infty} a_m \cos(m\varphi) \qquad a_m = (1/\pi) \int_0^{2\pi} d\varphi \cos(m\varphi)\, f(\varphi)$$

So we apply this to $f(\varphi) = b/r_1(\rho,\varphi)$ and we get

$$b/r_1(\rho,\varphi) = a_o/2 + \Sigma_{m=1}^{\infty} a_m \cos(m\varphi) \qquad a_m = (1/\pi) \int_0^{2\pi} d\varphi \cos(m\varphi)\,(b/r_1(\rho,\varphi)) = F_m(\alpha,\beta)/\pi$$

so

$$b/r_1(\rho,\varphi) = F_0(\alpha,\beta)/(2\pi) + \Sigma_{m=1}^{\infty} F_m(\alpha,\beta)/\pi \cos(m\varphi) = \Sigma_{m=0}^{\infty}(1+\delta_{m,0})^{-1}[\,F_m(\alpha,\beta)/\pi]\ \cos(m\varphi)$$

Then our first of the dual integral equations above becomes

$$q/r_1 + \Sigma_{m=0}^{\infty} \int dk\, A'_m(k)\, J_m(k\rho)\, H^{(1)}{}_m(ka) \cos(m\varphi) = V_1 \qquad \rho \le a$$

or (valid $\rho \le a$)

$$(q/b)\ \Sigma_{m=0}^{\infty}(1+\delta_{m,0})^{-1}[\,F_m(\alpha,\beta)/\pi]\ \cos(m\varphi) + \Sigma_{m=0}^{\infty} \int dk\, A'_m(k)\, J_m(k\rho)\, H^{(1)}{}_m(ka) \cos(m\varphi) = V_1$$

Doing a φ complete function analysis we then get

$$[\,F_0(\alpha,\beta)/2\pi] + \int dk\, A'_0(k)\, J_0(k\rho)\, H^{(1)}{}_0(ka) = V_1 \qquad \rho \le a$$

$$[\,F_m(\alpha,\beta)/\pi] + \int dk\, A'_m(k)\, J_m(k\rho)\, H^{(1)}{}_m(ka) = 0 \qquad \rho \le a \quad m = 1,2,3...$$

So we can how summarize our "dual integral equation" results:

$$\int dk\, k\, A'_m(k)\, J_m(ka)\, H^{(1)}{}_m(k\rho) = 0 \qquad \rho > a \quad m = 0,1,2,3...$$

$$[\, F_m(\alpha,\beta)/\pi] \; + \int dk\, A'_m(k)\, J_m(k\rho)\, H^{(1)}{}_m\,(ka) \;\; = 0 \qquad\qquad \rho \le a \qquad m = 1,2,3...$$

$$[\, F_0(\alpha,\beta)/2\pi] \; + \int dk\, A'_0(k)\, J_0(k\rho)\, H^{(1)}{}_0(ka) \;\; = V_1 \qquad\qquad \rho \le a$$

********** **more scraps** **********

The disk has some charge distribution $\sigma(\rho,\varphi)$ on it which creates a potential which in turn I think we can expand in terms of cylindrical atoms in this way:

$$V_\sigma(\rho,\varphi,z) \;= \Sigma_{m=0}{}^{\infty} \int dk\, f_m(k)\, e^{-k|z|}\, H^{(1)}{}_m(k\rho)\, \cos(m\varphi) \qquad\qquad \rho > a$$

But let's instead follow our historical method of doing these problems, and write the following

I choose this Hankel function because (1) we want $e^{-k|z|}$ and not $e^{-ik|z|}$ for the z dependence, since we know in the ± z direction the potential must go to 0, so we do not want an I or K functions; (2) This is the only J/N-world function which vanishes for large ρ. We can then write the total potential as

$$V_o(\rho,\varphi,z) \;= q/r_1 + \; \Sigma_{m=0}{}^{\infty} \int dk\, A_m(k)\, e^{-k|z|}\, H^{(1)}{}_m(k\rho)\, \cos(m\varphi) \qquad\qquad \rho > a$$

On the other hand, for $\rho < a$ away from the disk, we need J so that the potential does not blow up at the axis $\rho = 0$. So I then claim that

$$V_i(\rho,\varphi,z) \;= q/r_1 + \; \Sigma_{m=0}{}^{\infty} \int dk\, B_m(k)\, e^{-k|z|}\, J_m(k\rho)\, \cos(m\varphi) \qquad\qquad \rho < a$$

I think of these two equations as Smythian forms using the atoms appropriate to the problem. I have selected the origin to be on the disk, and we know that for spherical coordinates this is a big problem and negates an atomic sum (as outlined in some doc I wrote). But hopefully in cylindricals we are OK, we shall see.

So here then is our proposed potential in its first form:

$$V_o(\rho,\varphi,z) \;= q/r_1 + \; \Sigma_{m=0}{}^{\infty} \int dk\, A_m(k)\, e^{-k|z|}\, H^{(1)}{}_m(k\rho)\, \cos(m\varphi) \qquad\qquad \rho > a$$

$$V_i(\rho,\varphi,z) \;= q/r_1 + \; \Sigma_{m=0}{}^{\infty} \int dk\, B_m(k)\, e^{-k|z|}\, J_m(k\rho)\, \cos(m\varphi) \qquad\qquad \rho < a$$