**Disk Green Attempt #2** PhL 4.12.10

This is a cos(kz) attempt.


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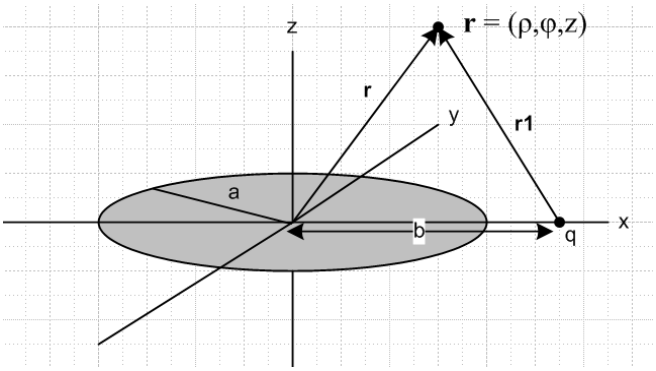
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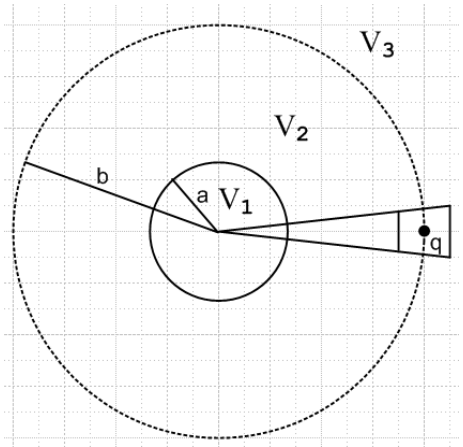
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## 1. Setting up the disk problem.

Our problem specifically is the Green's function where the point charge is in the plane of the disk or iris. Let's start with the disk and draw this picture showing where things are:




I have identified three cylindrically shaped regions that we need to worry about, as shown in the cross section on the right. For all three regions, we want $e^{-k|z|}$ and not $e^{-ik|z|}$ for the z dependence, since we know in the ± z direction the potential must go to 0, so we do not want any I or K functions. *However*, when I march down this path, I cannot solve for coefficients because I don't have a "complete set" of functions in the z direction. For "cylindrical harmonics" we really need cos(kz) cos(mφ) where we have oscillatory in both directions on our cylinder surface where the pillbox will be located. So even though

this seems wrong in terms of z decay, let's try it out and see what happens. This means we DO have I and K functions for the ρ dimension. So, for our Bessel functions I shall choose

| | | | |
|---|---|---|---|
| region 1 | $I_m(k\rho)$ | | since good at $\rho = 0$ |
| region 2 | $I_m(k\rho)$ ,$K_m(k\rho)$ | mixture | general mix |
| region 3 | $K_m(k\rho)$ | | since good at $\rho = \infty$ |

So we might then write a form for our potential in each region

$$V_1(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk \cos(kz) A_m(k) I_m(k\rho)$$

$$V_2(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk \cos(kz) [ B_m(k) I_m(k\rho) + C_m(k) K_m(k\rho)]$$

$$V_3(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk \cos(kz) D_m(k) K_m(k\rho)$$

**2. The potential matching condition between region 1 and region 2.**

The potential must be continuous at every point on this math cylinder, so we write

$$V_1(a,\varphi,z) = V_2(a,\varphi,z)$$

$$\Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk \cos(kz) A_m(k) I_m(ka)$$

$$= \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk \cos(kz) [ B_m(k) I_m(ka) + C_m(k) K_m(ka)]$$

Since the φ and z functions are both complete, this tells us that

$$A_m(k) I_m(ka) = B_m(k) I_m(ka) + C_m(k) K_m(ka)$$

which is one condition on our four sets of coefficients.

**3. The potential matching condition between region 2 and region 3.**

The potential must be continuous at every point on this math cylinder, so we write

$$V_3(a,\varphi,z) = V_2(a,\varphi,z)$$ // away from the single point z=0 and φ = 0 and ρ = b

$$\Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk \cos(kz) D_m(k) K_m(kb)$$

$$= \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk \cos(kz) [ B_m(k) I_m(kb) + C_m(k) K_m(kb)]$$

Since the φ and z functions are both complete, this tells us that

$$D_m(k)\, K_m(kb) = B_m(k)\, I_m(kb) + C_m(k)\, K_m(kb)$$

which is a second condition on our four coefficients.

**4. The Gaussian Pillbox condition between region 2 and region 3**

Using the M&M ordering for curvilinear coordinates we have

$$q^1, q^2, q^3 = z, \rho, \varphi \qquad h_1 = 1 \qquad h_2 = 1 \qquad h_3 = \rho$$

and from our oblate spheroid doc

$$\partial_\zeta V_i - \partial_\zeta V_o = (q/\varepsilon)\,(h_2/h_3h_1)\,\delta(\varphi-\varphi_0)\,\delta(\xi - \xi_0) \qquad q_1, q_2, q_3 = \xi, \zeta, \varphi$$

=>

$$\partial_\rho V_i - \partial_\rho V_o = (q/\varepsilon)\,(h_2/h_3h_1)\,\delta(\varphi-0)\,\delta(z - 0) = (q/\varepsilon)(1/b)\,\delta(\varphi)\delta(z) = \sigma/\varepsilon$$

and for our designated inner and outer regions this says

$$\partial_\rho V_2 - \partial_\rho V_3 = (q/\varepsilon)(1/b)\,\delta(\varphi)\delta(z)$$

I think I did this right, can come back later and fix it. Dimensions are correct $Q/L^2$ for σ.

Now we can compute our radial derivatives in the z = 0 plane. Start with the potentials at z=0:

$$V_2(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk \cos(kz)\,[\, B_m(k)\, I_m(k\rho) + C_m(k)\, K_m(k\rho)]$$

$$V_3(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk\, D_m(k) \cos(kz)\, K_m(k\rho)$$

Then we have, setting ρ = b in the last line,

$$\partial_\rho V_2(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk\, k \cos(kz)\,[\, B_m(k)\, I_m'(k\rho) + C_m(k)\, K_m'(k\rho)]$$

$$\partial_\rho V_3(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk\, k \cos(kz) D_m(k)\, K'_m(k\rho)$$

$$\partial_\rho V_2 - \partial_\rho V_3 = \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk\, k \cos(kz) \,*$$

$$\{\, B_m(k)\, I_m'(kb) + C_m(k)\, K_m'(kb) - D_m(k)\, K'_m(kb)\,\}$$

Then, using our pillbox condition above, we get

$$(q/\varepsilon)(1/b)\,\delta(\varphi)\delta(z) =$$

$$\Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk\, k \cos(kz)\{ B_m(k)\, I_m'(kb) + C_m(k)\, K_m'(kb) - D_m(k)\, K'_m(kb) \}$$

Now we first quote our result

$$\int_0^{2\pi} d\varphi \cos(m'\varphi) \cos(m\varphi) = \delta_{m,m'} 2\pi/(2-\delta_{m,0})$$

So apply $\int_0^{2\pi} d\varphi \cos(m'\varphi)$ to both sides of the above to get

$$(q/\varepsilon)(1/b)\,\delta(z) =$$

$$\Sigma_{m=0}^{\infty} \delta_{m,m'} 2\pi/(2-\delta_{m,0}) \int_0^{\infty} dk\, k \cos(kz)\{ B_m(k)\, I_m'(kb) + C_m(k)\, K_m'(kb) - D_m(k)\, K'_m(kb) \}$$

$$= 2\pi/(2-\delta_{m',0}) \int_0^{\infty} dk\, k \cos(kz)\{ B_{m'}(k)\, I_{m'}'(kb) + C_{m'}(k)\, K_{m'}'(kb) - D_{m'}(k) K'_{m'}(kb) \}$$

This must be true for every value of m'. Rewrite with m'→ m as usual

$$(q/\varepsilon)(1/b)\,\delta(z) =$$
$$2\pi/(2-\delta_{m,0}) \int_0^{\infty} dk\, k \cos(kz)\{ B_m(k)\, I_m'(kb) + C_m(k)\, K_m'(kb) - D_m(k)\, K'_m(kb) \}$$

**(a) Digression on the Fourier Cosine Transform**

The cos(kz) functions are complete. We can start from this point:

orthogonality on z in (-∞,∞)
$\int_{-\infty}^{\infty} dz\, e^{ikz} e^{-ik'z} = 2\pi\,\delta(k-k')$

completeness on same interval
$\int_{-\infty}^{\infty} dk\, e^{-ikz} e^{+ik'z} = 2\pi\,\delta(z-z')$

The usual Fourier Integral transform is then

$$f(z) = (1/2\pi) \int_{-\infty}^{\infty} dk\, e^{+ikz} F(k) \qquad F(k) = \int_{-\infty}^{\infty} dz\, e^{-ikz} f(z)$$

Here I am selecting signs as on page 1 of my Spectral Theory chapter 1. Now if f(z) is an even function of z, we can write instead (where we drop sine terms for obvious reasons in both)

$$f(z) = (1/2\pi) \int_{-\infty}^{\infty} dk \cos(kz)\, F(k) \qquad F(k) = \int_{-\infty}^{\infty} dz \cos(kz)\, f(z)$$

The right equation says F(k) is even in k. Therefore, both integrands in the above equations are even with respect to the integration variables, so we can fold both to get

$$f(z) = (1/\pi) \int_0^\infty dk \cos(kz) F(k) \qquad F(k) = 2 \int_0^\infty dz \cos(kz) f(z)$$

Note added 1.10.11. The transform appearing on the line above involves only $k > 0$ and $z > 0$ ! We don't even talk about f(z) if $z < 0$. *However*, if we were to use the expansion shown as the *definition* of f(z) for negative z, then we would say that the extended function f(z) was even in z, where we mean $f(-z) = f(z)$. Suppose we don't do this extension. Still, if f(z) came out to be, say, a power series in z, the expansion shown for $z > 0$ tells us that only even powers can occur in f(z). So we can then say the following: in the expansion and projection shown above, we have only $z \geq 0$ and $k \geq 0$ so f(z) is undefined for $z < 0$. If the expansion shown is valid for some f(z), then we know that if we were to somehow write f(z) as a power series in z, only even powers would appear, so *in that sense*, this transform is only valid for "even" functions.

So, if our domain of functions of interest f(z) consists only of even functions, then we can write orthogonality and completeness in this way:

$$f(z) = (1/\pi) \int_0^\infty dk \cos(kz) F(k) = f(z) = (1/\pi) \int_0^\infty dk \cos(kz)\, 2 \int_0^\infty dz' \cos(kz') f(z')$$

$$= \int_0^\infty dz' f(z') \int_0^\infty dk \cos(kz) \cos(kz') (2/\pi) = \int_0^\infty dz' f(z')\delta(z'-z)$$

[ Note added 1.10.11: if $z > 0$, the last integral equals f(z) unambiguously. If $z = 0$, we shall regard that as a limiting case $z = \varepsilon > 0$ where we take $\varepsilon \to 0$. So there is no issue of only picking up "half a delta function" or anything like that. ]

so that

$$\int_0^\infty dk \cos(kz) \cos(kz') = (\pi/2)\delta(z - z') \qquad \text{// completeness} \qquad z,z' \geq 0$$

If we did this the other way around, we have:

$$F(k) = 2 \int_0^\infty dz \cos(kz) f(z) = 2 \int_0^\infty dz \cos(kz) (1/\pi) \int_0^\infty dk' \cos(k'z) F(k')$$

$$= \int_0^\infty dk' F(k') \int_0^\infty dz \cos(kz) \cos(k'z)(2/\pi) = \int_0^\infty dk' F(k') \delta(k-k')$$

so that

$$\int_0^\infty dz\, \cos(kz)\cos(k'z) = (\pi/2)\,\delta(k-k') \qquad \text{// orthogonality} \qquad k,k' \geq 0$$

This stuff is just the Fourier Cosine transform as discussed Stak I page 293. As for our transform

$$f(z) = (1/\pi)\int_0^\infty dk\, \cos(kz)\, F(k) \qquad F(k) = 2\int_0^\infty dz\, \cos(kz)\, f(z)$$

[ Note added 1.10.11. Suppose we consider $f(z) = \delta(z)$ as the limit of a narrow Gaussian. Since we are only supposed to consider $f(z)$ for $z \geq 0$, when we write $\delta(z)$ in this context, we are referring to the right half of the Gaussian, albeit in the limit. It would seem then that we should say

$$F(k) = 2\int_0^\infty dz\, \cos(kz)\,\delta(z) = 2 * (1/2) = 1$$

so here we think of our integral as "picking up only the right half" of the full delta function. If we then put this into the expansion, we get

$$\delta(z) = (1/\pi)\int_0^\infty dk\, \cos(kz)\, 1 = (1/\pi)\int_0^\infty dk\, \cos(kz) \qquad z \geq 0$$

We can compare this to the Fourier Transform idea that

$$\delta(z) = (1/2\pi)\int_{-\infty}^\infty dk\; e^{-ikz} = (1/2\pi)\int_{-\infty}^\infty dk\; \cos(kz) = (1/\pi)\int_0^\infty dk\, \cos(kz) \qquad z \geq 0$$

and we get the same result! ]

We can make things symmetrical by defining $G(k) \equiv \sqrt{1/2\pi}\, F(k)$ so that

$$G(k) = \sqrt{1/2\pi}\; 2\int_0^\infty dz\, \cos(kz)\, f(z) = \sqrt{2/\pi}\int_0^\infty dz\, \cos(kz)\, f(z)$$

$$f(z) = (1/\pi)\int_0^\infty dk\, \cos(kz)\, F(k) = (1/\pi)\int_0^\infty dk\, \cos(kz)\, G(k)\sqrt{2\pi} = \sqrt{2/\pi}\int_0^\infty dk\, \cos(kz)\, G(k)$$

So here then are all our results:

For functions f(z) which are even and decay fast enough to converge the projection integral:

$$f(z) = \sqrt{2/\pi}\int_0^\infty dk\, \cos(kz)\, G(k) \qquad \text{// expansion}$$

$$G(k) = \sqrt{2/\pi}\int_0^\infty dz\, \cos(kz)\, f(z) \qquad \text{// projection}$$

$\int_0^\infty dz\, \cos(kz)\cos(k'z) = (\pi/2)\,\delta(k-k')$ // orthogonality k in $(0,\infty)$

$\int_0^\infty dk\, \cos(kz)\cos(kz') = (\pi/2)\delta(z-z')$ // completeness z in $(0,\infty)$

The transform now appears as in Stak p 293 (4.71) and completeness as in (4.70). [ See " confusion about completeness relations.doc" for more general comments about these cos cos integrals and their "second terms". Here we get to ignore the second terms since both k and z are on the half line. ]

**Resume Main Flow**

We had this pillbox condition:

$$(q/\varepsilon)(1/b)\,\delta(z) = 2\pi/(2-\delta_{m,0}) \int_0^\infty dk\, k \cos(kz)\{ B_m(k)\, I_m'(kb) + C_m(k)\, K_m'(kb) - D_m(k)\, K'_m(kb) \}$$

We now apply $\int_0^\infty dz \cos(k'z)$ to both sides. LHS integral is just $\cos(k'0) = 1$, so get

$$(q/\varepsilon)(1/b) = 2\pi/(2-\delta_{m,0}) \int_0^\infty dk\, k\, (\pi/2)\, \delta(k-k')\{ B_m(k)\, I_m'(kb) + C_m(k)\, K_m'(kb) - D_m(k)\, K'_m(kb) \}$$

$$= 2\pi/(2-\delta_{m,0})\, (\pi/2)\, k' \{ B_m(k')\, I_m'(k'b) + C_m(k')\, K_m'(k'b) - D_m(k')\, K'_m(k'b) \}$$

This must be valid for all k'. Replace $k' \rightarrow k$ to get

$$(q/\varepsilon)(1/b) = 2\pi/(2-\delta_{m,0})\, (\pi/2)\, k \{ B_m(k)\, I_m'(kb) + C_m(k)\, K_m'(kb) - D_m(k)\, K'_m(kb) \}$$

**5. Continuity of electric field $E_\rho$ between regions 1 and 2.**

This analysis is the same as the previous section, with these modifications:

$q \rightarrow 0$ $\quad b \rightarrow a$

$D_m(k)\, K_m(k\rho) \rightarrow A_m(k)\, I_m(ka)$ // in various forms

Thus we end up with

$$0 = 2\pi/(2-\delta_{m,0})\, (\pi/2)\, k \{ B_m(k)\, I_m'(kb) + C_m(k)\, K_m'(kb) - A_m(k)\, I_m'(ka) \}$$

which really says

$B_m(k)\, I_m'(kb) + C_m(k)\, K_m'(kb) = A_m(k)\, I_m'(ka)$

**6. Gather up all four conditions and solve for the coefficients**

Here they are from the above sections:

$A_m(k)\, I_m(ka) = B_m(k)\, I_m(ka) + C_m(k)\, K_m(ka)$ // match V between region 1 and 2

$D_m(k)\, K_m(kb) = B_m(k)\, I_m(kb) + C_m(k)\, K_m(kb)$ // match V between region 2 and 3

$B_m(k)\, I_m'(kb) + C_m(k)\, K_m'(kb) = A_m(k)\, I_m'(ka)$ // match E field between region 1 and 2

$(q/\varepsilon)(1/b) = 2\pi/(2-\delta_{m,0})\,(\pi/2)\,k\,\{ B_m(k)\, I_m'(kb) + C_m(k)\, K_m'(kb) - D_m(k)\, K'_m(kb) \}$
// match E field between region 2 and 3

Rewrite them all (change order too)

$[A_m(k) - B_m(k)]\, I_m(ka) = K_m(ka)$

$[A_m(k) - B(k)]\, I_m'(ka) = C_m(k)\, K_m'(kb)$

$[\, D_m(k) - C_m(k)\,]\, K_m(kb) = B_m(k)\, I_m(kb)$

$[\, D_m(k) - C_m(k)\,]\, K_m'(kb) = B_m(k)\, I_m'(kb) - (q/\varepsilon)(1/b)\{\, 2\pi/(2-\delta_{m,0})\,(\pi/2)\,k\}^{-1}$

Rewrite again

$[A_m(k) - B_m(k)] = K_m(ka)/\, I_m(ka)$

$[A_m(k) - B(k)] = C_m(k)\, K_m'(kb)/\, I_m'(ka)$

$[\, D_m(k) - C_m(k)\,] = B_m(k)\, I_m(kb)/\, K_m(kb)$

$[\, D_m(k) - C_m(k)\,] = B_m(k)\, I_m'(kb)/\, K_m'(kb) - (q/\varepsilon)(1/b)\{\, 2\pi/(2-\delta_{m,0})\,(\pi/2)\,k\, K_m'(kb)\}^{-1}$

Equate the first pair and equate the second pair to get

$K_m(ka)/\, I_m(ka) = C_m(k)\, K_m'(kb)/\, I_m'(ka)$

$B_m(k)\, I_m(kb)/\, K_m(kb) = B_m(k)\, I_m'(kb)/\, K_m'(kb) - (q/\varepsilon)(1/b)\{\, 2\pi/(2-\delta_{m,0})\,(\pi/2)\,k\, K_m'(kb)\}^{-1}$

We can solve these as follows:

$C_m(k) = [K_m(ka)/\, I_m(ka)]/[\, K_m'(kb)/\, I_m'(ka)]$

$B_m(k) = [I_m(kb)/\, K_m(kb) - I_m'(kb)/\, K_m'(kb)] - (q/\varepsilon)(1/b)\{\, 2\pi/(2-\delta_{m,0})\,(\pi/2)\,k\, K_m'(kb)\}^{-1}$

Then we get D and A from 3 and 1 above

$$D_m(k) = C_m(k) + B_m(k)\, I_m(kb)/ K_m(kb)$$

$$A_m(k) = B_m(k) + K_m(ka)/ I_m(ka)$$

**7. Clean things up with better presentation**

Now that we think we have solved for all the coefficients, we can start over and think more in terms of Smythian Forms. If we first think just about the potential continuity, we have these facts:

$$V_1(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk \cos(kz) A_m(k)\, I_m(k\rho)$$

$$V_2(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk \cos(kz) [\, B_m(k)\, I_m(k\rho) + C_m(k)\, K_m(k\rho)]$$

$$V_3(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk \cos(kz)\, D_m(k)\, K_m(k\rho)$$

$$A_m(k)\, I_m(ka) = B_m(k)\, I_m(ka) + C_m(k)\, K_m(ka)$$

$$D_m(k)\, K_m(kb) = B_m(k)\, I_m(kb) + C_m(k)\, K_m(kb)$$

Use the first line to get rid of A, and the second to get rid of D:

$$V_1(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk \cos(kz)\{ B_m(k)\, I_m(ka) + C_m(k)\, K_m(ka)\}\, I_m(k\rho)/ I_m(ka)$$

$$V_2(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk \cos(kz) [\, B_m(k)\, I_m(k\rho) + C_m(k)\, K_m(k\rho)]$$

$$V_3(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk \cos(kz)\{ B_m(k)\, I_m(kb) + C_m(k)\, K_m(kb)\}\, K_m(k\rho)/ K_m(kb)$$

So we have only B and C coefficients to worry about. Now define:

$$b_m(k) \equiv B_m(k) / [I_m(ka)\, K_m(kb)]$$

$$c_m(k) \equiv C_m(k) / [I_m(ka)\, K_m(kb)]$$

Then our potentials are these ( save space with $\cos(x) = C_x$)

$$V_1(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ b_m(k) K_m(kb) I_m(ka)\, I_m(k\rho) + c_m(k)\, K_m(ka) K_m(kb)\, I_m(k\rho)\}$$

$$V_2(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ b_m(k)\, I_m(ka)\, K_m(kb)\, I_m(k\rho) + c_m(k)\, I_m(ka)\, K_m(kb)\, K_m(k\rho)\}$$

$$V_3(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ b_m(k)\, I_m(kb)\, I_m(ka)\, K_m(k\rho) + c_m(k)\, K_m(kb)\, I_m(ka)\, K_m(k\rho)\}$$

Now try to put factors in some reasonable order

$$V_1(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ b_m(k)\, I_m(ka) K_m(kb)\, I_m(k\rho) + c_m(k)\, K_m(ka) K_m(kb)\, I_m(k\rho)\}$$

$$V_2(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ b_m(k)\, I_m(ka)\, K_m(kb)\, I_m(k\rho) + c_m(k)\, K_m(k\rho)\, K_m(kb)\, I_m(ka) \}$$

$$V_3(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ b_m(k)\, I_m(ka)\, K_m(k\rho)\, I_m(kb) + c_m(k)\, K_m(k\rho)\, K_m(kb)\, I_m(ka) \}$$

The red terms are equal, and so are the blue terms. At the 1-2 boundary, we see that the second terms of the first two expressions agree at $\rho = a$, and at the 2-3 boundary we see that the first terms of the last two expressions agree at $\rho = b$. Let's write this again, then, as our **Smythian Form** for this problem:

$$V_1(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ b_m(k)\, I_m(ka)\, K_m(kb)\, I_m(k\rho) + c_m(k)\, K_m(ka)\, K_m(kb)\, I_m(k\rho)\}$$

$$V_2(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ b_m(k)\, I_m(ka)\, K_m(kb)\, I_m(k\rho) + c_m(k)\, K_m(k\rho)\, K_m(kb)\, I_m(ka) \}$$

$$V_3(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ b_m(k)\, I_m(ka)\, K_m(k\rho)\, I_m(kb) + c_m(k)\, K_m(k\rho)\, K_m(kb)\, I_m(ka) \}$$

This is fancier that earlier forms we had because we have three regions instead of 2 to worry about. But the idea is the same: cause the boundary matches to be manifestly obvious, and don't have any factors in the denominator.

**Redo Section 4 for pillbox.** So let's take the above three expressions for the potential as our starting point, and re-do the electric field continuity calculations. This is probably faster that plugging things in, and I hope it will expose the Wronskian simplification which we have not seen yet. So here we go for the 2/3 boundary fields:

$$V_2(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ b_m(k)\, I_m(ka)\, K_m(kb)\, I_m(k\rho) + c_m(k)\, K_m(k\rho)\, K_m(kb)\, I_m(ka) \}$$

$$V_3(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ b_m(k)\, I_m(ka)\, K_m(k\rho)\, I_m(kb) + c_m(k)\, K_m(k\rho)\, K_m(kb)\, I_m(ka) \}$$

$$\partial_\rho V_2(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, k\, C_{kz} \{ b_m(k)\, I_m(ka)\, K_m(kb)\, I_m'(k\rho) + c_m(k)\, K_m'(k\rho)\, K_m(kb)\, I_m(ka) \}$$

$$\partial_\rho V_3(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, k\, C_{kz} \{ b_m(k)\, I_m(ka)\, K_m'(k\rho)\, I_m(kb) + c_m(k)\, K_m'(k\rho)\, K_m(kb)\, I_m(ka) \}$$

$$\partial_\rho V_2 - \partial_\rho V_3 = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\ k\ C_{kz} *$$

$$\{ b_m(k)\ I_m(ka)\ K_m(kb)\ I_m'(k\rho) + c_m(k)\ K_m'(k\rho)\ K_m(kb)\ I_m(ka) \}$$
$$- \{ b_m(k)\ I_m(ka)\ K_m'(k\rho)\ I_m(kb) + c_m(k)\ K_m'(k\rho)\ K_m(kb)\ I_m(ka) \}$$

Evaluate this thing at $\rho = b$ and we have

$$\partial_\rho V_2 - \partial_\rho V_3 = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\ k\ C_{kz} *$$

$$\{ b_m(k)\ I_m(ka)\ K_m(kb)\ I_m'(kb) + c_m(k)\ K_m'(kb)\ K_m(kb)\ I_m(ka) \}$$
$$- \{ b_m(k)\ I_m(ka)\ K_m'(kb)\ I_m(kb) + c_m(k)\ K_m'(kb)\ K_m(kb)\ I_m(ka) \}$$

The $c_m(k)$ terms cancel each other (as usual in this kind of process) and we have

$$\partial_\rho V_2 - \partial_\rho V_3 = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\ k\ C_{kz} *$$

$$b_m(k)\ I_m(ka)\ K_m(kb)\ I_m'(kb) - b_m(k)\ I_m(ka)\ K_m'(kb)\ I_m(kb)$$

$$= \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\ k\ C_{kz}\ b_m(k)\ I_m(ka)\ \{ K_m(kb)\ I_m'(kb) - K_m'(kb)\ I_m(kb) \}$$

$$= \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\ k\ C_{kz}\ b_m(k)\ I_m(ka)\ W[\ K_m(kb),\ I_m(kb)]$$

Bateman p 80 says this Wronskian is $+1/(kb)$, also Jackson p 86, first item not primed. So

$$\partial_\rho V_2 - \partial_\rho V_3 = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\ k\ C_{kz}\ b_m(k)\ I_m(ka)/(kb)$$

$$= (1/b)\ \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\ C_{kz}\ b_m(k)\ I_m(ka)$$

$$\partial_\rho V_2 - \partial_\rho V_3 = (q/\varepsilon)(1/b)\ \delta(\varphi)\delta(z)$$

Equating we get

$$(q/\varepsilon)\delta(\varphi)\delta(z) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\ C_{kz}\ b_m(k)\ I_m(ka)$$

Apply $\int_0^{2\pi} d\varphi\ C_{m'\varphi}$ to both sides to get

$$(q/\varepsilon)\delta(z) = \Sigma_{m=0}^{\infty} [\ \int_0^{2\pi} d\varphi\ C_{m'\varphi}\ C_{m\varphi}\ ] \int_0^{\infty} dk\ C_{kz}\ b_m(k)\ I_m(ka)$$

$$= \Sigma_{m=0}^{\infty}[\ \delta_{m,m'} 2\pi/(2-\delta_{m,0})\ ] \int_0^{\infty} dk\ C_{kz}\ b_m(k)\ I_m(ka)$$

$$= 2\pi/(2-\delta_{m,0}) \int_0^{\infty} dk\ C_{kz}\ b_m(k)\ I_m(ka)$$

where in the last line I replaced m' with m. Now apply $\int_0^{\infty} dz\ C_{k'z}$ to both sides. I think the LHS somehow sees the full δ, not just half, come back later if wrong. Then we have

$$(q/\varepsilon) = 2\pi/(2-\delta_{m,0}) \int_0^{\infty} dk\ [\int_0^{\infty} dz\ C_{k'z}\ C_{kz}\ ]\ b_m(k)\ I_m(ka) \qquad (*)$$

$$= 2\pi/(2-\delta_{m,0}) \int_0^{\infty} dk\ [(\pi/2)\ \delta(k-k')]\ b_m(k)\ I_m(ka)$$

$$= \pi^2/(2-\delta_{m,0})\ b_m(k)\ I_m(ka)$$

where in the last line I replaced k' by k. So we have found our result for one of the two coefficients:

$$b_m(k)\ I_m(ka) = (2-\delta_{m,0})\ (q/\pi^2\varepsilon)$$

IF we are only supposed to pick up half the delta, then we get 1/2 on the LHS of (*) so just replace q with q/2.

**Redo Section 5 for non-pillbox.**

$$V_1(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\ C_{kz}\ \{\ b_m(k)\ I_m(ka)\ K_m(kb)\ I_m(k\rho)\ + c_m(k)\ K_m(ka)\ K_m(kb)\ I_m(k\rho)\}$$

$$V_2(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\ C_{kz}\ \{\ b_m(k)\ I_m(ka)\ K_m(kb)\ I_m(k\rho)\ + c_m(k)\ K_m(k\rho)\ K_m(kb)\ I_m(ka)\ \}$$

$$\partial_\rho\ V_1(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\ k\ C_{kz}\ \{\ b_m(k)\ I_m(ka)\ K_m(kb)\ I_m'(k\rho)\ + c_m(k)\ K_m(ka)\ K_m(kb)\ I_m'(k\rho)\}$$

$$\partial_\rho\ V_2(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\ k\ C_{kz}\ \{\ b_m(k)\ I_m(ka)\ K_m(kb)\ I_m'(k\rho)\ + c_m(k)\ K_m'(k\rho)\ K_m(kb)\ I_m(ka)\ \}$$

$$\partial_\rho V_2 - \partial_\rho V_3 = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\ k\ C_{kz}\ *$$

$$\{\ b_m(k)\ I_m(ka)\ K_m(kb)\ I_m'(k\rho)\ + c_m(k)\ K_m(ka)\ K_m(kb)\ I_m'(k\rho)\}$$
$$-\ \{\ b_m(k)\ I_m(ka)\ K_m(kb)\ I_m'(k\rho)\ + c_m(k)\ K_m'(k\rho)\ K_m(kb)\ I_m(ka)\ \}$$

Evaluate this thing at ρ = a and we have

$$\partial_\rho V_2 - \partial_\rho V_3 = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\ k\ C_{kz}\ *$$

$$\{ b_m(k)\, I_m(ka)\, K_m(kb)\, I_m'(ka) + c_m(k)\, K_m(ka)\, K_m(kb)\, I_m'(ka)\}$$
$$- \{ b_m(k)\, I_m(ka)\, K_m(kb)\, I_m'(ka) + c_m(k)\, K_m'(ka)\, K_m(kb)\, I_m(ka) \}$$

This time the $b_m(k)$ terms cancel each other and we have

$$\partial_\rho V_2 - \partial_\rho V_3 = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, k\, C_{kz} *$$
$$c_m(k)\, K_m(ka)\, K_m(kb)\, I_m'(ka) - c_m(k)\, K_m'(ka)\, K_m(kb)\, I_m(ka)$$

$$= \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, k\, C_{kz}\, c_m(k)\, K_m(kb) [K_m(ka)\, I_m'(ka) - K_m'(ka)\, I_m(ka)]$$

$$= \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, k\, C_{kz}\, c_m(k)\, K_m(kb)\, W[ K_m(ka), I_m(ka)]$$

This Wronskian is +1/(ka) so we get

$$\partial_\rho V_2 - \partial_\rho V_3 = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, k\, C_{kz}\, c_m(k)\, K_m(kb)\, 1/(ka)$$

$$= (1/a)\, \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz}\, c_m(k)\, K_m(kb)$$

$$\partial_\rho V_2 - \partial_\rho V_3 = 0 \qquad \text{// no Green's charge on this boundary}$$

Equating we get

$$0 = (1/a)\, \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz}\, c_m(k)\, K_m(kb)$$

Completeness in both φ and z gives the surprising (to me) result that $c_m(k) = 0$.

<u>Put the pieces together</u> . Here was our Smythian form

$$V_1(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ b_m(k)\, I_m(ka)\, K_m(kb)\, I_m(k\rho) + c_m(k)\, K_m(ka)\, K_m(kb)\, I_m(k\rho)\}$$

$$V_2(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ b_m(k)\, I_m(ka)\, K_m(kb)\, I_m(k\rho) + c_m(k)\, K_m(k\rho)\, K_m(kb)\, I_m(ka) \}$$

$$V_3(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ b_m(k)\, I_m(ka)\, K_m(k\rho)\, I_m(kb) + c_m(k)\, K_m(k\rho)\, K_m(kb)\, I_m(ka) \}$$

and we found from our field continuity conditions that

$$b_m(k)\, I_m(ka) = (2-\delta_{m,0})\, (q/\pi^2\varepsilon)$$
$$c_m(k) = 0$$

Stuff these in to get:

$$V_1(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ (2-\delta_{m,0}) (q/\pi^2\varepsilon) K_m(kb) I_m(k\rho) \}$$

$$V_2(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ (2-\delta_{m,0}) (q/\pi^2\varepsilon) K_m(kb) I_m(k\rho) \}$$

$$V_3(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ (2-\delta_{m,0}) (q/\pi^2\varepsilon) K_m(k\rho) I_m(kb) \}$$

which we rewrite as:

$$V_1(z,\rho,\varphi) = (q/\pi^2\varepsilon)\Sigma_{m=0}^{\infty} (2-\delta_{m,0}) C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ K_m(kb) I_m(k\rho) \}$$

$$V_2(z,\rho,\varphi) = (q/\pi^2\varepsilon)\Sigma_{m=0}^{\infty} (2-\delta_{m,0}) C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ K_m(kb) I_m(k\rho) \}$$

$$V_3(z,\rho,\varphi) = (q/\pi^2\varepsilon)\Sigma_{m=0}^{\infty} (2-\delta_{m,0}) C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ K_m(k\rho) I_m(kb) \}$$

This says the potential in regions 1 and 2 is the same. This was not obvious to me at the start. I guess the reason is that on this math cylinder, continuity of potential and field forces the potential to be the same in both regions. We have not said much about what happens on the disk itself! I ignored that in the Smythian method, and probably now I am going to get into trouble with it. Indeed, if I set z = 0 in region 1, we should be getting V = constant on the disk. But we get the same result in regions 1 and 2 for z = 0 which we know is wrong. So I have "gone astray" by ignoring this fact.

The error is in the "field continuity between region 1 and 2". I think the other three conditions are OK. When we did the pillbox for the Green's point charge, we had to think about the fact that there was charge located somewhere on our math boundary between the two regions. For the 1/2 boundary, there is going to be some charge on the rim of the disk, and I have ignored it!

Let's try to maintain our Smythian form (which may be wrong, but maybe not) which is

$$V_1(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ b_m(k) I_m(ka) K_m(kb) I_m(k\rho) + c_m(k) K_m(ka) K_m(kb) I_m(k\rho) \}$$

$$V_2(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ b_m(k) I_m(ka) K_m(kb) I_m(k\rho) + c_m(k) K_m(k\rho) K_m(kb) I_m(ka) \}$$

$$V_3(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ b_m(k) I_m(ka) K_m(k\rho) I_m(kb) + c_m(k) K_m(k\rho) K_m(kb) I_m(ka) \}$$

We can install our conclusion from the 2/3 boundary,

$$b_m(k) I_m(ka) = (2-\delta_{m,0}) (q/\pi^2\varepsilon) = \varepsilon_m (q/\pi^2\varepsilon)$$ // see below on Neumann's factor

to get

$$V_1(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ (2-\delta_{m,0}) (q/\pi^2\varepsilon) K_m(kb) I_m(k\rho) + c_m(k) K_m(ka) K_m(kb) I_m(k\rho)\}$$

$$V_2(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ (2-\delta_{m,0}) (q/\pi^2\varepsilon) K_m(kb) I_m(k\rho) + c_m(k) K_m(k\rho) K_m(kb) I_m(ka) \}$$

$$V_3(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ (2-\delta_{m,0}) (q/\pi^2\varepsilon) K_m(k\rho) I_m(kb) + c_m(k) K_m(k\rho) K_m(kb) I_m(ka) \}$$

But now we have the dual integral equation situation which is, in our current context, ( all at z = 0)
( I wrongly had $V_o$ as in Jackson's disk, but here we have a Green's problem so V = 0 on the disk )

| | | |
|---|---|---|
| $V_1(z,\rho,\varphi) = 0$ | $\rho \leq a$ | Dirichlet |
| $\partial_z V_2(z,\rho,\varphi) = 0$ | $a < \rho \leq b$ | Neumann |
| $\partial_z V_3(z,\rho,\varphi) = 0$ | $b < \rho < \infty$ | Neumann, except for the Green's charge |

The last lines say there is no charge lying in the z plane outside the disk except for the Green's charge.

Comment: I have been wondering how the dual integral equation situation was going to appear here. I now am stating the mixed BC at the same time I am providing a guess at the Smythian atomic form. This is pretty close to what Jackson does for the charged disk. We know that $\partial_z C_{kz} = -kS_{kz} = 0$ at z = 0. This makes it appear that the two Neumann region conditions are already met (away from the Green's charge). The problem is then done if we can find $c_m(k)$ such that $V_1(z=0,\rho,\varphi) = 0$, which is to say,

$$\Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk \{ (2-\delta_{m,0}) (q/\pi^2\varepsilon) K_m(kb) I_m(k\rho) + c_m(k) K_m(ka) K_m(kb) I_m(k\rho)\} = 0$$

**Aside on Neumann's Number or Factor:** Various papers refer to $\varepsilon_m \equiv 2-\delta_{m,0}$ as "Neumann's number". I had to dig very hard to find the origin of this name. If you go to Watson's Bessel function treatise, you find

In view of the fact that the first terms in (1) and (3) are not formed according to the same law as the other terms, it is convenient to introduce *Neumann's factor*‡ $\epsilon_n$, which is defined to be equal to 2 when $n$ is not zero, and to be equal to 1 when $n$ is zero. The employment of this factor, which

* *Journal für Math.* xv. (1836), p. 12. [*Ges. Math. Werke*, vi. (1891), p. 101.]
† *Neueste Schriften der Naturf. Ges. in Danzig*, v. (1855), p. 2.
‡ Neumann, *Theorie der Bessel'schen Functionen* (Leipzig, 1867), p. 7.

So Watson calls it a Neumann's factor and gives an actual reference, last line above, which I look up:

NEUMANN, CARL GOTTFRIED†.

*Allgemeine Lösung des Problems über den stationären Temperaturzustand eines homogenen Körpers, welcher von zwei nicht concentrischen Kugelflächen begrenzt wird* (Halle, 1862).

Ueber das Gleichgewicht der Wärme und das der Electricität in einem Körper, welcher von zwei nicht concentrischen Kugelflächen begrenzt wird (1862). *Journal für Math.* LXII. (1863), pp. 36—49.

Ueber die Entwickelung beliebig gegebener Functionen nach den *Bessel*schen Functionen (March 28, 1867). *Journal für Math.* LXVII. (1867), pp. 310—314.

*Theorie der Bessel'schen Functionen. Ein Analogon zur Theorie der Kugelfunctionen* (Leipzig, 1867).

Ueber die Entwicklungen einer Function nach Quadraten und Producten der Fourier-Bessel'schen Functionen (Nov. 1869.) *Leipziger Berichte*, XXI. (1869), pp. 221—256. [*Math. Ann.* III. (1871), pp. 581—610.]

Ueber Producte und Quadrate der Bessel'schen Functionen. *Math. Ann.* II. (1870), p. 192.

*Über die nach Kreis-, Kugel- und Cylinder-functionen fortschreitenden Entwickelungen* (Leipzig, 1881).

Ueber gewisse particulare Integrale der Differentialgleichung $\Delta F' = F$ insbesondere über die Entwickelung dieser particularen Integrale nach Kugelfunctionen (March 8, 1886). *Leipziger Berichte*, XXXVIII. (1886), pp. 75—82.

*** This paper contains some Tables of Bessel functions which were computed by B. A. Smith.**
**† See also under Schläfli.**

So yes, it is our Bessel guy Carl involved here! On page 32 of Bateman II we see the $\varepsilon_n$ symbol also in use, it appears in the $J_n$ expansion of 1/(x-y). Does not appear in W&W. I see now that the web has I would say that Watson invented this symbol $\varepsilon_n$ just as a convenience, and did so in his 1922 first edition, though I have the second edition.

**... Resume Main Flow**

We can write this out as two conditions, the first line for m = 0, the second for m ≠ 0 :

$$\int_0^\infty dk\, I_0(k\rho)\, K_0(kb)\{ (q/\pi^2\varepsilon) \quad + c_0(k)\, K_0(ka) \} = 0$$

$$\int_0^\infty dk\, I_m(k\rho)\, K_m(kb)\{ (2)\, (q/\pi^2\varepsilon) + c_m(k)\, K_m(ka) \} = 0$$

These equations seem to have simple solutions:

$$c_m(k) = -\,(2)\,(q/\pi^2\varepsilon)/\, K_m(ka) \qquad m \neq 0$$

$$c_m(k) = -\,(1)\,(q/\pi^2\varepsilon)/\, K_m(ka) \qquad m = 0$$

which we can combine using Neumann's factor $\varepsilon_n$

$$c_m(k) = -\,\varepsilon_m(q/\pi^2\varepsilon)/\, K_m(ka)$$

Now let's back up and see where we are. I have this Smythian Form

$$V_1(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ b_m(k)\, I_m(ka)\, K_m(kb)\, I_m(k\rho) + c_m(k)\, K_m(ka)\, K_m(kb)\, I_m(k\rho)\}$$

$$V_2(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ b_m(k)\, I_m(ka)\, K_m(kb)\, I_m(k\rho) + c_m(k)\, K_m(k\rho)\, K_m(kb)\, I_m(ka) \}$$

$$V_3(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ b_m(k)\, I_m(ka)\, K_m(k\rho)\, I_m(kb) + c_m(k)\, K_m(k\rho)\, K_m(kb)\, I_m(ka) \}$$

and I think I have been able to compute the coefficients:

$b_m(k)\, I_m(ka) = (2-\delta_{m,0})\,(q/\pi^2\varepsilon) = \varepsilon_m\,(q/\pi^2\varepsilon)$ // maybe add factor of 1/2 see above

$c_m(k)\, K_m(ka) = -\varepsilon_m\,(q/\pi^2\varepsilon)$

Of special interest to me is the potential in the cylinder containing the disk. We then have

$$V_1(z,\rho,\varphi) = \varepsilon_m\,(q/\pi^2\varepsilon)\, \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ K_m(kb)\, I_m(k\rho) - K_m(kb)\, I_m(k\rho)\}$$

but oops, the {...} = 0 so the potential $V_1$ vanishes everywhere in this cylinder, not just on the disk. If the factor of 1/2 from δ(z) is supposed to be used, we get instead:

$$V_1(z,\rho,\varphi) = -(1/2)\varepsilon_m\,(q/\pi^2\varepsilon)\, \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz}\, K_m(kb)\, I_m(k\rho)$$

But then if we evaluate at z = 0 we must get

$$\Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, K_m(kb)\, I_m(k\rho) \quad \Rightarrow \quad \int_0^{\infty} dk\, K_m(kb)\, I_m(k\rho) = 0$$

which seems pretty unlikely.

I am not sure what my conclusion is at this point, but I feel the current document is a reasonable attempt on the disk and merits more study. [ 4.21.10]