Disk Green Attempt #3 PhL 4.22.10

This is another cos(kz) attempt.

1. Preliminaries concerning Region 1. ..... **1**

2. A Smythian three-region form **2**

3. The pillbox analysis at ρ = b: determination of $b_m(k)$ **3**

4. How do we find the remaining $c_m(k)$ coefficients? **4**

5. The "pillbox" analysis at the 1/2 region boundary: **5**

(5a) stay away from z = 0, it tries to make $c_m(k) = 0$ 5

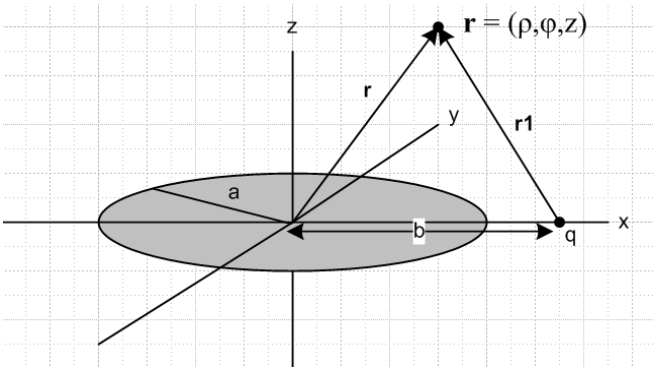
(5b) work at z = 0, it tries to make $c_m(k) = -(q\varepsilon_m/\pi^2\varepsilon)/ K_m(ka)$ 7

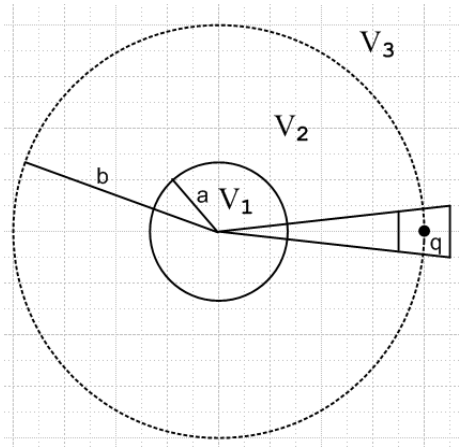
Summary of Section 5: 10

6. Try to obtain a dual integral equation. **10**

1. Preliminaries concerning Region 1.

First we have a picture showing the geometry and the three regions:





One idea for a region 1 expansion is this:

$$V_1(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk \cos(kz) A_m(k) I_m(k\rho) \qquad (1.1)$$

A requirement on this "form" is that the potential vanish on the disk, so we must have

$$\Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk\, A_m(k) I_m(k\rho) = 0$$

which in each partial wave becomes

$$\int_0^\infty dk\, A_m(k)\, I_m(k\rho) = 0 \qquad 0 \le \rho \le a \qquad (1.2)$$

This looks a little like Jackson p 91 3.173, so we don't give up at once. We selected I_m go get finite behavior at $\rho = 0$. It seems at least *possible* that as z increases, cos(kz) moves faster and chops up the integral causing $V_1 \to 0$. I think this is a viable Smythian form for the central cylinder region. The only other choice would be this: (I throw in a factor of k for reasons seen below, just definition of B)

$$U_1(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^\infty dk\, k\, e^{-k|z|} B_m(k)\, J_m(k\rho) \qquad (1.3)$$

and as with the above, this would lead to the requirement that

$$\int_0^\infty dk\, k\, B_m(k)\, J_m(k\rho) = 0 \qquad 0 \le \rho \le a \qquad (*) \qquad (1.4)$$

I cannot think of any other possible "forms" for this region 1. The only other choice would be that you cannot represent the potential in cylindrical atoms, as in the spherical case, but I think you can. The reason is that, unlike powers r^n, the functions $J_m(k\rho)$ may form an appropriate orthogonal set. We would *like* to use this Hankel Transform:

$$\int_0^\infty d\rho\, \rho J_m(k'\rho)\, J_m(k\rho) = \delta(k-k')/k \qquad \text{orthogonality}$$

$$\int_0^\infty dk\, k J_m(k\rho')\, J_m(k\rho) = \delta(\rho-\rho')/\rho \qquad \text{completeness} \qquad (1.5)$$

We would like to take (*) above and apply $\int_0^\infty d\rho\, \rho J_m(k'\rho)$ to both sides. IF (*) were valid on ρ in $(0,\infty)$, this is what would happen:

$$0 = \int_0^\infty d\rho\, \rho J_m(k'\rho) \int_0^\infty dk\, k\, B_m(k)\, J_m(k\rho) = \int_0^\infty dk\, k\, B_m(k)\, \delta(k-k')/k = B_m(k')$$

and we would conclude that our coefficient was 0. Although perhaps bad news, at least this is a well defined answer. Our problem of course is that we DON'T have (*) being true for all ρ. The upshot is that equation (*) could be true, but we *don't have enough information* to be able to invert it to find $B_m(k)$. So we have to regard (*) as a *condition* on $B_m(k)$, a boundary condition perhaps, but (*) does not fully determine $B_m(k)$. We cannot solve this problem, then, without information on what is happening in the other regions, I think that is the point here.

2. A Smythian three-region form

I argue elsewhere for the following form, using the cos(kz) model as in (1.1) above,

$$V_1(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^\infty dk\, C_{kz} \{ b_m(k)\, I_m(ka) K_m(kb)\, I_m(k\rho) + c_m(k)\, K_m(ka) K_m(kb)\, I_m(k\rho) \}$$

$$V_2(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ b_m(k)\, I_m(ka)\, K_m(kb)\, I_m(k\rho) + c_m(k)\, K_m(k\rho)\, K_m(kb)\, I_m(ka) \}$$

$$V_3(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ b_m(k)\, I_m(ka)\, K_m(k\rho)\, I_m(kb) + c_m(k)\, K_m(k\rho)\, K_m(kb)\, I_m(ka) \} \quad (2.1)$$

Although these things look a little complicated, it is really just this,

$$V_1(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk \cos(kz) A_m(k)\, I_m(k\rho)$$

$$V_2(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk \cos(kz) [\, B_m(k)\, I_m(k\rho) + C_m(k)\, K_m(k\rho)]$$

$$V_3(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk \cos(kz)\, D_m(k)\, K_m(k\rho) \quad (2.2)$$

but we have built in the requirements that the potentials match at the 1/2 and 2/3 cylinder boundaries. This matching is "manifestly clear" looking at the first three lines above. The 1/2 boundary is $\rho = a$, and the 2/3 boundary is $\rho = b$. So this form really is Smythian in its nature. The forms (2.2) are forced on us by the requirements on the Bessel functions at $\rho = 0$ and $\rho = \infty$.

3. The pillbox analysis at $\rho = b$: determination of $b_m(k)$.

I did all the details in my "other doc". I set $\partial_\rho V_2 - \partial_\rho V_3 = (q/\varepsilon)(1/b)\, \delta(\varphi)\delta(z)$ and compute the LHS using the forms above. The result is this:

$$(q/\varepsilon)\delta(\varphi)\delta(z) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz}\, b_m(k)\, I_m(ka)$$

I then apply $\int_0^{2\pi} d\varphi\, C_{m'\varphi}$ to both sides and this gives:

$$(q/\varepsilon)\delta(z) = 2\pi/(2-\delta_{m,0}) \int_0^{\infty} dk \cos(kz)\, b_m(k)\, I_m(ka) \qquad \varepsilon_m = (2-\delta_{m,0})$$

At this point, I would like to use the Fourier Cosine Transform outlined in "other doc". This analysis starts off dealing with z in range $(-\infty,\infty)$ but then we restrict to functions f(z) which are even in z and we can then fold the integral. The completeness relation is written this way:

$$\int_0^{\infty} dk \cos(kz) \cos(kz') = (\pi/2)\delta(z-z') \qquad \text{// completeness}$$

and if we set $z' = 0$ this says

$$\int_0^{\infty} dk \cos(kz) = (\pi/2)\delta(z)$$

So, let us use this expansion on the left of the above to get

$$(q/\varepsilon)\,(2/\pi)\ \int_0^\infty dk\ \cos(kz)\ =\ 2\pi/(2-\delta_{m,0})\ \int_0^\infty dk\ \cos(kz)\ b_m(k)\ I_m(ka)$$

We would argue that each "k spectral component" must match, so we get

$$(q/\varepsilon)\,(2/\pi) = 2\pi/(2-\delta_{m,0})\ b_m(k)\ I_m(ka)\ = (2\pi/\varepsilon_m)\ b_m(k)\ I_m(ka)$$

$$b_m(k)\ =\ (q/\varepsilon)\,(2/\pi)\,(\varepsilon_m/2\pi)\ /\ I_m(ka)\ =\ (q\varepsilon_m/\pi^2\varepsilon)/\ I_m(ka)$$

Therefore, it does seem that our "usual pillbox analysis" as used in Smythian form work has yielded up one of our two coefficients. The other coefficient could still be anything because it cancels out in this pillbox analysis, a standard fact we have seen many times before.

4. How do we find the remaining $c_m(k)$ coefficients?

Our Smythian forms with the $b_m(k)$ installed now look like this (no red or blue this time)

$$V_1(z,\rho,\varphi)\ =\ \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^\infty dk\ C_{kz}\ \{\ (q\varepsilon_m/\pi^2\varepsilon)\ K_m(kb)\ I_m(k\rho)\ +\ c_m(k)\ K_m(ka)\ K_m(kb)\ I_m(k\rho)\ \}$$

$$V_2(z,\rho,\varphi)\ =\ \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^\infty dk\ C_{kz}\ \{\ (q\varepsilon_m/\pi^2\varepsilon)\ K_m(kb)\ I_m(k\rho)\ +\ c_m(k)\ K_m(k\rho)\ K_m(kb)\ I_m(ka)\ \}$$

$$V_3(z,\rho,\varphi)\ =\ \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^\infty dk\ C_{kz}\ \{\ (q\varepsilon_m/\pi^2\varepsilon)\ K_m(k\rho)\ I_m(kb)\ +\ c_m(k)\ K_m(k\rho)\ K_m(kb)\ I_m(ka)\ \}$$

We should point out that these forms have "manifest" matching of the potentials at the 1/2 and 2/3 boundaries. But what about matching the electric fields at these boundaries? We know that anywhere on these two cylinder boundaries, at least away from the grounded disk, the E field has to be continuous at the boundary in all its components, since there is no charge floating around out there. One would think this requirement would have some significant effect on the above situation. And we also have our condition for potential on the plate, which is this:

$$V_1(0,\rho,\varphi)\ =\ \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^\infty dk\ \{\varepsilon_m\,(q/\pi^2\varepsilon)\ K_m(kb)\ +\ c_m(k)\ K_m(ka)\ K_m(kb)\}\ I_m(k\rho)\ =\ 0 \quad \rho\le a$$

where we now see that $\{\varepsilon_m\,(q/\pi^2\varepsilon)\ K_m(kb)\ +\ c_m(k)\ K_m(ka)\ K_m(kb)\}\ =\ A_m(k)$ of our opening discussion above. We can write (1.2) in this way:

$$\int_0^\infty dk\ \{\varepsilon_m\,(q/\pi^2\varepsilon)\ K_m(kb)\ +\ c_m(k)\ K_m(ka)\ K_m(kb)\}\ I_m(k\rho)\ =\ 0 \qquad (*)$$

As already noted in our first section, this condition is not enough to determine in this case $c_m(k)$, but it is a condition we have to make sure is satisfied.

So, what about the electric field matching stuff? We can think of the E field as having components related to $\partial_\varphi V$, $\partial_z V$ and $\partial_\rho V$ with some scale factors involved. If we just stare at the forms above, and in

particular, at the $C_{m\varphi}$ and C_{kz} factors, we can see what will happen if we require continuity of $\partial_\varphi V$ or $\partial_z V$: the condition of continuity will be the same as the condition for V to be continuous! Thus, continuity of the E field in these two directions is already built in. So we only need worry about $\partial_\rho V$. But at the 2/3 boundary, we already "worried about $\partial_\rho V$" when we did our pillbox analysis. So what remains is the 1/2 boundary. We must take care, because the metal disk rim lies *on* this boundary, and we expect this rim to have perhaps an infinite charge density.

Our hope then is that we can study this 1/2 $\partial_\rho V$ matching condition and maybe it will yield a condition which, combined with (*) above, will give us perhaps some "dual integral equations" which then will fully determine the $c_m(k)$, sort of in analogy with Jackson's charge disk problem.

So let's look into the 1/2 matching condition for $\partial_\rho V$.

5. The "pillbox" analysis at the 1/2 region boundary:

(5a) stay away from z = 0, it tries to make $c_m(k) = 0$

As long as we *stay away from* z=0 where the disk is located, we expect this to be true:

$$\partial_\rho V_1 - \partial_\rho V_2 = 0 \qquad z \neq 0 \qquad \text{at } \rho = a$$

From above we have

$$V_1(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ (q\varepsilon_m/\pi^2\varepsilon)\, K_m(kb)\, I_m(k\rho) + c_m(k)\, K_m(ka)\, K_m(kb)\, I_m(k\rho)\}$$

$$V_2(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ (q\varepsilon_m/\pi^2\varepsilon)\, K_m(kb)\, I_m(k\rho) + c_m(k)\, K_m(k\rho)\, K_m(kb)\, I_m(ka) \}$$

$$\partial_\rho V_1(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, k\, C_{kz} \{ (q\varepsilon_m/\pi^2\varepsilon)\, K_m(kb)\, I_m'(k\rho) + c_m(k)\, K_m(ka)\, K_m(kb)\, I_m'(k\rho)\}$$

$$\partial_\rho V_2(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, k\, C_{kz} \{ (q\varepsilon_m/\pi^2\varepsilon)\, K_m(kb)\, I_m'(k\rho) + c_m(k)\, K_m'(k\rho)\, K_m(kb)\, I_m(ka) \}$$

$$\partial_\rho V_1(z,a,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, k\, C_{kz} \{ (q\varepsilon_m/\pi^2\varepsilon)\, K_m(kb)\, I_m'(ka) + c_m(k)\, K_m(ka)\, K_m(kb)\, I_m'(ka)\}$$

$$\partial_\rho V_2(z,a,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, k\, C_{kz} \{ (q\varepsilon_m/\pi^2\varepsilon)\, K_m(kb)\, I_m'(ka) + c_m(k)\, K_m'(ka)\, K_m(kb)\, I_m(ka) \}$$

So we want to set the above two lines equal, at least for $z \neq 0$. Certainly the $C_{m\varphi}$ functions are orthogonal so we can get it quickly to this point:

$$\int_0^{\infty} dk\, k\, C_{kz} \{ (q\varepsilon_m/\pi^2\varepsilon)\, K_m(kb)\, I_m'(ka) + c_m(k)\, K_m(ka)\, K_m(kb)\, I_m'(ka)\}$$

$$= \int_0^{\infty} dk\, k\, C_{kz} \{ (q\varepsilon_m/\pi^2\varepsilon)\, K_m(kb)\, I_m'(ka) + c_m(k)\, K_m'(ka)\, K_m(kb)\, I_m(ka) \} \qquad (**)$$

At this point, we are *tempted* to use our Fourier Cosine Transform,

$$f(z) = \sqrt{2/\pi} \int_0^\infty dk \cos(kz)\, G(k) \qquad \text{// expansion}$$

$$G(k) = \sqrt{2/\pi} \int_0^\infty dz \cos(kz)\, f(z) \qquad \text{// projection}$$

$$\int_0^\infty dz \cos(kz) \cos(k'z) = (\pi/2)\, \delta(k-k') \qquad \text{// orthogonality}$$

$$\int_0^\infty dk \cos(kz) \cos(kz') = (\pi/2)\delta(z-z') \qquad \text{// completeness}$$

but I am worried that this is not valid because our f(z) is singular at z = 0.

Aside on this issue: What can I say about $\partial_\rho V(0,\rho,\varphi)$ as we approach the disk edge from region 2 ? If I sit just outside the disk edge in the z = 0 plane, I certainly expect to see some radial electric field component. In the charged disk problem, the potential in the z = 0 plane is given by $V(\rho) \sim \sin^{-1}(1/\rho)$ in region 2 if a=1 so

$$\partial_\rho V(\rho) \sim -1/[\rho\, \mathrm{sqrt}(\rho^2-1)]$$

As we approach ρ = 1 from the outside, we do in fact get $\partial_\rho V(\rho) = \infty$ so $E_\rho = \infty$ just outside the surface. I expect this might happen as well in our current problem. The point is that I expect $\partial_\rho V_2(z,\rho,\varphi)$ to be singular, but only at the location ρ = a and z = 0. This is very reminiscent of Stakgold's surface layers discussion. Meanwhile, it seems pretty clear that $\partial_\rho V(0,\rho,\varphi) = 0$ if we approach the disk edge from the inside, since this is in fact true for all $\rho \le a$. Since $\partial_\rho V_2(z,\rho,\varphi)$ is singular at z = 0 which makes up part of my 1/2 boundary, it seems likely that this singularity will find its way into some of not all the partial waves (maybe only m = 0 as in the charged disk problem). Therefore, when I end up with the condition

$$f(z) = \int_0^\infty dk\, k \cos(kz)\, \{\mathrm{stuff}(k)\} = 0 \qquad z \neq 0$$

I am hesitant to conclude that stuff(k) = 0.

For the moment, let's ignore this "cause for concern" and use our orthogonality anyway to conclude the next step:

$$\{ (q\varepsilon_m/\pi^2\varepsilon)\, K_m(kb)\, I_m'(ka) + c_m(k)\, K_m(ka)\, K_m(kb)\, I_m'(ka)\}$$
$$= \{ (q\varepsilon_m/\pi^2\varepsilon)\, K_m(kb)\, I_m'(ka) + c_m(k)\, K_m'(ka)\, K_m(kb)\, I_m(ka)\}$$

The first terms cancel so this says

$$c_m(k)\, K_m(ka)\, K_m(kb)\, I_m'(ka) = c_m(k)\, K_m'(ka)\, K_m(kb)\, I_m(ka)$$
$$c_m(k)\, K_m(ka)\, I_m'(ka) = c_m(k)\, K_m'(ka)\, I_m(ka)$$
$$c_m(k)\, W[K_m(ka), I_m(ka)] = 0$$

From our "other doc" we have $W[K_m(ka), I_m(ka)] = 1/(ka)$ so we get

$$c_m(k)/(ka) = 0$$

This is trying to fool us into accepting the solution $c_m(k) = 0$ which of course makes $\partial_\rho V_1 - \partial_\rho V_2 = 0$, but we know this is wrong for the following reason. If $c_m(k) = 0$, then $V_1(z,\rho,\varphi) = V_2(z,\rho,\varphi)$ identically everywhere. In other words, we would have:

$$V_{12}(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ (q\varepsilon_m/\pi^2\varepsilon) K_m(kb) I_m(k\rho) \}$$

This says that the expression on the right gives the potential for $0 \le \rho < b$. If we set $z = 0$ we get

$$V_{12}(0,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk \{ (q\varepsilon_m/\pi^2\varepsilon) K_m(kb) I_m(k\rho) \}$$

We know this has to be 0 for $\rho \le a$, so we have

$$0 = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk \{ (q\varepsilon_m/\pi^2\varepsilon) K_m(kb) I_m(k\rho) \} \qquad \rho \le a$$

Orthogonality of the $C_{m\varphi}$ then tells us that

$$\int_0^{\infty} dk\, K_m(kb) I_m(k\rho) = 0 \qquad \rho \le a \qquad (*)$$

But how can this be true where b is a free parameter with $b > a$? I think in fact this integral diverges at the high end. From Jackson page 75 we have

$$K_m(kb) \to (kb)^{-1/2} \exp(-kb)$$
$$I_m(k\rho) \to (k\rho)^{-1/2} \exp(+k\rho)$$

$$K_m(kb)\, I_m(k\rho) \to (1/k) \exp[-k(b-\rho)] \to 0$$

so I was wrong, the integral does converge. I cannot "disprove" (*) above because this integral does not appear in GR. Well, here is an easy disproof. A&S page 374 plot I_o and K_o and both are positive for all real positive argument. They state this to be true for both I_m and K_m for any $m > -1$. Therefore, my integrand above is always positive, so how can the integral of a positive definite integrand be 0! QED.

So this confirms that the $c_m(k) = 0$ solution is incorrect, as suspected.

(5b) work at z = 0, it tries to make $c_m(k) = -(q\varepsilon_m/\pi^2\varepsilon)/ K_m(ka)$

The ρ-radial E field will be 0 along the disk as you approach the edge (otherwise current would flow). So $\partial_\rho V = 0$ as you approach the edge at $z = 0$ from the inside, which is to say, in region 1. What does this fact tell us in terms of our form in region 1?

$$V_1(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ (q\varepsilon_m/\pi^2\varepsilon)\, K_m(kb)\, I_m(k\rho) + c_m(k)\, K_m(ka)\, K_m(kb)\, I_m(k\rho)\}$$

$$\partial_\rho V_1(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, k\, C_{kz} \{ (q\varepsilon_m/\pi^2\varepsilon)\, K_m(kb)\, I_m'(k\rho) + c_m(k)\, K_m(ka)\, K_m(kb)\, I_m'(k\rho)\}$$

$$\partial_\rho V_1(z,a,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, k\, C_{kz} \{ (q\varepsilon_m/\pi^2\varepsilon)\, K_m(kb)\, I_m'(ka) + c_m(k)\, K_m(ka)\, K_m(kb)\, I_m'(ka)\}$$

$$\partial_\rho V_1(0,a,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, k \{ (q\varepsilon_m/\pi^2\varepsilon)\, K_m(kb)\, I_m'(ka) + c_m(k)\, K_m(ka)\, K_m(kb)\, I_m'(ka)\}$$

Our "fact" is that $\partial_\rho V_1(0,a,\varphi) = 0$. So we have:

$$\Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, k \{ (q\varepsilon_m/\pi^2\varepsilon)\, K_m(kb)\, I_m'(ka) + c_m(k)\, K_m(ka)\, K_m(kb)\, I_m'(ka)\} = 0$$

As usual, we can apply the φ orthogonality and get

$$\int_0^{\infty} dk\, k \{ (q\varepsilon_m/\pi^2\varepsilon)\, K_m(kb)\, I_m'(ka) + c_m(k)\, K_m(ka)\, K_m(kb)\, I_m'(ka)\} = 0$$

$$\int_0^{\infty} dk\, k \{ (q\varepsilon_m/\pi^2\varepsilon) + c_m(k)\, K_m(ka) \}\, K_m(kb)\, I_m'(ka) = 0$$

This is a *new condition* I have not seen before. One way to make it be true is this:

$$\{ (q\varepsilon_m/\pi^2\varepsilon) + c_m(k)\, K_m(ka) \} = 0$$

$$c_m(k) = -(q\varepsilon_m/\pi^2\varepsilon)/\, K_m(ka)$$

But I am suspicious of this solution as well. If we install this into our region 1 potential we get

$$V_1(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ (q\varepsilon_m/\pi^2\varepsilon)\, K_m(kb)\, I_m(k\rho) + c_m(k)\, K_m(ka)\, K_m(kb)\, I_m(k\rho)\}$$

$$V_1(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ (q\varepsilon_m/\pi^2\varepsilon)\, K_m(kb)\, I_m(k\rho) - (q\varepsilon_m/\pi^2\varepsilon)\, K_m(kb)\, I_m(k\rho)\} \equiv 0$$

But we know that the potential does not vanish everywhere in region 1, so this is a bad "way" to make the above condition be true. This is the same "solution" for $c_m(k)$ that makes $A_m(k) \equiv 0$ and we say this same problem arise in our preliminary section above. So we seek some OTHER solution $c_m(k)$ of our conditions:

$$\int_0^{\infty} dk\, \{\varepsilon_m\, (q/\pi^2\varepsilon)\, K_m(kb) + c_m(k)\, K_m(ka)\, K_m(kb)\}\, I_m(k\rho) = 0 \qquad \text{// V=0 on disk}$$

$$\int_0^{\infty} dk\, k \{ (q\varepsilon_m/\pi^2\varepsilon) + c_m(k)\, K_m(ka) \}\, K_m(kb)\, I_m'(ka) = 0 \qquad \text{// } \partial_\rho V(\rho=a-) = 0$$

which we can rewrite as

$$\int_0^\infty dk\, K_m(kb)\{ (q\, \varepsilon_m /\pi^2\varepsilon) + c_m(k)\, K_m(ka) \}\, I_m(k\rho) = 0 \qquad \text{// V=0 on disk} \quad \rho \le a$$

$$\int_0^\infty dk\, K_m(kb) \{ (q\varepsilon_m/\pi^2\varepsilon) + c_m(k)\, K_m(ka) \}\, k\, I_m'(ka) = 0 \qquad \text{// } \partial_\rho V(\rho=a-) = 0$$

Notice that the same {} factor appears on both lines. Recall now our earlier definition:

$$K_m(kb)\{ (q\, \varepsilon_m /\pi^2\varepsilon) + c_m(k)\, K_m(ka) \} = A_m(k)$$

so our two conditions are really these:

$$\int_0^\infty dk\, A_m(k)\, I_m(k\rho) = 0 \qquad \text{// V=0 on disk}$$

$$\int_0^\infty dk\, A_m(k)\, k\, I_m'(ka) = 0 \qquad \text{// } \partial_\rho V(\rho=a-) = 0$$

I think I can make a more general statement of the second line here, but I need to back up and redo things:

The radial E field is not just 0 at the edge of the disk, it is zero for all $\rho \le a$ in region 1. So we have

$$V_1(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^\infty dk\, C_{kz} \{ (q\varepsilon_m/\pi^2\varepsilon)\, K_m(kb)\, I_m(k\rho) + c_m(k)\, K_m(ka)\, K_m(kb)\, I_m(k\rho)\}$$

$$\partial_\rho V_1(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^\infty dk\, k\, C_{kz} \{ (q\varepsilon_m/\pi^2\varepsilon)\, K_m(kb)\, I_m'(k\rho) + c_m(k)\, K_m(ka)\, K_m(kb)\, I_m'(k\rho)\}$$

$$\partial_\rho V_1(0,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^\infty dk\, k \{ (q\varepsilon_m/\pi^2\varepsilon)\, K_m(kb)\, I_m'(k\rho) + c_m(k)\, K_m(ka)\, K_m(kb)\, I_m'(k\rho)\}$$

So our statement really is this:

$$\Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^\infty dk\, k \{ (q\varepsilon_m/\pi^2\varepsilon)\, K_m(kb)\, I_m'(k\rho) + c_m(k)\, K_m(ka)\, K_m(kb)\, I_m'(k\rho)\} = 0 \qquad \rho \le a$$

which becomes

$$\int_0^\infty dk\, K_m(kb) \{ (q\varepsilon_m/\pi^2\varepsilon) + c_m(k)\, K_m(ka) \}\, k\, I_m'(k\rho) = 0 \qquad \rho \le a$$

$$\int_0^\infty dk\, A_m(k)\, k\, I_m'(k\rho) = 0 \qquad \rho \le a$$

So now we can restate our two conditions:

$$\int_0^\infty dk\, A_m(k)\, I_m(k\rho) = 0 \qquad \text{/ V=0 on disk} \qquad \rho \le a$$

$$\int_0^\infty dk\, A_m(k)\, k\, I_m'(k\rho) = 0 \qquad // \partial_\rho V = 0 \text{ on disk} \qquad \rho \le a$$

These conditions bear at least some resemblance to Jackson 3.173 p 91, but they are on the same range of ρ, a big difference. So these two integral equations do not fall into the "dual integral equation" class as discussed in Bateman at the end of the Bessel section.

Suppose we go way back and just look again at our region 1 all by itself:

$$V_1(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^\infty dk \cos(kz) A_m(k)\, I_m(k\rho) \qquad (1.1)$$

Once we postulate this "form" for region 1, we arrive immediately at our two conditions above, there is no need to think about any other "regions".

Imagine these "vectors", where I suppress the bystander label m:

$$A_k = A_m(k) \qquad B_k = I_m(k\rho) \qquad C_k = k\, I_m'(k\rho)$$

Then we have our two conditions as

$$\mathbf{A} \bullet \mathbf{B} = 0$$

$$\mathbf{A} \bullet \mathbf{C} = 0$$

We are saying that, in our infinite dimensional space of k, the vector **A** is perpendicular to both **B** and **C**. But I would think there would be lots of vectors **A** that satisfy these conditions. If N = 3, then maybe only one solution, but not if N = 4.

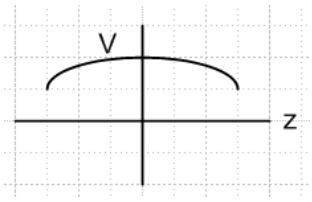
Summary of Section 5: I am unable to come up with a believable result for coefficient $c_m(k)$. If I look at the 1/2 boundary away from z = 0, and I do "bad math", exact matching on the cylinder could be achieved with $c_m(k) = 0$. But this says that regions 1 and 1 have the exact same "form" and this quickly leads to a contradiction, a positive integral being equal to 0. On the other hand, if I look at z = 0 and do another form of "bad math", I say that $A_m(k) = 0$ (by saying $c_m(k) = -(q\varepsilon_m/\pi^2\varepsilon)/ K_m(ka)$) and this then says that the potential is identically 0 in all of region 1, another nonsense result. Doing "good math", I am able to write two integral conditions for $A_m(k)$, but I don't think these are sufficient to determine $A_m(k)$, and they do not form a dual integral equation.

I note in passing that in all my Smythe problems, I never separated out the potential of the induced charge from that of the point charge. They were always treated together, as I have done here.

6. Try to obtain a dual integral equation.

First, here is some quotation from our attempt #1 doc:

"If we lower our point **r** onto the z = 0 plane outside the disk, we know that the potential will have this shape as we go up and down in z:



We don't know it is cupping down as I have drawn it, but we know that (1) V(z=0) = finite; (2) V(z) is symmetrical about z = 0. Therefore, we know that $\partial_z V = 0$ at z = 0, just as in Jackson's problem. But on the z = 0 plane, the quantity $\partial_z V$ is proportional to the charge density there which is of course zero (except near the Green's charge). So just as in Jackson's problem, we have these conditions:

$V(\rho,\varphi,0) = V_1$	$\rho \le a$	
$\partial_z V(\rho,\varphi,0) = 0$	$\rho > a$	but away from the point charge "

The idea here is that we know from symmetry alone that $\partial_z V = 0$ in the z = 0 plane outside the disk, except for the Green's charge, and this is very similar to what Jackson did. Later we might move the Green's charge off the z = 0 plane just to make this claim stronger. So in our triple model, we have to say something like this:

$V_1(0,\rho,\varphi) = 0$	$\rho \le a$	Dirichlet
$\partial_z V_2(0,\rho,\varphi) = 0$	$a < \rho \le b$	Neumann
$\partial_z V_3(0,\rho,\varphi) = 0$	$b < \rho < \infty$	Neumann, except for the Green's charge

With our forms above, where does this take us?

$$V_1(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ (q\varepsilon_m/\pi^2\varepsilon)\, K_m(kb)\, I_m(k\rho) + c_m(k)\, K_m(ka)\, K_m(kb)\, I_m(k\rho) \}$$

$$V_2(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ (q\varepsilon_m/\pi^2\varepsilon)\, K_m(kb)\, I_m(k\rho) + c_m(k)\, K_m(k\rho)\, K_m(kb)\, I_m(ka) \}$$

$$V_3(z,\rho,\varphi) = \Sigma_{m=0}^{\infty} C_{m\varphi} \int_0^{\infty} dk\, C_{kz} \{ (q\varepsilon_m/\pi^2\varepsilon)\, K_m(k\rho)\, I_m(kb) + c_m(k)\, K_m(k\rho)\, K_m(kb)\, I_m(ka) \}$$

We know that $\partial_z C_{kz} = -kS_{kz}$ so we find that $\partial_z V_2(0,\rho,\varphi) = 0$ automatically as part of our form, and the same for $V_3(z,\rho,\varphi)$. That is to say, $S_{kz} = 0$ at z = 0. So we don't GET any "second" dual integral equation. This fact might be pointing out the "flaw" in our taking a cos(kz) approach in the first place.