Disk Green Attempt #4 PhL 4.22.10

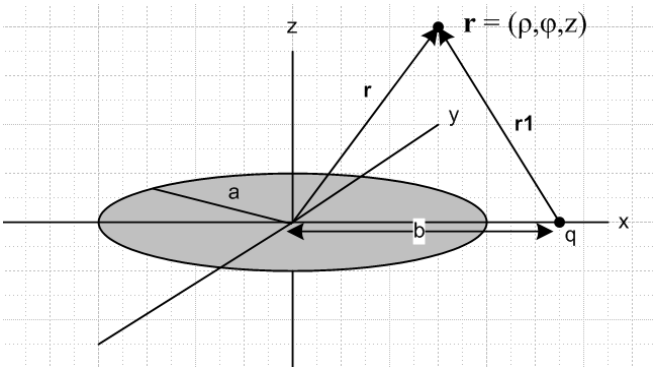
This is an $e^{-k|z|}$ attempt.

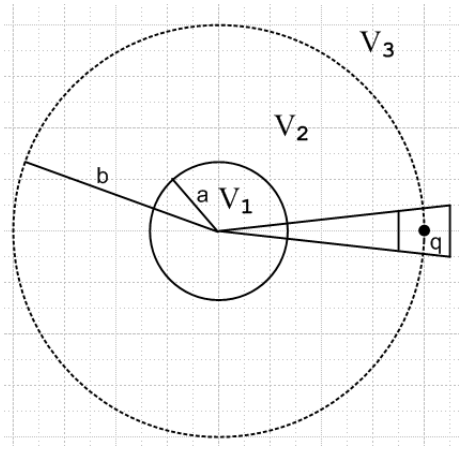
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Let's open this new attempt exactly repeating Section 1 from Attempt #1

1. Setting up the disk problem.

Our problem specifically is the Green's function where the point charge is in the plane of the disk or iris. Let's start with the disk and draw this picture showing where things are:





I have identified three cylindrically shaped regions that we need to worry about, as shown in the cross section on the right. For all three regions, we want $e^{-k|z|}$ and not $e^{-ik|z|}$ for the z dependence, since we know in the ± z direction the potential must go to 0, so we do not want any I or K functions. For our Bessel functions I shall choose

region 1	$J_m(k\rho)$	since good at $\rho = 0$
region 2	$J_m(k\rho)$, $H^{(1)}{}_m(k\rho)$ mixture	general mix
region 3	$H^{(1)}{}_m(k\rho)$	since good at $\rho = \infty$

So we might then write a form for our potential in each region

$$V_1(z,\rho,\varphi) = q/r_1 + \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk\, e^{-k|z|} A_m(k)\, J_m(k\rho)$$

$$V_2(z,\rho,\varphi) = q/r_1 + \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk\, e^{-k|z|} [\, B_m(k)\, J_m(k\rho) + C_m(k)\, H^{(1)}{}_m(k\rho)]$$

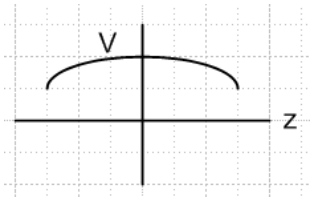
$$V_3(z,\rho,\varphi) = q/r_1 + \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk\, e^{-k|z|}\, D_m(k)\, H^{(1)}{}_m(k\rho)$$

Here I am doing something very different from Smythian work in the past. I am showing the point charge explicitly so that the fancy expressions are really the potential due to the induced charge only.

2. Can we find a dual integral equation situation?

First, here is some quotation from our attempt #1 doc:

"If we lower our point **r** onto the z = 0 plane outside the disk, we know that the potential will have this shape as we go up and down in z:



We don't know it is cupping down as I have drawn it, but we know that (1) V(z=0) = finite; (2) V(z) is symmetrical about z = 0. Therefore, we know that $\partial_z V = 0$ at z = 0, just as in Jackson's problem. But on the z = 0 plane, the quantity $\partial_z V$ is proportional to the charge density there which is of course zero (except near the Green's charge). So just as in Jackson's problem, we have these conditions:

$V(\rho,\varphi,0) = V_1$	$\rho \le a$	
$\partial_z V(\rho,\varphi,0) = 0$	$\rho > a$	but away from the point charge "

The idea here is that we know from symmetry alone that $\partial_z V = 0$ in the z = 0 plane outside the disk, except for the Green's charge, and this is very similar to what Jackson did. Later we might move the Green's charge off the z = 0 plane just to make this claim stronger. So in our triple model, we have to say something like this:

$V_1(0,\rho,\varphi) = 0$	$\rho \le a$	Dirichlet
$\partial_z V_2(0,\rho,\varphi) = 0$	$a < \rho \le b$	Neumann
$\partial_z V_3(0,\rho,\varphi) = 0$	$b < \rho < \infty$	Neumann, except for the Green's charge

With our forms above, where does this take us? Let's think of the $z \ge 0$ region for the moment. Using our forms above, we get:

$$V_1(0,\rho,\varphi) = q/r_1 + \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk\, A_m(k)\, J_m(k\rho) = 0 \qquad \rho \le a$$

$$\partial_z V_2(0,\rho,\varphi) = q\,\partial_z(1/r_1) - \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk\, k\, [\, B_m(k)\, J_m(k\rho) + C_m(k)\, H^{(1)}{}_m(k\rho)] = 0$$

$$a < \rho \le b$$

$$\partial_z V_3(0,\rho,\varphi) = q\,\partial_z(1/r_1) - \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk\, k\, D_m(k)\, H^{(1)}{}_m(k\rho) = 0 \qquad b < \rho < \infty$$

Looking at the above picture, if **r** is in the z plane, then $\partial_z(1/r_1) = 0$ because the r_1 vector has its tip lying in the z plane so a plot of $r_1(z)$ near z = 0 would show zero slope at z = 0. We can this simplify the above to write:

$$V_1(0,\rho,\varphi) = q/r_1 + \Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk\, A_m(k)\, J_m(k\rho) = 0 \qquad \rho \le a$$

$$\partial_z V_2(0,\rho,\varphi) = -\Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk\, k\, [\, B_m(k)\, J_m(k\rho) + C_m(k)\, H^{(1)}{}_m(k\rho)] = 0 \qquad a < \rho \le b$$

$$\partial_z V_3(0,\rho,\varphi) = -\Sigma_{m=0}^{\infty} \cos(m\varphi) \int_0^{\infty} dk\, k\, D_m(k)\, H^{(1)}{}_m(k\rho) = 0 \qquad b < \rho < \infty$$

We know from drawing a triangle that

$$r_1^2 = \rho^2 + b^2 - 2\rho b \cos\varphi$$

and we know from elsewhere how to do the messy cos(mφ) expansion of $1/r_1$. So doing a partial wave analysis on the first line above is a bit messy. We will get a result of this form

$$\int_0^{\infty} dk\, A_m(k)\, J_m(k\rho) = F(\rho,m,b)$$

which is a horrible condition to meet for all $\rho \le a$. But the other two conditions look nicer, and we might like to claim that we have these three conditions,

$$\int_0^{\infty} dk\, A_m(k)\, J_m(k\rho) = F(\rho,m,b) \qquad \rho \le a$$

$$\int_0^{\infty} dk\, k\, [\, B_m(k)\, J_m(k\rho) + C_m(k)\, H^{(1)}{}_m(k\rho)] = 0 \qquad a < \rho \le b$$

$$\int_0^{\infty} dk\, k\, D_m(k)\, H^{(1)}{}_m(k\rho) = 0 \qquad b < \rho < \infty$$

Unlike the Jackson case on page 91, although we have covered the full radial range, we have *different* Bessel functions and *different* coefficient functions in our three pieces. So this is nothing at all like the simple situation Jackson faces in (3.173). I have tried to produce something like Jackson, but this is the best I know how to do.

3. The potential matching condition between region 1 and region 2.

I did this detail in Sec 2 of Attempt #1 and I will just quote the results. Although in #1 I was not "breaking out" the potential of the Green's point charge, it makes no difference in the results here because the potential q/r_1 is the same on both sides of our cylindrical boundary. The first thing we find is this:

$$\int_0^\infty dk\, e^{-k|z|} \{ B_m(k) J_m(ka) + C_m(k) H^{(1)}_m(ka) - A_m(k) J_m(ka) \} = 0$$

Although the $e^{-k|z|}$ are not a complete set, "one way" to make the equation be true is this:

$$A_m(k) J_m(ka) = B_m(k) J_m(ka) + C_m(k) H^{(1)}_m(ka)$$

4. The potential matching condition between region 3 and region 2.

Doing the analogous thing to the above, we will get

$$\int_0^\infty dk\, e^{-k|z|} \{ B_m(k) J_m(kb) + C_m(k) H^{(1)}_m(kb) - D_m(k) H^{(1)}_m(kb) \} = 0$$

and "one way" to make this work is to say

$$D_m(k) H^{(1)}_m(kb) = B_m(k) J_m(kb) + C_m(k) H^{(1)}_m(kb)$$

5. Writing a better Smythian Form.

So let's use the above two facts (which hopefully are true) to construct a Smythian form that is manifestly smooth at the two boundary regions. This is a standard thing I do so just do it. First we have the starting forms

$$V_1(z,\rho,\varphi) = q/r_1 + \Sigma_{m=0}^\infty \cos(m\varphi) \int_0^\infty dk\, e^{-k|z|} A_m(k) J_m(k\rho)$$

$$V_2(z,\rho,\varphi) = q/r_1 + \Sigma_{m=0}^\infty \cos(m\varphi) \int_0^\infty dk\, e^{-k|z|} [B_m(k) J_m(k\rho) + C_m(k) H^{(1)}_m(k\rho)]$$

$$V_3(z,\rho,\varphi) = q/r_1 + \Sigma_{m=0}^\infty \cos(m\varphi) \int_0^\infty dk\, e^{-k|z|} D_m(k) H^{(1)}_m(k\rho)$$

Next, replace the A and D coefficients as per above:

$$V_1(z,\rho,\varphi) \sim \Sigma_{m=0}^\infty C_{m\varphi} \int_0^\infty dk\, e^{-k|z|} [B_m(k) J_m(ka) + C_m(k) H^{(1)}_m(ka)] J_m(k\rho)/ J_m(ka)$$

$$V_2(z,\rho,\varphi) \sim \Sigma_{m=0} C_{m\varphi} \int_0^\infty dk\, e^{-k|z|} [B_m(k) J_m(k\rho) + C_m(k) H^{(1)}_m(k\rho)]$$

$$V_3(z,\rho,\varphi) \sim \Sigma_{m=0}^\infty C_{m\varphi} \int_0^\infty dk\, e^{-k|z|} [B_m(k) J_m(kb) + C_m(k) H^{(1)}_m(kb)] H^{(1)}_m(k\rho)/ H^{(1)}_m(kb)$$

where ~ means I have temporarily omitted the point charge to get things on one line each. Let's focus just on the integrand expressions:

$$[\, B_m(k)\, J_m(ka) \;+ C_m(k)\, H^{(1)}{}_m(ka)]\;\; J_m(k\rho)/\, J_m(ka)$$
$$[\, B_m(k)\, J_m(k\rho) \;+ C_m(k)\, H^{(1)}{}_m(k\rho)]$$
$$[\, B_m(k)\, J_m(kb) \;+ C_m(k)\, H^{(1)}{}_m(kb)]\;\; H^{(1)}{}_m(k\rho)/\, H^{(1)}{}_m(kb)$$

Rewrite these, making no change, as

$$[\, B_m(k)\, J_m(ka)\, H^{(1)}{}_m(kb) \;+ C_m(k)\, H^{(1)}{}_m(ka)\, H^{(1)}{}_m(kb)]\;\; J_m(k\rho)/\, [J_m(ka)\, H^{(1)}{}_m(kb)]$$
$$[\, B_m(k)\, J_m(k\rho)\, J_m(ka) \;+ C_m(k)\, H^{(1)}{}_m(k\rho)\, J_m(ka)]\, H^{(1)}{}_m(kb)/\, [J_m(ka)\, H^{(1)}{}_m(kb)]$$
$$[\, B_m(k)\, J_m(kb)\, J_m(ka) \;+ C_m(k)\, H^{(1)}{}_m(kb)\, J_m(ka)]\;\; H^{(1)}{}_m(k\rho)/\, [J_m(ka)\, H^{(1)}{}_m(kb)]$$

Now make these definitions:

$$B_m(k)/\, [J_m(ka)\, H^{(1)}{}_m(kb)] \;\equiv b_m(k)$$
$$C_m(k)/\, [J_m(ka)\, H^{(1)}{}_m(kb)] \;\equiv c_m(k)$$

Then we have

$$[\, b_m(k)\, J_m(ka)\, H^{(1)}{}_m(kb) \;+ c_m(k)\, H^{(1)}{}_m(ka)\, H^{(1)}{}_m(kb)]\;\; J_m(k\rho)$$
$$[\, b_m(k)\, J_m(k\rho)\, J_m(ka) \;+ c_m(k)\, H^{(1)}{}_m(k\rho)\, J_m(ka)]\, H^{(1)}{}_m(kb)$$
$$[\, b_m(k)\, J_m(kb)\, J_m(ka) \;+ c_m(k)\, H^{(1)}{}_m(kb)\, J_m(ka)]\;\; H^{(1)}{}_m(k\rho)$$

or

$$[\, b_m(k)\, J_m(ka)\, H^{(1)}{}_m(kb)\, J_m(k\rho) \;+ c_m(k)\, H^{(1)}{}_m(ka)\, H^{(1)}{}_m(kb)\, J_m(k\rho)]$$
$$[\, b_m(k)\, J_m(k\rho)\, J_m(ka)\, H^{(1)}{}_m(kb) \;+ c_m(k)\, H^{(1)}{}_m(k\rho)\, J_m(ka)\, H^{(1)}{}_m(kb)]$$
$$[\, b_m(k)\, J_m(kb)\, J_m(ka)\, H^{(1)}{}_m(k\rho) \;+ c_m(k)\, H^{(1)}{}_m(kb)\, J_m(ka)\, H^{(1)}{}_m(k\rho)]$$

or

$$[\, b_m(k)\, J_m(ka)\, H^{(1)}{}_m(kb)\, J_m(k\rho) \;+ c_m(k)\, H^{(1)}{}_m(kb)\, J_m(k\rho)\, H^{(1)}{}_m(ka)]$$
$$[\, b_m(k)\, J_m(ka)\, H^{(1)}{}_m(kb)\, J_m(k\rho) \;+ c_m(k)\, H^{(1)}{}_m(kb)\, J_m(ka)\, H^{(1)}{}_m(k\rho)]$$
$$[\, b_m(k)\, J_m(kb)\, H^{(1)}{}_m(k\rho)\, J_m(ka) \;+ c_m(k)\, H^{(1)}{}_m(kb)\, J_m(ka)\, H^{(1)}{}_m(k\rho)]$$

and this then makes the matching clear by inspection, in Smythe mode. So we then have

$$V_1(z,\rho,\varphi) \;= q/r_1 + \;\; \Sigma_{m=0}^{\infty}\, C_{m\varphi} \int_0^{\infty} dk\; e^{-k|z|} \ast$$
$$[\, b_m(k)\, J_m(ka)\, H^{(1)}{}_m(kb)\, J_m(k\rho) \;+ c_m(k)\, H^{(1)}{}_m(kb)\, J_m(k\rho)\, H^{(1)}{}_m(ka)]$$
$$V_2(z,\rho,\varphi) \;= q/r_1 + \;\; \Sigma_{m=0}^{\infty}\, C_{m\varphi} \int_0^{\infty} dk\; e^{-k|z|} \ast$$
$$[\, b_m(k)\, J_m(ka)\, H^{(1)}{}_m(kb)\, J_m(k\rho) \;+ c_m(k)\, H^{(1)}{}_m(kb)\, J_m(ka)\, H^{(1)}{}_m(k\rho)]$$
$$V_3(z,\rho,\varphi) \;= q/r_1 + \;\; \Sigma_{m=0}^{\infty}\, C_{m\varphi} \int_0^{\infty} dk\; e^{-k|z|} \ast$$
$$[\, b_m(k)\, J_m(kb)\, H^{(1)}{}_m(k\rho)\, J_m(ka) \;+ c_m(k)\, H^{(1)}{}_m(kb)\, J_m(ka)\, H^{(1)}{}_m(k\rho)]$$

I am not sure what I have gained by doing all this, but there it is. We have two sets of coefficients that we don't know, the $b_m(k)$ and $c_m(k)$. Our three conditions from above are now these:

$$\int_0^\infty dk[\, b_m(k)\, J_m(ka)\, H^{(1)}{}_m(kb)\, J_m(k\rho) + c_m(k)\, H^{(1)}{}_m(kb)\, J_m(k\rho)\, H^{(1)}{}_m(ka)] = F(\rho,m,b)$$

$$\int_0^\infty dk\, k\, [\, b_m(k)\, J_m(ka)\, H^{(1)}{}_m(kb)\, J_m(k\rho) + c_m(k)\, H^{(1)}{}_m(kb)\, J_m(ka)\, H^{(1)}{}_m(k\rho)] = 0$$

$$\int_0^\infty dk\, k\, [\, b_m(k)\, J_m(kb)\, H^{(1)}{}_m(k\rho)\, J_m(ka) + c_m(k)\, H^{(1)}{}_m(kb)\, J_m(ka)\, H^{(1)}{}_m(k\rho)] = 0$$

where each equation is valid in its own range of ρ.

6. Is there a Dirichlet problem here?

Except for the fact that the point charge is in the z plane, we could claim that we have a Dirichlet problem and we want to solve it for let's call it U, which is the potential just due to the induced charge on the disk. The closed boundary is the Great Sphere at U = 0, and the disk on which $U = -q/r_1$ so that the total potential will be V = 0 on the disk. This might be flyable, but it just sounds very painful to me.

7. Comments.

Somehow, I am just plain approaching this problem wrong. I just don't have any clear view of how a solution is to be obtained. I wanted to try the "Smythian form" approach because that is what Jackson did for his charged disk problem, and I wanted to do this in cylindricals for the same reason. The Smythian forms might be OK, but I just don't know how to solve for the coefficients. There MUST be a better way to go about this problem. Somehow, this problem is harder than all the Smythe problems he and I have done to date.

So I will refer to this series of Attempts (now 1,2,3,4) as my 3-region cylindrical-coordinates Smythian Form method. I tried pretty hard for several weeks at least to make this method fly, but I could not make it fly.