June 2010 comments re on-axis Green's function for a disk PhL 6.23.10

Overview (1/2 page, written 12.7.10)

In **Part 1** I just comment that my on-axis Green's result for the disk, as obtained in oblates, is a messy sum which I quote. My interest in this on-axis Green's is that it is the inversion of a charged bowl and I was hoping to learn about the bowl by solving this problem.

In **Part 2** I write the potential of disk with unknown σ plus Green's point charge on axis. One term is the point charge, the other I write as integral σdA/R . Since unknown σ depends only on ρ, I can do one of these integrals (the dφ one) and I find that

$$V(\rho,\varphi,z) = 4 \int_0^B \rho' d\rho' \, \sigma(\rho') \, K(\sqrt{2\rho\rho'} / R_1) / R_1 + q \, [\rho^2 + (z-d)^2]^{-1/2} \qquad R_1 = \sqrt{(\rho+\rho')^2 + z^2}$$

I then evaluate this on the disk, and its derivative as well, and end up with some dual integral equations

$$\int_0^B \rho' d\rho' \, \sigma(\rho') \, K[\sqrt{2\rho\rho'} / (\rho+\rho')] / (\rho+\rho') = - q \, [\rho^2 + d^2]^{-1/2} = 0 \qquad \rho \le B$$

$$4 \int_0^B \rho' d\rho' \, \sigma(\rho') \, \partial_z \{K(\sqrt{2\rho\rho'} / R_1) / R_1\}^{z=0} = - q \, \partial_z \{[\rho^2 + (z-d)^2]^{-1/2}\}^{z=0} \qquad \rho > B$$

where the unknown is σ. Might be in Polyanin for all I know. Interesting for several reasons: (1) here we get a dual integral equation not from a Smythian form with some unknown coefficient like $A_m(k)$, but rather it comes just from integral σdA/R as in Stak. (2) I now know the σ on a charged bowl, both surfaces, and could map those back to this on-axis Green's problem. The two sides obviously have different σ, so perhaps the ∂_z object above is different on the two sides -- sides are not mentioned in my doc below!

It has been a while since I thought about this problem.

1. The "early brief comments" indicate that I have in fact solved this problem in the oblate world and got a single-sum solution there for the potential due only to the induced charge on the disk

$$V_i = \frac{q}{2\pi^2 \epsilon a} \sum_{n=0}^{\infty} (4n+1) Q_{2n}(j\zeta_0) Q_{2n}(j\zeta) P_{2n}(\xi)$$

and to this you could add the potential of the on-axis point charge and then you have a statement of the Green's solution. Yes, I have done all that in "On-axis Green's Function for an Oblate Spheroid META.doc". It does give a direct expression for the Green's function in question, it is this (point charge is at ζ_0, 1 which is the on-axis point)

$$g(\zeta, \xi \mid \zeta_0, 1) = q/[4\pi^2 c_1 \varepsilon] \Sigma_{n=0}^{\infty}(2n+1) \ [\{1+(-1)^n\} Q_n(j\zeta_<) - P_n(j\zeta_<)] Q_n(j\zeta_>) \ P_n(\xi)$$

2. I have seen starting with Smythe problem 38 that sometimes a problem has a complex summation answer like the one shown above, and a simple algebraic answer as well. I still wonder if there is not some simple algebraic answer to this on-axis disk Green's function problem. [since it is the inversion of a charged bowl, I think there IS in fact a relatively simple answer 7/10.]

First, recall the distance between two points in cylindrical coordinates (see "A study of (R)-1 expansions in cylindrical coordinates.doc" in cylindricals):

$$\rho^2 + \rho'^2 + (z-z')^2 - 2\rho\rho'\cos(\varphi-\varphi')$$

Our interest will have the primed point on the z' = 0 plane, so we will define

$$R^2 = \rho^2 + \rho'^2 + z^2 - 2\rho\rho'\cos(\varphi-\varphi')$$

and as a special case, if $\varphi = \varphi'$ for our two points, we will have

$$R_0^2 = \rho^2 + \rho'^2 + z^2 - 2\rho\rho' = (\rho-\rho')^2 + z^2$$

and if $\varphi-\varphi' = \pi$ we have

$$R_1^2 = \rho^2 + \rho'^2 + z^2 + 2\rho\rho' = (\rho+\rho')^2 + z^2$$

We can draw a simple picture with a horizontal grounded disk of radius B and a point charge q hovering a distance d above it on the z axis. We then find that

$$V(\rho,\varphi,z) = \int_0^B \rho' d\rho' \ \sigma(\rho') \int_0^{2\pi} d\varphi'/\sqrt{R} + q\,[\rho^2 + (z-d)^2]^{-1/2}$$

where $\sigma(\rho')$ is the unknown charge density on the disk, which due to symmetry, is a function only of the variable ρ' and not also φ'. The φ' integral here is one we have seen very many times before. It comes as no surprise that it is not a function of φ. This is obvious one way because we know the resultant $V(\rho,\varphi,z)$ cannot depend on φ, and another way in that we can just change the integration variable in the integral. My usual way of doing this integral is this:

$$I = \int_0^{2\pi} d\varphi' \ [\ \rho^2+\rho'^2+z^2 - 2\rho\rho'\cos(\varphi-\varphi')]^{-1/2}$$

$$= \int_0^{2\pi} d\varphi' \ [\ \rho^2+\rho'^2+z^2 - 2\rho\rho'\cos(\varphi')]^{-1/2}$$

$$= \int_0^{2\pi} dx \ [\ a - b\cos x]^{-1/2} \qquad a = \rho^2+\rho'^2+z^2 \qquad b = 2\rho\rho'$$

Then I call upon my just-created integrals doc on this integral (found GR error, Wolfram Alpha etc),

$$\int_0^{\pi} dx(1/\sqrt{a \pm b\cos x}) = (2/\sqrt{a+b})\, K(\sqrt{2b/(a+b)}) \qquad a > b > 0$$

so this then tells us that (note that $a+b = R_1^2$)

$$V(\rho,\varphi,z) = 4 \int_0^{B} \rho' d\rho' \, \sigma(\rho') \, K(\sqrt{2\rho\rho'} / R_1) / R_1 + q\,[\rho^2 + (z-d)^2]^{-1/2}$$

where $R_1 = \sqrt{(\rho+\rho')^2 + z^2}$

If we evaluate this on the disk plane z = 0, we get

$$V(\rho,\varphi,0) = 4 \int_0^{B} \rho' d\rho' \, \sigma(\rho') \, K[\sqrt{2\rho\rho'} / (\rho+\rho')] / (\rho+\rho') + q\,[\rho^2 + d^2]^{-1/2}$$

The claim then is that this must be 0 when $\rho \le B$, so we have this condition on the disk

$$\int_0^{B} \rho' d\rho' \, \sigma(\rho') \, K[\sqrt{2\rho\rho'} / (\rho+\rho')] / (\rho+\rho') = -\, q\,[\rho^2 + d^2]^{-1/2} = 0 \qquad \rho \le B$$

Thus, when we try to directly attack our on-axis Green's problem, we are faced with this unpleasant integral equation for the unknown charge density σ'. Our other condition is going to be this:

$$[\partial_z V(\rho,\varphi,z)]|^{z=0} = 0$$

saying there is no charge density outside the disk. This then reads

$$4 \int_0^{B} \rho' d\rho' \, \sigma(\rho') \, \partial_z \{K(\sqrt{2\rho\rho'} / R_1) / R_1\}^{z=0} = -\, q\, \partial_z \{[\rho^2 + (z-d)^2]^{-1/2}\}^{z=0} \qquad \rho > B$$

We are then faced with the infamous "dual integral equations" which are a pain in the neck to do anything with, at least as far as I know. Here they are

$$\int_0^{B} \rho' d\rho' \, \sigma(\rho') \, K[\sqrt{2\rho\rho'} / (\rho+\rho')] / (\rho+\rho') = -\, q\,[\rho^2 + d^2]^{-1/2} = 0 \qquad \rho \le B$$

$$4 \int_0^{B} \rho' d\rho' \, \sigma(\rho') \, \partial_z \{K(\sqrt{2\rho\rho'} / R_1) / R_1\}^{z=0} = -\, q\, \partial_z \{[\rho^2 + (z-d)^2]^{-1/2}\}^{z=0} \qquad \rho > B$$

3. **Conclusion**: Even for the very symmetric on-axis point charge location, I have no non-oblate method of solving for the potential, and charge density on the disk, for the disk Green's function.

4. **Idea**: Interestingly, I do have a detailed expression for the potential and charge distribution for the Green's function for a bowl where the point charge lies on the complementary part of the sphere, and it involves no sums at all. Suppose I take the limit that the bowl becomes very small (but not TOO small). I think a small bowl is a disk, and maybe in this way I can get the result, in fact for an arbitrary location of

the point charge! This is the first time I have had this idea. // But no, this is a no go. It only gives you a disk result in the case that the disk occupies a tiny solid angle from the point charge, just draw a picture.