The charged disk problem PhL 11.5.09

Note: This doc was formally named "potential of half-spherical shell v3.doc", but it has nothing to do with a shell. It has to do with solving the charged DISK problem using the Stak integral equation method. So the doc got a new name today 9.26.10 and moved from bowl world to disk world. [When I wrote this doc, I thought the charged disk and charged bowl were simply related by inversion. I now know this is not the case, because a point charge is involved in the inversion process, see elsewhere.]

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Overview (added 9.15.10, 3 pages)

This is a very old doc which now gets a happy ending.

In **Part A,** I set up to solve the charged disk problem using the basic Stakgold "integral equation method" which says this [then once you find I(ξ), you use same LHS with arbitrary **s** to get V(**s**).]

$$\int_\sigma E(s|\xi)I(\xi)dS_\xi = 1.$$

This means that if we integrate q/R over the entire disk and set **s** to be a point *on* the disk, we should get potential V = 1 for any such point **s**. Normally I would now regard this as one of a pair of dual integral equations, the other saying that $\partial_z V = 0$ outside the disk. But it turns out, as Stakgold of course implies, that this one equation is all you really need to solve the problem.

In **Section 1**, I define polar coordinates (ρ,φ) all in the plane of the disk, meaning z = 0 all the time, so we essentially have a 2D problem to solve:

$$\xi = (\rho_\xi, \varphi_\xi) \qquad \mathbf{s} = (\rho_s, \varphi_s)$$

I show that the above integral equation becomes [I am using the E = 1/4πR Stak convention]

$$\int \rho d\rho d\varphi\, I(\rho,\varphi)\, [\rho^2 + s^2 - 2\,\rho\, s\cos(\varphi)]^{-1/2} = 4\pi$$

But I know I(ρ,φ) does not depend on φ, so this becomes

$$\int \rho d\rho\, I(\rho) \int d\varphi\, [\rho^2 + s^2 - 2\,\rho\, s\cos(\varphi)]^{-1/2} = 4\pi$$

I then show that the dφ integral is equal to $4\,(\rho+s)^{-1}\,K[\,2\sqrt{\rho s}\,/(\rho+s)]$, so I end up with this,

$$\int_0^b \rho d\rho\, I(\rho)\,(\rho+s)^{-1}\,K[\,2\sqrt{\rho s}\,/(\rho+s)] = \pi$$

I had no idea how to solve such an integral equation. I complained to myself that you can often "look up" a differential equation to find its solution, but we have no place really to look up integral equations. [But see Part D!]

In **Section 2** I changed variable to $x = 2\sqrt{\rho s}\,/(\rho+s)$ to get K(x) in the integral. But this significantly complicated the integral equation which then became

$$s^{-1/2} \int_0^{x(b)} dx\ \left(1 \pm \sqrt{1-x^2} \right) / \left(1-x^2 \pm \sqrt{1-x^2}\right) \rho^{3/2}\, I(\rho(x))\, K(x) = \pi$$

So this was not a great thing to do, but I leave it in the notes. As before, I gave up on finding a solution.

In **Part B** I review **Jackson's solution** of the charged disk problem (although I review this as well in at least one other doc). As I wrote these notes, I found and downloaded the Smythe book based on it being a Jackson reference. Thus, this was before I ever talked about "Smythian forms" and "atoms" and such. The Hankel Transform was new, everything was new. I am now an old hand at all this stuff.

In **Part C** I thought I was going to transform Jackson's disk result to a charged bowl result, but I only got about one line into this section when I must have realized this doesn't work by inversion because I neglected the point charge element. This section is at least consistent with the title of this doc! Probably I went off and digressed into the inversion method and never came back.

In **Part D** we have the Big Payoff for our current doc, and I only wrote this section today, 9.15.10. Recall in Section 1 above we arrived at this integral equation for the charge density I in Stak convention,

$$\int_0^b \rho d\rho\, I(\rho)\,(\rho+s)^{-1}\,K[\,2\sqrt{\rho s}\,/(\rho+s)] = \pi$$

Since my original work here, I discovered the Polyanin "handbook of integral equations" which is probably the only such in the world, AND I was able to download it. Today, while just reviewing this doc and writing this overview, I found the following entry in Polyanin:

3.7-18. Kernels Containing Other Special Functions.

97. $$\int_0^a K\left(\frac{2\sqrt{xt}}{x+t}\right)\frac{y(t)\,dt}{x+t} = f(x).$$

Here $K(z) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-z^2t^2)}}$ is the complete elliptic integral of the first kind (see Supplement 11.13-1).

Solution:

$$y(x) = -\frac{4}{\pi^2}\frac{d}{dx}\int_x^a \frac{tF(t)\,dt}{\sqrt{t^2-x^2}}, \qquad F(t) = \frac{d}{dt}\int_0^t \frac{sf(s)\,ds}{\sqrt{t^2-s^2}}.$$

⦿ Reference: P. P. Zabreyko, A. I. Koshelev, et al. (1975).

This is a generalization of my integral equation in which the RHS is a general f(x), whereas my RHS is simply π. It did not then take me very long to find that the solution to my integral equation is

$$I(\rho) = (1/\pi^2)\ 1/\sqrt{b^2-\rho^2}$$

which agrees with Jackson's result when we account for Stak convention and V = 1.

This is, I think, the first time I have *ever* used the Stakgold "integral equation method" to solve any significant problem. In previous attempts, I kept ending up with an insoluble integral equation. Of course the whole "solution" was *finding* the above integral equation solution. The solution has a very similar "look" to those of Sneddon for his *dual* integral equations. I'll bet I could derive the above solution using Sneddon's methods. He never dealt with special functions like K, but I suspect you insert an elegant integral representation for K, perhaps even the defining one. Then everything involves elementary factors of the kind Sneddon dealt with.

The reference for the above entry is this 1975 earlier "handbook" :

Zabreyko, P. P., Koshelev, A. I., et al., *Integral Equations: A Reference Text*, Noordhoff Int. Publ., Leyden, 1975.

I don't see this book on the Russian site. Amazon is $150 and up, and I see no sites. So this was an earlier version of the Polyanin effort. The book sits right now at Marriott QA431 .I513, amazing! I could go take it out today and review it.

A. Potential and charge distribution on an ungrounded metal disk in 3D.

1. Setting up and doing the dφ integral. We might as well start in our usual fashion, setting the constant potential arbitrarily to 1. This is Stakgold notation where E = 1/(4πR). So:

$$\int_\sigma E(s|\xi)I(\xi)dS_\xi \quad = 1$$

where I(ξ) is our unknown charge density. Imagine the picture with the disk in the x-y plane and our observation point is out at an r in spherical coordinates (r,θ,φ). Yes, oblates might make this problem even simpler, but for now let's not get involved in that!

A point on the disk is ξ = (ρ,π/2,φ) with ρ ≤ b, the assumed radius of the disk. So ξ runs over the disk and we have s being some other point on the disk. We have

$$\xi = (\rho_\xi, \varphi_\xi)$$
$$s = (\rho_s, \varphi_s)$$

Law of cosines tell us $R^2 \equiv |s-\xi|^2 = \rho_\xi^2 + \rho_s^2 - 2\,\rho_\xi\,\rho_s \cos(\varphi_s - \varphi_\xi)$. So

$$\int \rho_\xi d\rho_\xi d\varphi_\xi \; I(\xi)\,(1/4\pi)\; [\rho_\xi^2 + \rho_s^2 - 2\,\rho_\xi\,\rho_s \cos(\varphi_s - \varphi_\xi)]^{-1/2} = 1$$

Let's choose $\varphi_s = 0$ so our point x lies on the x axis of the disk. Then

$$\int \rho_\xi d\rho_\xi d\varphi_\xi\, I(\xi)\, [\rho_\xi^2 + \rho_s^2 - 2\,\rho_\xi\,\rho_s \cos(\varphi_\xi)]^{-1/2} = 4\pi$$

To compact down the notation, set $\rho_s = s$, the distance of point s from the origin, and remove ξ subscript on the ξ coordinates

$$\int \rho d\rho d\varphi\, I(\rho,\varphi)\, [\rho^2 + s^2 - 2\,\rho\, s \cos(\varphi)]^{-1/2} = 4\pi$$

Now we know I(ρ,φ), the final charge density on our disk, is azimuthally symmetric, so

$$\int \rho d\rho\, I(\rho) \int d\varphi\, [\rho^2 + s^2 - 2\,\rho\, s \cos(\varphi)]^{-1/2} = 4\pi$$

Now examine the integral

$$I_1 = \int d\varphi\; 1/\sqrt{a - b\cos\varphi} \qquad a = \rho^2 + s^2 \qquad b = 2\rho s$$

you see it is (of course) the same one we encountered in v2 of this effort, and we found there that

$$I_1 = 2 \int_0^{\pi} d\varphi\, [\, a - b\cos(\varphi)]^{-1/2} =$$

$$= 4\, c^{-1/2}\, K(i\sqrt{2b/c}\,) = 4\, c^{-1/2} \sqrt{c/d}\; K(\sqrt{2b/d}\,) = 4\, d^{-1/2}\, K(\sqrt{2b/d}\,)$$

where c = a-b and d = c + 2b = a+b. In our current context, we need only b and d

$$b = 2\rho s \qquad d = a+b = \rho^2 + s^2 + 2\rho s = (\rho+s)^2$$

So it appears that

$$I_1 = 4\,(\rho+s)^{-1}\,K[\,2\sqrt{\rho s}\,/(\rho+s)]$$

If I did all this correctly, then our integral equation is now

$$\int_0^b \rho d\rho\; I(\rho)\,(\rho+s)^{-1}\,K[\,2\sqrt{\rho s}\,/(\rho+s)] = \pi$$

2. What to do next? This is an integral equation, how are we going to solve it? Both ρ and s are in the range 0,b. We need to expand things in a complete set of eigenfunctions on this interval. But we are going to have to face this K function one way or another. Integrals in GR are going to be of K(x) so let's try to change variables to dimensionless x,

$$x = 2\sqrt{\rho s}\,/(\rho+s) \qquad dx = 2\sqrt{s}\;[\,\rho^{1/2}\,(-1)(\rho+s)^{-2} + (1/2)\,\rho^{-1/2}\,(\rho+s)^{-1}]\,d\rho$$

$$= \sqrt{s}\,(\rho+s)^{-2}\,\rho^{-1/2}\;[\,-2\rho\; + (\rho+s)]\,d\rho$$

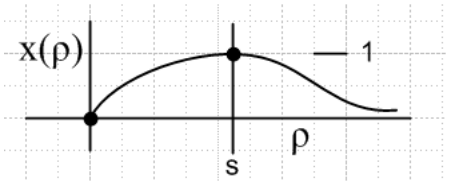
$$= \{\;\sqrt{s}\,(\rho+s)^{-2}\,\rho^{-1/2}\;\;(s-\rho)\;\}\,d\rho$$

$$= \sqrt{s}\,(\rho+s)^{-2}(s-\rho)\,\rho^{-1/2}\,d\rho$$

$$\Rightarrow \qquad d\rho\; = s^{-1/2}\,(\rho+s)^{2}(s-\rho)^{-1}\,\rho^{1/2}\,dx$$

$$\rho d\rho/(\rho+s) = s^{-1/2}\,(\rho+s)(s-\rho)^{-1}\,\rho^{3/2}\,dx$$

If you plot x(ρ), you see that slope is 0 when s=ρ and I think that is a maximum and at that point we have that x(ρ=s) = 2s/2s = 1. The plot of x(ρ) is roughly this



so the good news is that this is a "perfect" argument for the K function, this is the range we like.

Well I guess we have to solve for ρ(x) at some point

$$(\rho+s)x = 2\sqrt{\rho s}$$

$$(\rho+s)^2x^2 = 4\rho s$$

$$(x^2\,\rho^2 + x^2\,2\rho s + x^2\,s^2) = 4\rho s$$

$(x^2)\,\rho^2 + (2sx^2 - 4s)\rho + (x^2 s^2) = 0$

$\rho = \{ - (2sx^2 - 4s) \pm [(2sx^2 - 4s)^2 - 4\,x^4 s^2]^{1/2} \} /(2x^2)$

The argument of the radical is

$(2sx^2 - 4s)^2 - 4\,x^4 s^2 = 4\{ sx^2 - 2s)^2 - x^4 s^2 \} = 4\{ s^2 x^4 - 4s^2 x^2 + 4\,s^2 - x^4 s^2 \}$

$= 4\{ - 4s^2 x^2 + 4\,s^2 \} = 16 s^2 (1 - x^2)$

so back to our solution

$\rho = \{ - (2sx^2 - 4s) \pm [16 s^2 (1 - x^2)]^{1/2} \} /(2x^2)$
$= \{ - (2sx^2 - 4s) \pm 4s\sqrt{1-x^2} \} /(2x^2)$

$= s\{ - (x^2 - 2) \pm 2\sqrt{1-x^2} \} /x^2$

$\rho + s = s\{ - (x^2 - 2) \pm 2\sqrt{1-x^2} + x^2 \} /x^2 = s\{ 2 \pm 2\sqrt{1-x^2} \} /x^2$

$= 2s (1 \pm \sqrt{1-x^2})/x^2$

$\rho - s = s\{ - (x^2 - 2) \pm 2\sqrt{1-x^2} - x^2 \} /x^2 = 2s\{ (1-x^2) \pm \sqrt{1-x^2} \} /x^2$

$= 2s\sqrt{1-x^2}\{ \sqrt{1-x^2} \pm 1 \} /x^2 = \mp 2s\sqrt{1-x^2} (1 - \sqrt{1-x^2})$

$\rho - s = \rho + s - 2s = 2s (1 \pm \sqrt{1-x^2} - x^2)/x^2$ = same result

$(\rho+s)/(\rho-s) = 2s (1 \pm \sqrt{1-x^2})/x^2 \;/\; 2s (1-x^2 \pm \sqrt{1-x^2}) /x^2$

$= (1 \pm \sqrt{1-x^2}) / (1-x^2 \pm \sqrt{1-x^2})$

The plot shows that for given x, there are in fact two values of ρ.

Finally, when ρ = 0 we have x = 0 (even if s = 0) and when ρ = b we have x(b) = $2\sqrt{bs}$ /(b+s). And we will also need

$\rho d\rho/(\rho+s) = \rho$

So here is our integral equation now:

$s^{-1/2} \int_0^{x(b)} dx\, (\rho+s)(s-\rho)^{-1}\, \rho^{3/2}\, I(\rho(x))\; K(x) = \pi$

$$s^{-1/2} \int_0^{x(b)} dx \; (1 \pm \sqrt{1-x^2}) / (1-x^2 \pm \sqrt{1-x^2}) \rho^{3/2} I(\rho(x)) K(x) = \pi$$

Well, as expected, we end up with another horrible mess, similar to our v2 approach. This is bad enough trying to verify I(ρ) but now we want to treat this as an integral equation we solve for I(ρ). The argument of K is nice, but everything else is a mess.

B. Jackson on the subject, and discovery of the Smythe book (11.5.09)

These are very early notes when I knew little about cylindricals. I am now an "expert" on this stuff, but I keep these notes anyway. You see me learning things for the first time (current era), and this is where I found and downloaded the Smythe book!

He treats exactly this problem starting on page 89, but we have to go way back to page 69 where he constructs the "general form" of a solution to Laplace in cylindrical coordinates. We shall see soon why this is a good choice. When we do the separation, we get two separation constants he calls ν and k. They both then appear in Bessel's equation 3.75 which then has the solution of interest to us of $J_\nu(k\rho)$. When you form-fit a solution, you have these ingredients [which I later started calling "atoms"] :

$$u(\rho,\varphi,z) = e^{\pm kz} \quad e^{\pm i\nu\varphi} \quad J_\nu(k\rho)$$

Stakgold never did this separation which is why it is unfamiliar to me, but totally reasonable. Jackson then in 3.170 puts our disk of radius a in the x-y plane, as I did. Since the z direction goes forever, we will have an integral over parameter k in our "fit". We know the solution is independent of φ, so can only have ν = 0, and we also know it will be $e^{-k|z|}$ since it is symmetrical in z obviously, and drops off in both directions. So we get this thing [which I later refer to as a "Smythian Form"]

$$u(\rho,z) = \int_0^\infty dk \; f(k) \, e^{-k|z|} \, J_0(k\rho) \qquad\qquad 3.170$$

Just *writing* it this way is a HUGE step forward. This is a very "smart" form fit for the solution. Here of course f(k) are the "Fourier coefficients" for the problem. There is a transform that goes with this, which he calls the **Hankel transform** shown on page 77. We know that ν = integer in a full azimuth problem, so that gets called m. We consider only $z \geq 0$ and so put in a decaying expo. Then 3.110 is the "expansion" part of the transform. The coefficients are given by the "projection" part which is 3.113. The crucial completeness relation is shown in 3.112. It seems odd that it has the name Hankel when no Hankel functions appear, but I guess he came up with this transform. This would solve our problem if it were a Dirichlet problem, but we only have V specified on a part of the boundary (the finite disk). Jackson points out that outside the disk symmetry tells us that $\partial_n u = 0$ in the z=0 plane, so we really have mixed boundary conditions. These are expressed in 3.171, I totally agree. Then our "form fit" equation 3.170 gives us 3.173 which is that same "dual integral equation" think in my big PDF canonical doc. Jackson then pulls out of his hat a solution to this dual integral equation without telling us how on earth you would find that solution, but of course we could verify it. Here is the solution:

$$f(k) = (2/\pi)Va \sin(ka)/(ka) \qquad V = (\pi/2) \, q/a \qquad \text{// Jackson charge units}$$

If you throw this f(k) into our fitted form, you get

$$\Phi(\rho,z) = q \sin^{-1} [\, 2a / \{ \sqrt{(\rho-a)^2 + z^2} + \sqrt{(\rho+a)^2 + z^2} \} \,]$$

He differentiates then to get the charge density on the surface

$$\sigma(\rho) = (q/2\pi a)\, 1/\sqrt{a^2 - \rho^2}$$

which is amazingly simple. I will soon transform this back to the spherical cap and see what we get there. Jackson then makes the Richard Price / Jim comment that the right coordinate system for this problem is oblate sphericals. Jackson nicely gives a Smythe reference. I found Smythe on my download site! Amazing. I have found everything Jackson talks about in this book, so that can be another project if I want, to do this problem in the right coordinate system.

C. Transforming back to the spherical cap

In Jackson units, the charge distribution on a metal disk of radius b is this:

$$\sigma(\rho) = (q/2\pi b)\, 1/\sqrt{b^2 - \rho^2}$$

$(r_o + r_o\cos\theta_o\, ,\, r_o\sin\theta_o)$

D. How to solve the integral equation (this section was added 9.15.10)

[Note added 3.20.11. I think when you use the K(z) expansion shown, you can cast this integral equation into a "double Abel transform" situation, like the case I did in my charged toroidal bowl doc. But I have not pursued how this works, I just use the quoted result below. But here is how things would start off:

$$K[\, 2\sqrt{xt}\,/(x+t)] \,/\, (x+t) \;=\; \int_0^1 dt' \,/\, [\sqrt{1-t'^2}\;\sqrt{([x+t]^2 - 4xtt'^2)}$$

You then have

$$f(x) = \int_0^a dt\, y(t) \int_0^1 dt' \,/\, [\sqrt{1-t'^2}\;\sqrt{([x+t]^2 - 4xtt'^2)} \qquad \text{// and I stop here.]}$$

When I wrote this doc, I did not know about Polyanin's large table (book) of integral equations. Here is an interesting entry in that book from page 277:

3.7-18. Kernels Containing Other Special Functions.

97. $$\int_0^a K\left(\frac{2\sqrt{xt}}{x+t}\right)\frac{y(t)\,dt}{x+t} = f(x).$$

Here $K(z) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-z^2t^2)}}$ is the complete elliptic integral of the first kind (see Supplement 11.13-1).

Solution:

$$y(x) = -\frac{4}{\pi^2}\frac{d}{dx}\int_x^a \frac{tF(t)\,dt}{\sqrt{t^2-x^2}}, \qquad F(t) = \frac{d}{dt}\int_0^t \frac{sf(s)\,ds}{\sqrt{t^2-s^2}}.$$

⦿ Reference: P. P. Zabreyko, A. I. Koshelev, et al. (1975).

and stunningly this exactly matches the integral equation I am trying here to solve !!! I can make these identifications:

$t = \rho$	integration variable
$a = b$	upper endpoint
$y(t) = \rho I(\rho)$	unknown solution
$x = s$	constant in the equation, argument of f(x)
$f(x) = \pi$	the right hand side

Let's now try to evaluate Polyanin's solution. First we need to know (Schaum p 69)

$$\int_0^t ds\ \ s\,\pi/\sqrt{t^2-s^2} \ = \ \pi \int_0^t ds\ s/\sqrt{t^2-s^2} \ = \pi\left[-\sqrt{t^2-s^2}\ \right]\Big|^{s=t}_0 \ = \ +\pi\sqrt{t^2-1}$$

Then we compute

$$F(t) = \partial_t \{\ \pi\sqrt{t^2-1}\ \} = \pi\, t/\sqrt{t^2-1}$$

Next, we compute the integral [and then set $w = t^2$]

$$\int_s^b dt\ \ t\ \{\ \pi\, t/\sqrt{t^2-1}\ \}/\sqrt{t^2-s^2} \ = \pi \int_s^b dt\ \ t^2/[\sqrt{t^2-1}\,\sqrt{t^2-s^2}\ ]$$

$$= (\pi/2) \int_{s^2}^{b^2} dw\,\sqrt{w}\ /\ [\sqrt{w-1}\,\sqrt{w-s^2}\ ]$$

Let's now define $s^2 = a$ and $b^2 = u$ so this thing is then (also, replace w by x)

$$= (\pi/2) \int_a^u dx\,\sqrt{x}\ /\ [\sqrt{x-a}\,\sqrt{x-1}\ ]$$

I now have this in a form that GR7 shows,

$$12.\quad \int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)}}\,dx = \frac{2(a-b)}{\sqrt{a-c}}\,F(\mu,q) - 2\sqrt{a-c}\,E(\mu,q) + 2\sqrt{\frac{(u-a)(u-c)}{u-b}} \qquad [u>a>b>c] \qquad \text{BY (237.04)}$$

In our application, we have b = 0 and c = 0 and the inequality conditions are met! We then have

int = $2a/\sqrt{a}\ F(\mu,q) - 2\sqrt{a}\ E(\mu,q) + 2\sqrt{u-a}\ = 2\sqrt{a}\ [F(\mu,q) - E(\mu,q)\ ] + 2\sqrt{u-a}$

Now μ and q are given by

$$\mu = \arcsin\sqrt{\frac{u-a}{u-b}},\quad q = \sqrt{\frac{b-c}{a-c}}.$$

so we then have $\mu = \sin^{-1}[\sqrt{u-a}/u]$ and q = 0. This we have

int = $2\sqrt{a}\ [F(\mu,0) - E(\mu,0)\ ] + 2\sqrt{u-a}$

where F and E are the usual elliptic integrals. But from AS p 594 we know

$F(\mu,0) = \mu$ and $E(\mu,0) = \mu$

so these terms cancel, and we get

int ≡ $\int_a^u dx\ \sqrt{x}\ /\ [\sqrt{x-a}\ \sqrt{x-1}\ ] = 2\sqrt{u-a}$

We have now shown that

$$\int_s^b dt\ t\ \{\pi\, t/\sqrt{t^2-1}\}/\sqrt{t^2-s^2} = (\pi/2)\,2\sqrt{u-a} = \pi\sqrt{b^2-s^2} = \int_s^b dt\ t\ F(t)/\sqrt{t^2-s^2}$$

and we change s back to x to match Polyanin so we have

$$\int_x^b dt\ t\ F(t)/\sqrt{t^2-x^2} = \pi\sqrt{b^2-x^2}$$

Our solution function to Polyanin's integral equation is then

$$y(x) = -(4/\pi^2)\,\partial_x[\pi\sqrt{b^2-x^2}\,] = -(4/\pi^2)\,\pi\,(1/2)\,(-2x)/\sqrt{b^2-x^2} = (4/\pi)\,x/\sqrt{b^2-x^2}$$

The solution to our integral equation is then this:

$$I(\rho) = y(\rho)/\rho = [(4/\pi)\,\rho/\sqrt{b^2-\rho^2}\,]/\rho = (4/\pi)\,1/\sqrt{b^2-\rho^2}$$

If we want to change from Stakgold $E = 1/(4\pi R)$ to Jackson $E = 1/R$, we need to divide our Stak result by 4π and this tells us that

$$I(\rho) = (1/\pi^2)\ 1/\sqrt{b^2-\rho^2}$$

This agrees with green Jackson p 93 except for the constant! But we assumed V = 1, and we know the capacitance of our disk is $C = 2b/\pi$ (Jackson p 92) which is $C = q/V = q$, so $q = 2b/\pi$. Jackson's result has the constant $q/(2\pi a)$ which then becomes

$$q/(2\pi b) = (2b/\pi)\ /(2\pi b) = 1/\pi^2$$

Thus, we have shown that, for V = V on our disk of radius b (and charge q), we have

$$I(\rho) = \sigma(\rho) = (q/2\pi b)\ 1/\sqrt{b^2-\rho^2}$$

This is the sum of both sides because our starting integral equation integrated q/R for both sides at once.