Superposition Example - Uncharged metal sphere approached by a point charge. PhL 1.15.10

Review 4.3.10. Everything below seems right today. I solve the problem of a point charge brought near an uncharged metal sphere by superposition. The sphere ends up with constant potential V = q/d. Superposition seems to work here because only one piece of metal is involved in the 1+2=3 superposition idea. Two days later I wrote another document talking about two pieces of metal. [Later this developed into a more general Red Flag Theorem about an arbitrary collection of metal pieces.]

Today I realized I don't know the answer to this simple problem.

First, park an uncharged metal sphere at the origin, nothing else around. The potential of the sphere's surface is V= 0, the same as the potential at infinity.

Now, imagine a positive point charge q located very far away from this sphere. We slowly move this point charge toward the sphere. The sphere starts to polarize and acquires a charge distribution σ on its surface, which is negative toward the point charge. The total charge on the sphere remains 0.

Suppose the sphere has radius a, and suppose the point charge q is located d > a from the sphere center. The sphere must be at a constant potential A. What is A ? [I realize now 12.9.10 that this is a simple example of Green's Reciprocity Theorem for 2 conductors, see Smythe Problems 38-42 section.]

Preliminary Remarks about "superposition" of "problems".

I should use the word "situation" or "configuration" or "system" or "scenario". But I will just use the word "problem". This implies that the problem has a "solution".

(a) Suppose " problem 1" involves some set of charges and has a Laplace solution u_1. Assume there are no BC's at all. Suppose " problem 2" has some *different* set of charges and some Laplace solution u_2 and, again, no BC's. We can "superpose" these two problem to get " problem 3" which has the union of the charges of the first two situations, and the solution will be $u_3 = u_1 + u_2$. We say that we have "superposed" two problems to get a new problem whose solution is the superposition of the two problems' solutions. This seems a viable thing to do. You might say that you require that both problems have the BC that $V(\infty) = 0$ which really means both problems 1 and 2 involve only localized charges. So let's summarize what we have said:

Problem #	Description	PDE	BC's	solution
1	charge set #1	Laplace	none	u_1
2	charge set #2	Laplace	none	u_2
3	charge sets #1 + #2	Laplace	none	$u_3 = u_1 + u_2$

(b) Now let's start throwing in some BC's. Suppose problem 1 is a Laplace one with some Dirichlet BC on some surface. We know the solution is unique, so there is no possibility of "superposing" solutions to get a new solution to the *same* PDE with the *same* BC. You can only superpose the trivial solution.

(c) But, we can do the following

Problem #	Description	PDE	BC's	solution
1	---	Laplace	Dirichlet 1	u_1
2	---	Laplace	Dirichlet 2	u_2
3	---	Laplace	Dirichlet 1+2	$u_3 = u_1 + u_2$

Here we are able to superpose two different Dirichlet solutions for two different problems, and we can add the solutions and that will in fact be the solution of a Dirichlet problem whose BC's are the sum of the other two.

We could do the same with Neumann or Mixed BC's. The main idea is that you have to add the BC's, and all three problems are really *different* "boundary value problems". Despite the fact that they are different PDE systems, the solutions are additive as described above.

(d) Now can we do this "superposition"

Problem #	Description	PDE	BC's	solution
1	---	Green's	V_1	u_1
2	---	Laplace	V_2	u_2
3	---	Green's	$V_1 + V_2$	$u_3 = u_1 + u_2$

Here I am extended the meaning of "Green's" to mean that the surface involved is held at a constant potential V which might not be 0. And of course I mean it is a *Laplace* Green's problem which means Laplace driven by a delta. To see is this is OK, let's just check:

$\nabla^2 u_3 = \nabla^2 u_1 + \nabla^2 u_2 = \delta + 0$ // so satisfies the problem 3 PDE

$u_3(\sigma) = u_1(\sigma) + u_2(\sigma) = V_1 + V_2 = V_3$ // so satisfies the problem 3 BC

So I think this "superposition" is OK.

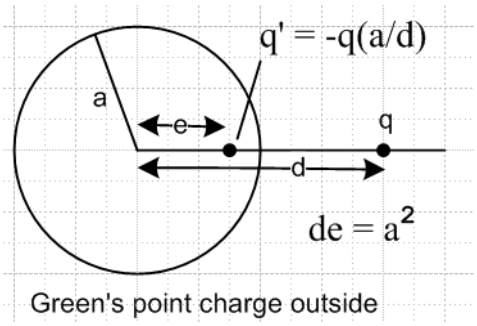
What is A for the problem posed at the start?

Problem 1 is this: sphere held at potential V = 0 in presence of a Green's point charge. Think of this as the first line of case (d) discussed above, where $V_1 = 0$. The solution here is just the Green's Function for the sphere which is the potential of the point charge plus the image charge:

$u_1(\mathbf{r}) = V_{point}(\mathbf{r}) + V_{image}(\mathbf{r})$ outside the sphere

$u_1(\mathbf{r}) = 0$ on and inside the sphere

The image charge is located at the usual symmetry point and has magnitude $q' = -q(a/d)$. Here is the picture for this situation:



If we observe these two objects (sphere + point charge) from far away, we see a point charge of this magnitude

$q_{\mathbf{eff}} = q\,(1\text{-}a/d)$

The sphere must therefore have an integrated induced charge equal to the image charge, q' = -q(a/d).

Problem 2: Suppose we have an isolated metal sphere loaded up with charge q" = +q(a/d). In this electrostatics problem, the potential will be

$u_2(\mathbf{r}) = q(a/d)/r$ outside the sphere
$u_2(\mathbf{r}) = q/d$ on and inside the sphere

Think of this as the second problem of case (d) above with $V_2 = q/d$ on the sphere.

Problem 3: Superpose problems 1 and 2 in the sense of case (d) above. We end up with a sphere having no charge, and being at potential V = q/d. The Green's point charge is still there. So this is then our original problem, and we have found the answer.

A = q/d

and we are a little amazed to find that the result is independent of sphere radius a. Here is our table

Problem #	Description	PDE	BC's	solution
1	---	Green's	0	u_1
2	---	Laplace	q/d	u_2
3	---	Green's	q/d	$u_3 = u_1 + u_2$

The solution for Problem 3 is this:

$u_3(\mathbf{r}) = V_{\mathbf{point}}(\mathbf{r}) + V_{\mathbf{image}}(\mathbf{r}) + q(a/d)/r$ outside the sphere
$u_3(\mathbf{r}) = q/d$ on and inside the sphere

Comment 1: If the sphere were microscopically tiny compared to distance d, we would say that V ≈ q/d everywhere in the vicinity of the uncharged sphere due to the Green's charge, and we would suppose that V = q/d *on* that sphere in the tiny limit. Our result above is that this is true even for the sphere being very

large and very close to Green's charge q. It is only the distance from d to the center of the sphere that matters.

Comment 2: When we apply ∇^2 to u_3 shown above, we only get a δ from $V_{\mathbf{point}}(\mathbf{r})$. The other two charges are inside the sphere, but the potential inside the sphere is just q/d which has $\nabla^2 = 0$.

Question: What is the charge distribution on the sphere in Problem 3?

The charge distribution for Problem 2 is going to be $q(a/d)/ [4\pi a^2]$ = constant on the sphere. So the Problem 3 result will be this added to the Problem 1 result.

So what is the charge distribution in Problem 1? You just compute it from the outside potential!

$$V_1(\mathbf{r}) = V_{\mathbf{point}}(\mathbf{r}) + V_{\mathbf{image}}(\mathbf{r})$$

$$\sigma = -\partial_{\mathbf{r}} V_1(\mathbf{r}) = -\partial_{\mathbf{r}} g \qquad g = \text{Green's Function}$$

since the normal on the sphere is outbound. So specifically we have

$$V_1(\mathbf{r}) = q/|\mathbf{r}-\mathbf{d}| - q(a/d) / |\mathbf{r}-\mathbf{e}|$$

Stakgold calculates the derivative on page 151 and I could do it here, but don't think it is necessary at the moment, there is some well-defined answer. And you could certainly think of the result as being decomposable a sum of the surface charge due to the Green's point charge and the surface charge due to the image charge

$$\sigma = \sigma_{\mathbf{point}} + \sigma_{\mathbf{image}}$$

This is all on the outer surface of the sphere. Inside, V = constant in Problem 3 so $\sigma = 0$ there.

Summary of the Problem 3 Solution. We have a point charge q located a distance d from the center of a radius-a sphere that is metal and uncharged and has constant potential V = q/d. The charge distribution on the outer sphere surface is given by

$$\sigma = \sigma_{\mathbf{point}} + \sigma_{\mathbf{image}} + q(a/d)/ [4\pi a^2]$$

If $q > 0$, the first and last terms are positive, and the $\sigma_{\mathbf{image}}$ term is negative. The potential is this

$$u_3(\mathbf{r}) = V_{\mathbf{point}}(\mathbf{r}) + V_{\mathbf{image}}(\mathbf{r}) + q(a/d)/r \qquad \text{outside the sphere}$$
$$u_3(\mathbf{r}) = q/d \qquad \text{on and inside the sphere}$$

Problem 4: Let this situation be just the Green's point charge of size -q. The potential is

$$u_4(\mathbf{r}) = -q /r$$

Problem 5: What happens if I superpose problems 3 and 4 ? How would we draw the chart?

Problem #	Description	PDE	BC's	solution
3	---	Green's	q/d	u_3
4	---	Laplace	f(**r**)	u_4
5	---	Green's	q/d + f(**r**)	$u_5 = u_3 + u_4$??

This seems a reasonable superposition, BUT, problems 4 and 5 cannot involve a *metal* sphere. So Problem 5 is one where we have some variable Dirichlet potential on the sphere. This would then not really be a Green's problem per se where we assume a constant potential on a surface.

The potential for Problem 5 would be this:

$u_5(\mathbf{r}) = V_{\mathtt{image}}(\mathbf{r}) + q(a/d)/r$	outside the sphere
$u_5(\mathbf{r}) = q/d - V_{\mathtt{point}}(\mathbf{r})$	on and inside the sphere

Since this solution does not involve a metal sphere, it is of no great interest to me. Just an example.