Explanation of σ near an edge PhL 12.4.10

I *thought* I had a quick and dirty way to show σ behaves as $1/\sqrt{a^2-x^2}$ near the edge of a 2D surface in 3 space. But naturally it bogged down, and I don't want to invest the time to push it through. It was only worth something to me if it really was quick and easy. I thought the strip would be "easy", but then all that old complexity came back, I finally remembered I have Stak notes on this (the 2D wire segment), and I end up with no quick and easy method. BUT, in Plan D I do argue that σ has to blow up monotonically at the edge, and still be integrable, and that $1/\sqrt{a^2-x^2}$ does *have* these properties.

The Idea. Consider an extruded strip which we plot in the z = 0 plane. It runs from -a to a. I worked on this problem elsewhere, folder "the 2D wire. We treat the z = 0 projection as a 2D problem. We assume some unknown σ(x) which runs from -a to a and which will be symmetrical about x = 0. At some point x_1, the potential due to σ(x) at dx will be $dV(x) = - \sigma(x)dx \ln(|x-x_1|)$, since - ln(1/r) is the 2D fundamental solution for a 2D point charge that far away. The force on a unit test charge at x_1 will be $\sigma(x)/|x-x_1|$. So in the 2D world, force is a 1/R thing, not a $1/R^2$ thing. We can write the equation of **force balance** on test charge at x_1 this way: (assume $x_1 > 0$)

$$\int_{-a}^{x_1} dx\, \sigma(x) / (x_1-x) = \int_{x_1}^{a} dx\, \sigma(x) / (x-x_1)$$

where everything is positive.

Plan A. Put everything on the LHS and we have

$$\left(\int_{-a}^{x_1-\varepsilon} + \int_{x_1+\varepsilon}^{a} \right) dx\, \sigma(x) / (x_1-x) = 0$$

This looks like a principle part integral to me. So we are looking for a solution σ(x) which satisfies this equation.

$$PP \int_{-a}^{a} dx\, \sigma(x) / (x_1-x) = 0$$

I guess that could be called an integral equation for σ. Two comments:

(1) I don't know off hand how to solve such an integral equation. (probably just Abel or some such)
(2) Given the solution $\sigma(x) = 1/\sqrt{a^2-x^2}$, I don't know how to *verify* it is the solution without doing more work than I want to do on this today.

Plan B.

Each side blows up at the x_1 end of the integral. So we imagine that a tiny segment on each side ($x_1-\varepsilon$, $x_1+\varepsilon$) can be excluded from our force balance, since we argue that the two halves of this segment will balance each other since σ(x) is approximately a constant on this segment. Then our force balance equation becomes this

$$\int_{-a}^{x_1-\varepsilon} dx\ \sigma(x) / (x_1-x) = \int_{x_1+\varepsilon}^{a} dx\ \sigma(x) / (x-x_1) \qquad (*)$$

Now process the left integral. Let x' = -x so it reads

$$\text{LHS} = \int_{a}^{-x_1+\varepsilon} (-dx')\ \sigma(-x') / (x_1+x') = \int_{-x_1+\varepsilon}^{a} dx'\ \sigma(x') / (x_1+x')$$

where we use the symmetry of σ(x). Then we have for (*)

$$\int_{-x_1+\varepsilon}^{a} dx\ \sigma(x) / (x_1+x) = \int_{x_1+\varepsilon}^{a} dx\ \sigma(x) / (x-x_1)$$

Suppose we just guess that the solution is this

$$\sigma(x) = 1/\sqrt{a^2-x^2}$$

Then we would need to have this be true

$$\int_{-x_1+\varepsilon}^{a} dx\ /\ [(x_1+x)\sqrt{a^2-x^2}\ ] = \int_{x_1+\varepsilon}^{a} dx/\ [(x-x_1)\sqrt{a^2-x^2}\ ]$$

$$I =?= J$$

To do integral I, change variables to x' = x + x_1 and get

$$I = \int_{+\varepsilon}^{a+x_1} dx'\ /\ [\ x'\sqrt{a^2-(x'-x_1)^2}\ ] = \int_{+\varepsilon}^{a+x_1} dx\ /\ [\ x\sqrt{a^2-(x-x_1)^2}\ ]$$

$$= \int_{+\varepsilon}^{a+x_1} dx\ /\ [\ x\sqrt{R}\ ] \qquad R = -x^2 + 2x_1x + a^2-x_1^2 \quad = "\ cx^2 + bx + a\ " \quad a>0$$

$$\Delta = "4ac-b^2" = 4(a^2-x_1^2)(-1) - 4x_1^2 = -\ 4a^2 \quad < 0$$

This is the famous page 97 GR integral that one can write many ways.

$$= \frac{1}{\sqrt{a}} \operatorname{arccosh}\left(\frac{2a+bx}{x\sqrt{-\Delta}}\right) \qquad [a>0, \Delta<0]$$

$$= 1/\sqrt{a^2-x_1^2}\ \operatorname{arccosh}[\ (2(a^2-x_1^2) + 2x_1x)/2ax] = 1/\sqrt{a^2-x_1^2}\ \operatorname{arccosh}[\ ((a^2-x_1^2) + x_1x)/ax]$$

I was hoping to show that the integrals are the same, but I don't want to spend the time now.

Plan C:

Maple was not very useful: I enter these two integrals into Maple

```
f := Int(1/((x1+x)*sqrt(a^2-x^2)),x =-x1+epsilon..a );
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$$f := \int_{-x1+\varepsilon}^{a} \frac{1}{(x+x1)\sqrt{a^2-x^2}}\,dx$$

```
g := Int(1/((x-x1)*sqrt(a^2-x^2)),x =x1+epsilon..a );
```

$$g := \int_{x1+\varepsilon}^{a} \frac{1}{(x-x1)\sqrt{a^2-x^2}}\,dx$$

I then have Maple DO these integrals, and it gets

```
f := int(1/((x+x1)*sqrt(a^2-x^2)),x =x1-epsilon..a );
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$$f := \lim_{x \to a-} \frac{-\operatorname{arctanh}\left(\frac{a^2+x\,x1}{\sqrt{-x1^2+a^2}\sqrt{a^2-x^2}}\right)+\operatorname{arctanh}\left(\frac{x1^2+a^2-x1\,\varepsilon}{\sqrt{-\varepsilon^2-x1^2+a^2+2\,x1\,\varepsilon}\sqrt{-x1^2+a^2}}\right)}{\sqrt{-x1^2+a^2}}$$

```
g := int(1/((x-x1)*sqrt(a^2-x^2)),x =x1+epsilon..a );
```

$$g := \lim_{x \to a-} -\frac{-\operatorname{arctanh}\left(\frac{-a^2+x\,x1}{\sqrt{-x1^2+a^2}\sqrt{a^2-x^2}}\right)+\operatorname{arctanh}\left(\frac{x1^2-a^2+x1\,\varepsilon}{\sqrt{-\varepsilon^2-x1^2+a^2-2\,x1\,\varepsilon}\sqrt{-x1^2+a^2}}\right)}{\sqrt{-x1^2+a^2}}$$

Now I need to know $\tanh^{-1}(1/x)$ for small x in order to do the limits here. Too much work!

Plan D. I can see that as you move closer to the left end of the strip, the charge to the left of a selected point x_1 has to keep getting larger because it has to fight against more charge to the right of x_1, so we know it will be monotonic increasing. As we get VERY close to the edge, little tiny $\sigma(x)dx$ to the left has to fight back the finite hordes from the right, so I think this does argue that we need $\sigma(x) \to \infty$ in some manner. And it has to be integrable. We imagine as we approach the edge, $\sigma(x) \to f(x-a)$ where f blows up. We know that $\sigma(x) = 1/\sqrt{x-a}$ will do the trick : it is integrable and it blows up. But any power would have this property near exponent -1/2, so we don't have a proof it is this special power. And of course other functions blow up as well.

We say this behavior in the case of the disk, but there was never any "simple argument" as to why σ had this form. It may be that in my strip problem there is some other power behavior, I really have no idea.

Plan E. I see now that I have actually calculated the potential for this strip situation, see "the 2D wire". The solution is this when a = 1:

$$u(x,y) = -(1/4\pi)\ln(2) - (1/4\pi)\ln\left(z + \sqrt{z^2-1}\right) + \text{c.c.} \qquad z = x + iy$$

I guess I could compute $\partial_{\mathbf{y}}$u to get the charge density.

$$\partial_{\mathbf{z}} \ln(z + \sqrt{z^2-1}) = 1/[z + \sqrt{z^2-1}] * (1 + z/\sqrt{z^2-1}) = 1/\sqrt{z^2-1}$$

$$\partial_{\mathbf{y}} \ln(z + \sqrt{z^2-1}) = i /\sqrt{z^2-1} = 1/\sqrt{1-z^2}$$

So yes, the solution really is

$$\sigma = 1/\sqrt{a-x^2}$$

My notes on the 1D wire show all this stuff. Again, you had to know how to solve a certain integral equation which Stak talks about to rescue this problem.