

An application of toroidal functions in electrostatics

Jerry Selvaggi,^{a)} Sheppard Salon,^{b)} and M. V. K. Chari^{c)}

Electrical, Computer and Systems Engineering Department, Rensselaer Polytechnic Institute,
Troy, New York 12180-3590

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A method employing the use of toroidal functions is introduced for calculating the scalar potential and the electric field from a charged conducting ring. This method is an alternative to the well-known elliptic integral formulation and is usually easier to formulate than the elliptic integral solution. © 2007 American Association of Physics Teachers.

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I. INTRODUCTION

The calculation of the electric scalar potential and the electric field at some arbitrary point in space, not coincident with the charged filamentary ring and not on the symmetry axis, usually involves the use of elliptic integrals.¹⁻⁴ We show that this use is not necessary and solve this problem by employing toroidal functions.⁵⁻⁷ This approach, unlike the elliptic integral formulation, is not discussed in the physics curriculum, but is easy to apply. Toroidal functions come from the solution of Laplace's equation in toroidal coordinates.⁸ A special class of these functions, called the Legendre functions of the second kind, are uniquely suited for handling finite circular cylindrical geometries.

The calculation of the scalar potential and the electric field due to a charged ring at an arbitrary point in space not coincident with the ring is but one application of the use of toroidal functions. The toroidal expansion has applications to coil design used in MRI magnets,⁹ transformer coils, and circular cylindrical coils with a rectangular cross-section.¹⁰ Also, it has been used for calculating the external field from permanent magnet motors.¹¹⁻¹⁴ It has recently been used for determining the gravitational potential.¹⁵⁻¹⁷

II. SCALAR POTENTIAL PRODUCED BY A CHARGED RING

Figure 1 illustrates the geometry of the problem. To calculate the electric scalar potential at an arbitrary point in space, P , not coincident with the charged ring, we start with the differential scalar potential given by

$$d\Phi_P = \frac{1}{4\pi\epsilon_0} \frac{\lambda(\phi') a d\phi'}{|\mathbf{R} - \mathbf{a}|}, \quad (1)$$

where $\mathbf{R}$ is the position vector of the observation point and $\mathbf{a}$ is the position vector of an element of the circular filamentary line charge. The distance in circular cylindrical coordinates between the source point, (ρ', ϕ', z') , and an arbitrary observation point or field point, (ρ, ϕ, z) , is given by

$$\frac{1}{|\mathbf{R} - \mathbf{r}'|} = \frac{1}{\sqrt{\rho^2 + \rho'^2 + (z - z')^2 - 2\rho\rho' \cos(\phi - \phi')}}. \quad (2)$$

In Fig. 1, $r' = \rho' = |\mathbf{a}| = a$ and $z' = 0$. This coordinate system allows us to write Eq. (2) as

$$\frac{1}{|\mathbf{R} - \mathbf{a}|} = \frac{1}{\sqrt{\rho^2 + a^2 + z^2 - 2\rho a \cos(\phi - \phi')}}. \quad (3)$$

We employ Eqs. (3) and (1) and write the electric scalar potential as

$$\Phi_P = \frac{a}{4\pi\epsilon_0} \int_0^{2\pi} \frac{\lambda(\phi') d\phi'}{\sqrt{\rho^2 + a^2 + z^2 - 2\rho a \cos(\phi - \phi')}}. \quad (4)$$

Equation (4) can be converted into an elliptic integral, which is the approach taken in most advanced textbooks. However, Eq. (4) can be used directly without any mathematical manipulations. The Green's function^{2,4} expansion for Eq. (2) is given by¹⁸⁻²¹

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{\pi\sqrt{\rho\rho'}} \sum_{m=0}^{\infty} \epsilon_m Q_{m-1/2}(\beta) \cos[m(\phi - \phi')], \quad (5)$$

where $\beta = (\rho^2 + \rho'^2 + (z - z')^2)/2\rho\rho' > 1$, ϵ_m is Neumann's factor,⁵ which is 1 for $m=0$ and 2 otherwise. $Q_{m-1/2}(\beta)$ is the Legendre function of the second kind and of half-integral degree or a toroidal function of zeroth order.^{7,21,22} These functions are also referred to as Q -functions¹⁸ and can be represented by the formula¹²

$$Q_{m-1/2}(\beta) = \frac{\pi}{(2\beta)^{m+1/2} 2^m} \sum_{n=0}^{\infty} \frac{(4n + 2m - 1)!!}{2^{2n}(n+m)! n!} \frac{1}{(2\beta)^{2n}}. \quad (6)$$

Snow²² has given the most complete account of the mathematical properties of the toroidal functions. This in-depth reference should be consulted by anyone who wants to better understand toroidal functions and their properties. However, to apply the Q -function successfully, Eqs. (5) and (6) are all that are necessary.

Equation (5) represents a Fourier cosine series expansion of the inverse distance in cylindrical coordinates whose weighting coefficients are the toroidal functions given by Eq. (6). This expansion is reminiscent of the expansion of an arbitrary periodic function in terms of a Fourier series whose weighting coefficients are found from certain orthogonality conditions.

By employing Eqs. (5) and (6), we can write Eq. (4) as

$$\Phi_P = \frac{a}{4\pi^2\epsilon_0} \frac{1}{\sqrt{\rho a}} \sum_{m=0}^{\infty} \epsilon_m Q_{m-1/2}(\beta) \int_0^{2\pi} \lambda(\phi') \cos[m(\phi - \phi')] d\phi', \quad (7)$$

where $\beta = (\rho^2 + a^2 + z^2)/2\rho a > 1$.

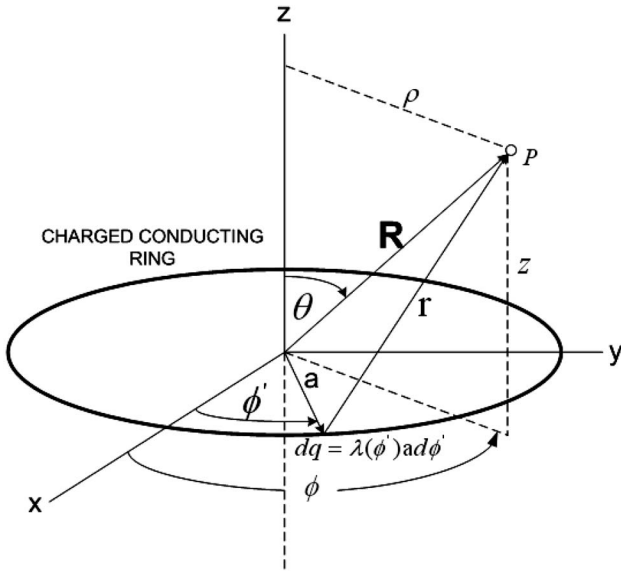


Fig. 1. Charged ring.

For a uniformly charged ring, $\lambda(\phi') = \lambda_0 = q/2\pi a$ and the $m=0$ term in Eq. (7) is the only term that survives the integration. As a result, we obtain a simple-looking expression for the electric scalar potential at an arbitrary point in space not coincident with the charged ring,

$$\Phi_P = \frac{q}{4\pi^2 \epsilon_0 \sqrt{\rho a}} Q_{-1/2}(\beta). \quad (8)$$

From Eq. (6), $Q_{-1/2}(\beta)$ is given by

$$Q_{-1/2}(\beta) = \pi \sum_{n=0}^{\infty} \frac{(4n-1)!!}{2^{2n}(n!)^2} \left(\frac{\rho a}{\rho^2 + a^2 + z^2} \right)^{2n+1/2}. \quad (9)$$

If we employ Eq. (9), we can write Eq. (8) as

$$\Phi_P = \frac{q}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{(4n-1)!!}{2^{2n}(n!)^2} \frac{(\rho a)^{2n}}{(\rho^2 + a^2 + z^2)^{2n+1/2}}. \quad (10)$$

Equation (10) is the infinite series solution for the scalar potential of a uniformly charged ring of radius a and is valid at an arbitrary point that is not coincident with the charged ring.

Likewise, for a harmonic charge density given by

$$\lambda(\phi') = \lambda_0 \cos(p\phi'), \quad (11)$$

where $p=1, 2, 3, \dots$, the integral in Eq. (7) is easily evaluated. A few three-dimensional surface plots have been created to visualize the potential and electric field components. The total charge, q , used for all the plots is $1 \mu\text{C}$ and the radius of the charged ring $a=1 \text{ m}$. The observation cylinder includes both the cylindrical hull and the circular caps. The cylindrical hull, which surrounds the source, has a length of 2 m and has a radius $\rho=2 \text{ m}$. The circular caps are located at $z=\pm 1 \text{ m}$ and the radial coordinate of the caps varies from $\rho=\frac{1}{20} \text{ m}$ to $\rho=2 \text{ m}$. The cylindrical surface plots were created using Tecplot²³ by importing a data file that includes the Cartesian coordinates and its associated scalar potential or electric field component. The data file was created by Maple,²⁴ but other packages of this type will work.

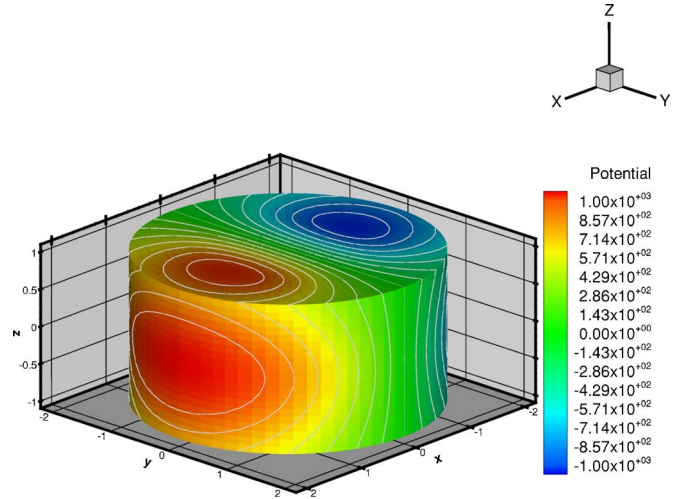


Fig. 2. Scalar potential if $\lambda(\phi') = \lambda_0 \cos \phi'$.

For a uniform charge density the total charge on the source is nonzero and, therefore, the total scalar potential is dominated in the far field by its monopole contribution. If $\lambda(\phi') = \lambda_0 \cos(p\phi')$ with integer $p \geq 1$, the total charge on the circular ring is zero. If $p=1$, then the total scalar potential is dominated in the far field by its dipole contribution. The reason is that if $p=1$, the $m=1$ term is the only term that survives the integration in Eq. (7) and the scalar potential is

$$\Phi_P = \frac{q Q_{1/2}(\beta)}{4\pi^2 \epsilon_0 \sqrt{\rho a}} \cos(\phi). \quad (12)$$

The $Q_{1/2}(\beta)$ term includes a spherical dipole contribution, which is the major contribution to the scalar potential. Figure 2, which is a plot of Eq. (12), illustrates the dipole pattern.

If $\lambda(\phi') = \lambda_0 \cos(2\phi')$, the total charge on the circular ring is zero and the total scalar potential is dominated by its quadrupole contribution. This is true because only the $m=2$ term survives the integration in Eq. (7). The scalar potential for this case is

$$\Phi_P = \frac{q Q_{3/2}(\beta)}{4\pi^2 \epsilon_0 \sqrt{\rho a}} [2 \cos^2(\phi) - 1]. \quad (13)$$

Figure 3 shows a plot of the scalar potential represented by

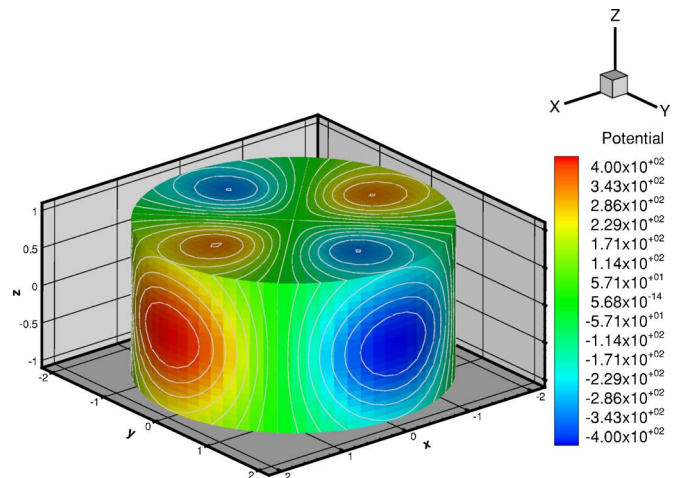


Fig. 3. Scalar potential if $\lambda(\phi') = \lambda_0 \cos(2\phi')$.

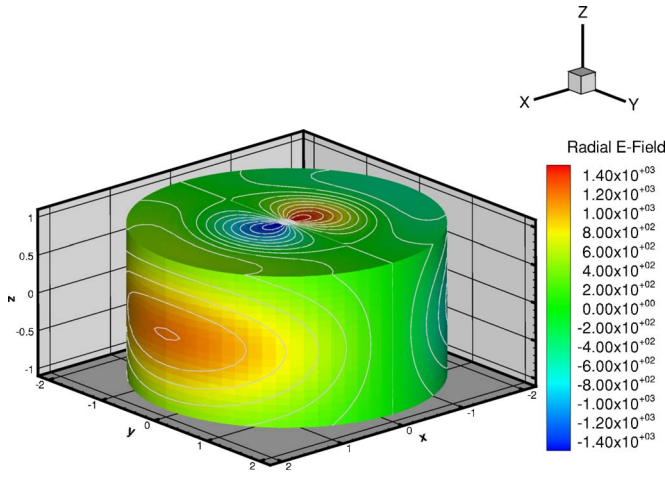


Fig. 4. Radial component of E -field for $\lambda(\phi') = \lambda_0 \cos(\phi')$.

Eq. (13) and illustrates the quadrupole pattern, which dominates the scalar potential.

Note that if the linear charge density of a charged ring is harmonic in $\cos(p\phi)$, then Eq. (7) acts like a filter, filtering out only those specific harmonic components where $m=p$.

III. ELECTRIC FIELD PRODUCED BY A CHARGED RING

The electric field components due to a ring of charge with a harmonic charge density, $\lambda(\phi') = \lambda_0 \cos(\phi')$, calculated from $\mathbf{E} = -\nabla\Phi_P$, are

$$E_\rho = \frac{q\{g_1 Q_{-1/2}(\beta) + g_2 Q_{1/2}(\beta)\}}{4\pi^2 \epsilon_0 \rho \sqrt{\rho a D}} \cos(\phi), \quad (14a)$$

$$E_z = \frac{q\{g_3 Q_{-1/2}(\beta) + g_4 Q_{1/2}(\beta)\}}{4\pi^2 \epsilon_0 \sqrt{\rho a D}} \cos(\phi), \quad (14b)$$

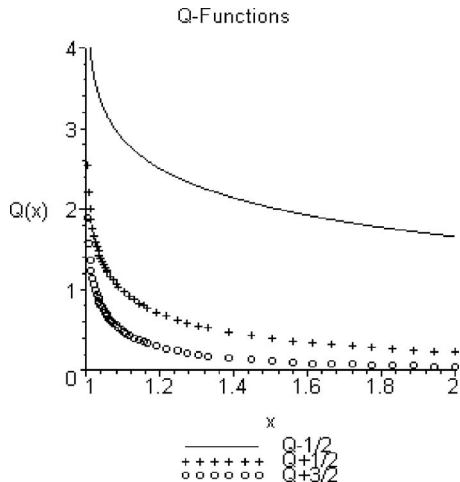


Fig. 5. A few Q -functions.

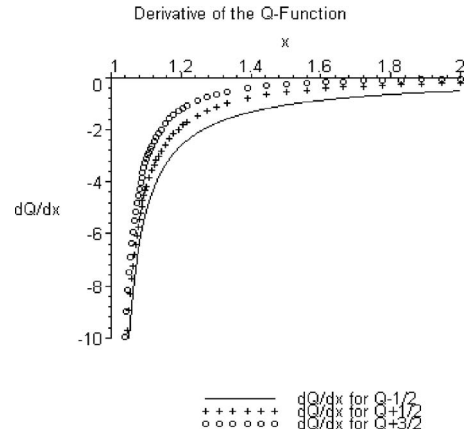


Fig. 6. A few Q -function derivatives.

$$E_\phi = \frac{q Q_{1/2}(\beta)}{4\pi^2 \epsilon_0 \rho \sqrt{\rho a}} \sin(\phi), \quad (14c)$$

where $g_1 = \rho a(\rho^2 - a^2 - z^2)$, $g_2 = (-\rho^2 a^2 + \rho^2 z^2 + a^4 + 2z^2 a^2 + z^4)$, $g_3 = 2\rho z a$, $g_4 = -z(\rho^2 + a^2 + z^2)$, and $D = (\rho^2 + a^2 - 2\rho a + z^2) \times (\rho^2 + a^2 + 2\rho a + z^2)$. Equation (14) is easily computed using Maple, in which it is easily shown that

$$\frac{dQ_{1/2}(\lambda)}{d\lambda} = \frac{\lambda Q_{1/2}(\lambda) - Q_{-1/2}(\lambda)}{2(\lambda^2 - 1)}. \quad (15)$$

The three-dimensional surface plot of the radial component given by Eq. (14a) is shown in Fig. 4. We can continue this process for any value of p in Eq. (11) to produce higher multipole patterns.

IV. DISCUSSION

Another method of attack, which is widely employed for calculating the electric field from a charged ring but not discussed in this paper, is spherical harmonic analysis. This method requires two solutions; one valid for $R > a$ and one valid for $R < a$. A spherical harmonic solution suffers from slow convergence as the observation point moves closer to the source and, therefore, the series solution developed from this method is not very useful for accurately calculating near-field solutions. With a toroidal expansion, only one series solution is needed, which is valid for observation points inside or outside the ring's radius. Equations (8) and (10) or (12)–(14) are valid at an arbitrary point in space not coincident with the source.

It is useful to know what the toroidal functions and their derivatives look like. Figures 5 and 6 illustrate a few of the Q -functions and their corresponding derivatives. We see from Figs. 5 and 6 that the Q -functions, unlike the Legendre polynomials,²⁵ are monotonic functions and highly convergent.

^{a)}Electronic address: jpsst44@bellatlantic.net

^{b)}Electronic address: salons@rpi.edu

^{c)}Electronic address: charim@rpi.edu

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