Try to compute the potential of a 3D strip extrusion PhL 10.13.09

Stakgold computes the charge distribution I(ξ) on page 179 6.153. I thought it would be "interesting" to plug this into 6.146 to compute the potential u(**x**) for this problem. I have found this integral to be highly recalcitrant, hence I am breaking this work off in yet another separate doc.

I include my opening raw 2 notes and then wander off to try to compute the integral.

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Overview (most written earlier, header written 12.9.10, 2 pages)

This doc got moved to my physics electrostatics world, but it is very closely tied to Stakgold Chapter 6 in which Stak treats this problem as an example, and where this doc first lived. The Stak Chap 2 raw notes (page 38 of that 60 page doc) talk about the 2D wire problem, and in fact those raw notes contain overviews of the both this v1 and the next v2 docs, which I will replicate below.

Stak writes the integral equation for σ(ξ) on our 2D wire (he calls this I(ξ), not σ(ξ)) , and then he has to quote the solution of a pattern integral equation to conclude that $\sigma(\xi) = \pi/\sqrt{a^2-\xi^2}$. Whether you think of this V = $\int \sigma$ E = constant as the integral equation, or of a corresponding force balance integral equation, there is no way to avoid the fact that this σ(ξ) conclusion is the result of the "global" effect of all charge on the wire. You cannot localize the problem to one point on the wire as if there were some ODE at that point you could solve to get the result. It is intrinsically an integral equation situation!!!

Now, once we know σ, we can learn the potential anywhere as this same V = $\int \sigma$ E integral, which is always the second phase of Stak's "integral equation method". In the notes below, this potential is called u(**x**) where **x** is a point in 2D space off the wire, and our problem is to do this integral! I decide here to express **x** = at**R** where 2a is the length of the wire, **R** I guess is a point on a unit circle, and t is a scaling parameter. When I insert this into the Stak integral equation, I end up wanting to do a certain integral which I call I(t), which is a little confusing since I is also the letter Stak uses for σ.

With this as introduction, I now copy here the overview of this v1 doc from the Stak raw notes:

" This document really has no useful notes. Skip to v2.doc summary below. [see v2 doc for this]

Since I(t) was being recalcitrant, I moved flailing notes off to this separate document. After flailing with Plans A,B,C, in Plan D I made some headway as follows:

$$I(t) \equiv \int_0^1 d\tau \ln\left([tX-\tau]^2 + t^2Y^2 \right) / \sqrt{1-\tau^2} + \int_0^1 d\tau \ln\left([tX+\tau]^2 + t^2Y^2 \right) / \sqrt{1-\tau^2}$$

$$I(t) = I_1(t) + I_1(-t)$$
$$= I_{11}(t) + I_{11}(t)^* + I_{11}(-t) + I_{11}(-t)^*$$

$$I_{11}(t,\alpha) = \int_0^1 (d\tau / \sqrt{1-\tau^2}) \ln(\tau-\alpha) \qquad \alpha = tX+itY$$

$$= \int_{-\pi/2}^{\pi/2} d\theta \ln(\sin\theta-\alpha)$$

$$= \int_0^{\pi/2} d\theta \ln(\sin\theta-\alpha) + \int_0^{\pi/2} d\theta \ln(-\sin\theta-\alpha)$$

$$= J(t,\alpha) + i\pi^2/2 + J(t,-\alpha)$$

$$J(t,\alpha) = (\pi/2) \ln(-\alpha) + \int_0^{\pi/2} d\theta \ln(1 + a \sin\theta) \qquad a = -1/\alpha \qquad \alpha = tX+iY$$

and GR p 527 #11 have at least something to say about this. At this point I paused this thread and flailed on Plan E where I tried to convert to a contour integral. Then in Plan F I am back pondering the above integral which I called K,

$$K(t,\alpha) \equiv \int_0^{\pi/2} d\theta \ln(1 + a \sin\theta) \quad =?= (GR): \quad (1/2)\,\pi \ln\left((1 + \sqrt{1-a^2})/2 \right)$$

which appears as noted in GR but with some strangeness, one result as shown above. I could not explain the "strangeness" which reduced confidence a lot. But assuming the above is right, I then wrote out the final result for I(t) and it was big mess. In frustration, I terminated my "v1" document at that point. But I think this result was in fact correct, but appears in much simple form below."

So basically, I tried in this v1 doc, but was unable, to do the required integral I(t).

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raw notes:

Example 2: Strip of width 2a (meaning a 2D line segment on x axis -a to a) (178)

So integral instead of being around C is now -a to a as shown in 6.152 [which is 6.147]. Now

$$\int_{-a}^{a} d\xi \ln|x-\xi| I(\xi) = \int_{-1}^{1} a\,d\tau \ln|at-a\tau| I(a\tau) = \int_{-1}^{1} a\,d\tau (\ln a + \ln|t-\tau|) I(a\tau)$$

so that

$$2\pi c_0/a = \ln a \int_{-1}^{1} d\tau\, I(a\tau) + \int_{-1}^{1} d\tau\, \ln|t-\tau|\, I(a\tau) \qquad \text{// agrees with p 179 A}$$

At this point, he quotes a result of a future Exercise (with wrong number) which result is p 179 B. I suspect this is the key step and you see in A that sort of "double integral equation" Jackson found for his disk in 3D problem. These charge distribution problems tend to have a lot of "internal feedback". I then followed all the remaining steps on page 179, finding one other error, and I agree with result H.

Here is what that Exercise 6.47 on page 181 deals with. It addresses the problem of solving the following integral equation for f(t), given g(t).

$$\int_{-1}^{1} d\tau\, \ln|t-\tau|\, f(t) = g(t)$$

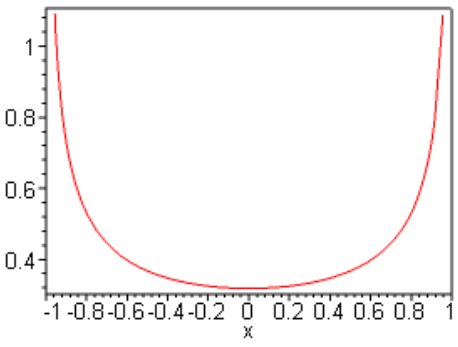
It then applies the solution found to g(t) = 1, and that is the result we use here.

So this answers one of the questions I posed above: what is the charge distribution on a line segment in 2D! The charge runs off mostly to the ends, blowing up at the ends, but being integrable.

```
restart;
f := (1/Pi)/((1 - x^2))^(1/2);
```

$$f := \frac{1}{\pi\sqrt{1-x^2}}$$

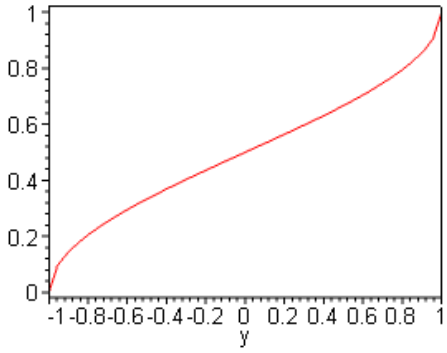
```
plot(f,x=-1..1);
```




```
g := int(f,x=-1..y);
```

$$g := \frac{1}{2}\,\frac{2\arcsin(y)+\pi}{\pi}$$

```
plot(g,y=-1..1);
```



I find this result fascinating. I suspect there is some much simpler way to arrive at this result.

But first, what is the u(x) for this problem? He does not state the result! It is given by 6.146 and we already know that $c_0 = \ln(a/2)/2\pi$ from p 179 F. So

$$u(\mathbf{x}) = \ln(a/2)/2\pi + \int_{-a}^{a} d\xi\, [\, -(1/2\pi)\ln|\, \mathbf{x}-\xi\,|\,]\, [\, 1/\pi\sqrt{a^2-\xi^2}\,]$$

As before, let $\xi = a\tau$, but now let let $\mathbf{x} = a\mathbf{t}\mathbf{R}$

$$u(at\mathbf{R}) = \ln(a/2)/2\pi + \int_{-1}^{1} a\, d\tau\, [\, -(1/2\pi)\ln|\, at\mathbf{R} - a\tau|\,]\; [\; 1/\pi\sqrt{a^2 - a^2\tau^2}\;]$$

$$= \ln(a/2)/2\pi + \int_{-1}^{1} d\tau\, [\, -(1/2\pi)\ln|\, at\mathbf{R} - a\tau|\,][\; 1/\pi\sqrt{1-\tau^2}\;]$$

$$= \ln(a/2)/2\pi + \int_{-1}^{1} d\tau\, [\, -(1/2\pi)\ln(a)]\; [\; 1/\pi\sqrt{1-\tau^2}\;] \;+\; \int_{-1}^{1} d\tau\, [\, -(1/2\pi)\ln|t\mathbf{R} - \tau|]\; [\; 1/\pi\sqrt{1-\tau^2}\;]$$

$$= \ln(a/2)/2\pi \;-(1/2\pi^2)\ln(a) \int_{-1}^{1} d\tau\; [\; 1/\sqrt{1-\tau^2}\;] \;\; -(1/2\pi^2) \int_{-1}^{1} d\tau\, [\ln|\, t\mathbf{R} - \tau|]\; [\; 1/\sqrt{1-\tau^2}\;]$$

$$= \ln(a/2)/2\pi \;-(1/2\pi)\ln(a) \;-(1/2\pi^2) \int_{-1}^{1} d\tau\, [\ln|\, t\mathbf{R} - \tau|]\; [\; 1/\sqrt{1-\tau^2}\;] \qquad \text{// using p 179 integral}$$

$$= -\,(1/2\pi)\ln(2) \;\;\; -(1/2\pi^2) \int_{-1}^{1} d\tau\, \ln|\, t\mathbf{R} - \tau| \,/\, \sqrt{1-\tau^2}$$

Now set $\mathbf{R} = X\,\hat{\mathbf{x}} + Y\,\hat{\mathbf{y}}$ so we get

$$\ln|\, t\mathbf{R} - \tau| \;= (1/2)\ln\,(t\mathbf{R} - \tau\hat{\mathbf{x}})^2 \;=\; (1/2)\ln\,(\;\, t\,[X\,\hat{\mathbf{x}} + Y\,\hat{\mathbf{y}}] - \tau\hat{\mathbf{x}}\;\,)^2$$

$$= \;(1/2)\ln\,(\,[tX-\tau]\,\hat{\mathbf{x}} + tY\,\hat{\mathbf{y}}]\,)^2 \;=\; (1/2)\ln\,(\,[tX-\tau]^2 + t^2Y^2\,)$$

So we can write our result as

$$u(atX,atY) = u(x,y) = -\,(1/2\pi)\ln(2) \;-(1/4\pi^2) \int_{-1}^{1} d\tau\, \ln\,(\,[tX-\tau]^2 + t^2Y^2\,) \,/\, \sqrt{1-\tau^2}$$

$$u(ax,ay) = -\,(1/2\pi)\ln(2) \;-(1/4\pi^2) \int_{-1}^{1} d\tau\, \ln\,(\,[x-\tau]^2 + y^2\,) \,/\, \sqrt{1-\tau^2}$$

Just a **check** at this point: as x and y both get very large, this thing becomes

$$u(ax,ay) = -\,(1/2\pi)\ln(2) \;-(1/4\pi^2)\, \ln\,(\,r^2\,) \int_{-1}^{1} d\tau \,/\, \sqrt{1-\tau^2}$$

$$= \;-\,(1/2\pi)\ln(2) \;-(1/4\pi^2)\, \ln\,(\,r^2\,)\pi$$

$$= \;-\,(1/2\pi)\ln(2) \;-(1/4\pi)\, \ln\,(\,r^2\,)$$

$$= -\,(1/2\pi)\ln(2) \;-(1/2\pi)\, \ln\,(\,r\,)$$

$$= \;-(1/2\pi)\, \ln\,(\,r\,) \;+\{\,-\,(1/2\pi)\ln(2)\;\}$$

which is the potential of a point charge Q = 1 plus a certain constant. If we scale both x and y down by a, we get this:

$$u(x,y) = -(1/2\pi)\ln(r/a) + \{ -(1/2\pi)\ln(2) \} = -(1/2\pi)\ln(2r/a)$$

$$= -(1/2\pi)\ln(r) + (1/2\pi)\ln(a/2)$$

GR does not list definite integrals of "logs of more complicated arguments and algebraic functions". It turns out this integral is non-trivial, and I am going to undertake several attack plans. By the way, he never states the answer to this problem in this chapter including in the ending exercises. This book http://books.google.com/books?id=8tXUCl48XJ8C&printsec=frontcover#v=onepage&q=&f=false might have the answer. But preview is incomplete.

Plan A. So let's try setting $\tau = \sin\theta$ so $d\tau = \cos\theta d\theta$ and $\sqrt{1-\tau^2} = \cos\theta$ so our integral is

$$I \equiv \int_{-1}^{1} d\tau \ln([tX-\tau]^2 + t^2Y^2) / \sqrt{1-\tau^2} = \int_{-\pi/2}^{\pi/2} d\theta \ln([tX-\sin\theta]^2 + t^2Y^2)$$

$$= \int_{0}^{\pi/2} d\theta \ln([tX-\sin\theta]^2 + t^2Y^2) + \int_{-\pi/2}^{0} d\theta \ln([tX-\sin\theta]^2 + t^2Y^2)$$

$$= \int_{0}^{\pi/2} d\theta \ln([tX-\sin\theta]^2 + t^2Y^2) + \int_{0}^{\pi/2} d\theta \ln([tX+\sin\theta]^2 + t^2Y^2)$$

$$= \int_{0}^{\pi/2} d\theta \ln\{([tX-\sin\theta]^2 + t^2Y^2)/([tX+\sin\theta]^2 + t^2Y^2)\}$$

$$= \int_{-\pi/2}^{\pi/2} d\theta \ln(\sin^2\theta - 2tX\sin\theta + [t^2X^2 + t^2Y^2])$$

So sorry, no GR on any of this stuff. My trig substitution did not pan out.

Plan B. Try first doing derivative with respect to a parameter (t). Go back to

$$I(t) \equiv \int_{-1}^{1} d\tau \ln([tX-\tau]^2 + t^2Y^2) / \sqrt{1-\tau^2}$$

$$\partial_t I = \int_{-1}^{1} d\tau (1/\sqrt{1-\tau^2}) \{ 2(tX-\tau)X / ([tX-\tau]^2 + t^2Y^2) \} \quad \text{// Maple did derivative}$$

$$= 2X \int_{-1}^{1} d\tau (1/\sqrt{1-\tau^2}) \{ (tX-\tau) / ([tX-\tau]^2 + t^2Y^2) \}$$

At least we have gotten rid of trig functions and logs. Let's put Plan B on hold for the moment.

Plan C.

$$I(t) \equiv \int_0^1 d\tau \ln\left([tX-\tau]^2 + t^2Y^2 \right) / \sqrt{1-\tau^2} + \int_0^1 d\tau \ln\left([tX+\tau]^2 + t^2Y^2 \right) / \sqrt{1-\tau^2}$$

$$= I_1(t) + I_2(t) = I_1(t) + I_1(-t)$$

so we only need to do I_1. So

$$I_1(t) = \int_0^1 d\tau \ln\left([tX-\tau]^2 + t^2Y^2 \right) / \sqrt{1-\tau^2} = \int_0^1 (d\tau / \sqrt{1-\tau^2}) \ln(\tau^2 + A\tau + B)$$

This seems pretty simple but no listing. So write it this way

$$I_1(t) = \int_0^1 d\tau \ln \{([tX-\tau] + itY)([tX-\tau] - itY)\} / \sqrt{1-\tau^2}$$

$$= \int_0^1 d\tau \ln \{(\tau - (tX+itY))(\tau - (tX- itY))\} / \sqrt{1-\tau^2}$$

$$= \int_0^1 d\tau \ln [(\tau - (tX+itY)] / \sqrt{1-\tau^2} + cc$$

$$= I_{11}(t) + I_{11}(t)^*$$

So now we only need compute $I_{11}(t)$ where $\alpha = tX+itY$

$$I_{11}(t,\alpha) = \int_0^1 (d\tau / \sqrt{1-\tau^2}) \ln(\tau-\alpha)$$

Now finally Maple has something to say that is maybe useful:

```
f := ln(tau-alpha)/(1-tau^2)^(1/2);
```

$$f := \frac{\ln(\tau - \alpha)}{\sqrt{1-\tau^2}}$$

```
simplify(int(f,tau = 0..1));
```

$$\frac{1}{8}\,\frac{4\ln(-\alpha)\,\pi\,\alpha^2 - 8\,\mathrm{hypergeom}\left(\left[1, 1, \frac{1}{2}\right], \left[\frac{3}{2}, \frac{3}{2}\right], \frac{1}{\alpha^2}\right)\alpha - \pi\,\mathrm{hypergeom}\left(\left[\frac{3}{2}, 1, 1\right], [2, 2], \frac{1}{\alpha^2}\right)}{\alpha^2}$$

showing some kind of weird generalized hypergeometric functions. Is it really this bad?? [No!] Our complete answer is this:

$$I(t) = I_1(t) + I_1(-t) = I_{11}(t) + I_{11}(t)^* + I_{11}(-t) + I_{11}(-t)^*$$

Plan D. Let's now look harder at this integral

$$I_{11}(t,\alpha) = \int_0^1 (d\tau / \sqrt{1-\tau^2}) \ln(\tau-\alpha) \qquad \alpha = tX+itY$$

Go back to $\tau = \sin\theta$ so $d\tau = \cos\theta d\theta$ and $\sqrt{1-\tau^2} = \cos\theta$ so we get

$$= \int_{-\pi/2}^{\pi/2} d\theta \ln(\sin\theta-\alpha) = \int_0^{\pi/2} d\theta \ln(\sin\theta-\alpha) + \int_0^{\pi/2} d\theta \ln(-\sin\theta-\alpha)$$

But write

$$\ln(-\sin\theta-\alpha) = \ln[(-1)(\sin\theta+\alpha)] = i\pi + \ln(\sin\theta+\alpha)$$

So now define

$$J(t,\alpha) = \int_0^{\pi/2} d\theta \ln(\sin\theta-\alpha)$$

Then we have

$$I_{11}(t,\alpha) = J(t,\alpha) + \int_0^{\pi/2} d\theta (i\pi) + \int_0^{\pi/2} d\theta \ln(\sin\theta+\alpha)$$

$$= J(t,\alpha) + i\pi^2/2 + J(t,-\alpha)$$

so now all we need compute is this

$$J(t,\alpha) = \int_0^{\pi/2} d\theta \ln(\sin\theta-\alpha)$$

But now write

$$(\sin\theta-\alpha) = (-\alpha)(1 + a\sin\theta) \qquad -\alpha a = 1 \qquad a = -1/\alpha$$

$$\ln(\sin\theta-\alpha) = \ln(-\alpha) + \ln(1 + a\sin\theta)$$

Then

$$J(t,\alpha) = \int_0^{\pi/2} d\theta \ln(-\alpha) + \int_0^{\pi/2} d\theta \ln(1 + a\sin\theta)$$

$$= (\pi/2)\ln(-\alpha) + \int_0^{\pi/2} d\theta \ln(1 + a\sin\theta) \qquad a = -1/\alpha \qquad \alpha = tX+iY$$

Now try to locate this integral

$$K(t,\alpha) \equiv \int_0^{\pi/2} dx \ln(1 + a \sin x) = (1/2) \int_0^{\pi/2} dx \ln(1 + a \sin x)^2$$

But now finally GR p 527 #11 have something at last to say, but it depends on nature of a. They seem to suggest that a has to be real, but in my case a is complex. There must be a nice contour integral solution to this problem, so ...

Plan E. Go back to

$$I_{11}(t,\alpha) = \int_{-\pi/2}^{\pi/2} d\theta \ln(\sin\theta - \alpha)$$

Let $z = e^{2i\theta}$ so the integral becomes a complete unit circle. Then $dz = z(2id\theta)$ and

$\sin\theta = (e^{i\theta} - e^{-i\theta})/(2i) = (z^{1/2} - z^{-1/2})/(2i)$ so we get

$$I_{11}(t,\alpha) = \int_C dz/(2iz) \ln [(z^{1/2} - z^{-1/2})/(2i) - \alpha]$$

But write

$$\ln [(z^{1/2} - z^{-1/2})/(2i) - \alpha] = \ln \{ (1/2i) ((z^{1/2} - z^{-1/2}) - 2i\alpha) \}$$

$$= \ln(1/2i) + \ln(z^{1/2} - z^{-1/2}) - 2i\alpha)$$

The constant gives rise to something easy to compute, so look at the second term which involves this integral

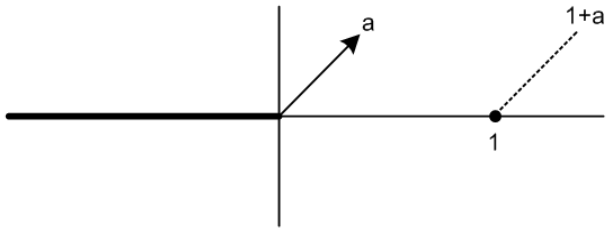
$$\int_C dz/z \ln [(z^{1/2} - z^{-1/2})/(2i) - \alpha]$$

This is getting very ugly.

Plan F. So go back to Plan D.

$$K(t,\alpha) \equiv \int_0^{\pi/2} dx \ln(1 + a \sin x) = (1/2) \int_0^{\pi/2} dx \ln(1 + a \sin x)^2$$

Consider the following picture of the z plane where $z = 1+a \sin(x)$. During this integration, z moves along the dotted line segment. Notice that we stay far away from the branch cut of ln(z), and this is true for any a that is not on the negative real axis. So I would say this integral is "non-singular" in nature.



So nothing strange should happen for "general a". Let's start with a being positive real and a < 1. The integral according to GR p 527 should then be

$$K(t,\alpha) = (1/2)\,\pi \ln\left((1 + \sqrt{1-a^2})/2 \right)$$

So I am perplexed that GR say something unusual happens when a is real and passes through a = 1.

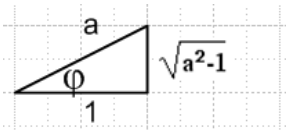
If a = 0 we get ln(1) = 0 so K=0. If a = 1 we get ln(1/2) which is fine. Suppose a > 1. Then

$$\ln\left((1 + \sqrt{1-a^2})/2 \right) = \ln\left((1 + i\sqrt{a^2-1})/2 \right).$$

Let $z = (1 + i\sqrt{a^2-1})/2 = Re^{i\varphi} = (1/2)\,[1\,,\sqrt{a^2-1}\,]$. Then

$$R^2 = (1/2)^2\,[\,1 + (a^2-1)] = (1/4)\,a^2 = (a/2)^2 \qquad R = a/2$$

This triangle



shows then that $\cos\varphi = 1/a$. Then

$$\ln(z) = \ln(Re^{i\varphi}) = \ln(R) + i\varphi = \ln(a/2) + i\cos^{-1}(1/a)$$

Then we seem to get

$$K(t,\alpha) = (1/2)\,\pi\,\{\ln(a/2) + i\cos^{-1}(1/a)\}$$

which does not agree with the claim GR make for $a^2 > 1$

My inclination is to take their result for a<1 and treat it as the analytic continuation for general a. If this is correct, then here is my answer:

$$K(t,\alpha) = (1/2)\,\pi \ln\left((1 + \sqrt{1-a^2})/2 \right) \qquad a = -1/\alpha \qquad \alpha = tX+itY$$

$$J(t,\alpha) = (\pi/2)\ln(-\alpha) + K(t,\alpha)$$

$I_{11}(t,\alpha) = J(t,\alpha) + i\pi^2/2 + J(t,-\alpha)$

$I(t) = I_1(t) + I_1(-t) = I_{11}(t) + I_{11}(t)^* + I_{11}(-t) + I_{11}(-t)^*$

So let's construct the answer based on the above assumption

$J(t,\alpha) = (\pi/2)\ln(-\alpha) + (1/2)\,\pi \ln((1 + \sqrt{1-a^2})/2)$

$= (\pi/2)\ln(-\alpha) + (\pi/2)\ln(1 + \sqrt{1-a^2}) - (\pi/2)\ln(2)$

$= (\pi/2)[\ \ln(-\alpha/2) + \ln(1 + \sqrt{1-a^2})]$

Now as $\alpha \rightarrow -\alpha$, $a \rightarrow -a$ in some manner we could clearly define. So it would seem that

$J(t,-\alpha) = (\pi/2)[\ \ln(\alpha/2) + \ln(1 + \sqrt{1-a^2})]$

Then

$I_{11}(t) = i\pi^2/2 + (\pi/2)[\ \ln(-\alpha/2) + \ln(1 + \sqrt{1-a^2})] + (\pi/2)[\ \ln(\alpha/2) + \ln(1 + \sqrt{1-a^2})]$

$= (\pi/2)\{ i\pi + \ln(-\alpha/2) + \ln(\alpha/2) + 2\ln(1 + \sqrt{1-a^2})$

Let's now define these four quantities

$a = -1/(tX+iY)$ $\quad b = a^*$ $\quad c = -1/(-tX+iY)$ $\quad d = c^*$

then

$I_{11}(t) = (\pi/2)\{ i\pi + \ln(-\alpha/2) + \ln(\alpha/2) + 2\ln(1 + \sqrt{1-a^2})$
$I_{11}(t)^* = (\pi/2)\{- i\pi + \ln(-\alpha^*/2) + \ln(\alpha^*/2) + 2\ln(1 + \sqrt{1-b^2})$
$I_{11}(-t) = (\pi/2)\{ i\pi + \ln(-\alpha/2) + \ln(\alpha/2) + 2\ln(1 + \sqrt{1-c^2})$
$I_{11}(-t)^* = (\pi/2)\{- i\pi + \ln(-\alpha/2) + \ln(\alpha/2) + 2\ln(1 + \sqrt{1-d^2})$

Adding these all up we get

$I(t) = \pi[\ \ln(-\alpha/2) + \ln(-\alpha^*/2) + \ln(\alpha/2) + \ln(\alpha^*/2)\ ] +$

$\pi \ln\{ (1 + \sqrt{1-a^2})(1 + \sqrt{1-b^2})(1 + \sqrt{1-c^2})(1 + \sqrt{1-d^2}) \}$

But $(1 + \sqrt{1-a^2})(1 + \sqrt{1-b^2}) = |1 + \sqrt{1-a^2}|^2$ etc. Also,

$\ln(-\alpha/2) + \ln(-\alpha^*/2) + \ln(\alpha/2) + \ln(\alpha^*/2) = \ln\{ (-\alpha/2)(-\alpha^*/2)(\alpha/2)(\alpha^*/2) \}$

$= \ln \{ |\alpha|^4/2^4 \} = 4 \ln(|\alpha|/2)$

so our answer is now

$I(t) = 4\pi \ln(|\alpha|/2) + \pi \ln \{ |1 + \sqrt{1-a^2}\,|^2 \, |1 + \sqrt{1-c^2}\,|^2 \}$ $= u(atX,atY) = u(x,y)$

$a = -1/(tX+itY)$ $c = -1/(-tX+itY)$

Restate as

$u(x,y) = -4\pi \ln(2|a|) + 2\pi \ln \{ |1 + \sqrt{1-a^2}\,| \, |1 + \sqrt{1-c^2}\,| \}$

$a = -1/(x+iy)$ $c = -1/(-x+iy)$

Now write $z = x+iy = r\, e^{i\varphi}$. Then

$a = - [r\, e^{i\varphi}]^{-1} = - (1/r)\, e^{-i\varphi}$

Then

$c = -1/(-x+iy) = -1/(-x-iy)^* = 1/(x+iy)^* = (-a)^* = (1/r)\, e^{i\varphi}$

So we then have

$u(x,y) = -4\pi \ln(2|a|) + 2\pi \ln \{ |1 + \sqrt{1-a^2}\,| \, |1 + \sqrt{1-c^2}\,| \}$

$a = - (1/r)\, e^{-i\varphi}$ $c = (1/r)\, e^{i\varphi}$ (r,φ) = polar coordinates of x+iy

Then $-4\pi \ln(2|a|) = -4\pi \ln(2/r)$. And

$|1 + \sqrt{1-a^2}\,|^2 = (1 + \sqrt{1-a^2}\,)(1 + \sqrt{1-b^2}\,)$

$= (1 + \sqrt{1-e^{-2i\varphi}/r^2}\,)(1 + \sqrt{1-e^{+2i\varphi}/r^2}\,)$

So here is our final answer:

$u(x,y) = -4\pi \ln(2|a|) + \pi \ln \{ |1 + \sqrt{1-a^2}\,|^2 \, |1 + \sqrt{1-c^2}\,|^2 \}$

$= -4\pi \ln(2/r) + 2\pi \ln \{ (1 + \sqrt{1-e^{-2i\varphi}/r^2}\,)(1 + \sqrt{1-e^{+2i\varphi}/r^2}\,) \}$

I have very little confidence in this result, but I wanted to force my way through to some kind of answer. The thing is of course real. As $r \to \infty$ we get this result

$-4\pi \ln(2/r) + 2\pi \ln(4) = -4\pi \ln(2/r) + 4\pi \ln(2) = 4\pi \ln(r)$

which is not totally unreasonable. If we treat the strip as a point charge with Q = 1, the answer ought to be this: $u = (-1/2\pi) \ln(r)$, so off by a factor of (-2).