Jackson Chapter 16 Notes PhL 1.31.03 → 2.5.03

Chapter 16: Multipole Fields

This chapter shows how to solve Maxwell's equations in full glory in spherical coordinates. The usual $e^{i\omega t}$ time dependence is assumed. The general idea is that you expand the fields in each region of your problem in full generality (which means all ℓm values and both electric and magnetic modes), then you require that field boundary conditions be met at the boundaries. Several examples are given, and comments are made regarding both atomic and nuclear emissions. When sources are present, it is shown how to compute all the moments. In the general case there are 4 such moments in each ℓm harmonic. As an example of a problem with "sources", the center-fed antenna is treated with the usual King Smile assumption about the current shape. The result is compared with a full treatment in Chapter 9 that did all the multipoles at once, but again in the King Smile limit. The second example is scattering from a conducting sphere where we get heavily involved in the formalism. I have done a third example of scattering from a dielectric sphere in a separate document, a somewhat harder problem that the conducting sphere.

The reason that the multipole expansion can help solve problems is the usual one. The rotation group has "good quantum numbers" ℓ and m since each represents a conserved quantity. If the corresponding operators commute with the Hamiltonian, the solution can be diagonalized into the individual partial waves ℓm and there is no interaction between different partial waves. So the problem is simplified into a set of smaller problems, one in each partial wave. Within a partial wave, the general form of a solution is quite limited, there are only two radial function possibilities. If you do something like scatter off a cube, the boundary conditions will be functions of spherical angles, and this will mix adjacent harmonics of different ℓm, I think, this corresponding to that fact that the "potential" that this cube represents is no longer spherically symmetrical. For problems with radiating sources, the expansion is useful in long λ limit because the terms drop off quickly.

16.1 Solutions of the scalar wave equation in (r,θ,ϕ). Back on page 54 where we had the Laplace equation (wave equation with $k^2 = 0$), we assumed U(r)/r as the radial form and got (3.7) as the radial equation which led to r^{ℓ} and $r^{-\ell-1}$ as the solutions. We ended up with an expansion as in 3.61 on page 67. Here, we have that extra k^2 term, so the radial functions are going to be different. The expansion of a scalar wave equation solution is given in 16.4 and the radial functions turn out to be the *spherical* Bessel functions of various types. So one way to show the expansion is in 16.16 using the Hankels. Naturally, Jackson takes this opportunity to tell us *all about* these Bessel functions. Note that the Hankels are just those linear combinations of the j and n functions which match outgoing or incoming spherical waves at large argument.

Now, the Green's function in the presence of $k^2 \neq 0$ is e^{ikR}/R as in 16.18. Of course we want to expand this in the usual double-spherical-harmonic type formula and that turns out to be 16.22 which we can compare to the Laplace version in 3.70 page 69. We now have non-trivial radial functions in this thing, they are of course the new spherical Bessel functions. For $r_>$ we have the Hankel which dies as e^{ix}/x, and for $r_<$ we have the j type which is well-behaved at the origin.

We are now top of page 542. Here, we look again at the ODE for the $Y_{\ell m}$ functions. Jackson defines a vector of operators which include $\partial/\partial\phi$ and $\partial/\partial\theta$. He calls this vector by the name **L**, and identifies this thing squared as the LHS of that ODE! I know that this is just a specific representation of these operators,

and that they have properties that are representation-independent. We get the raising and lower L operators as shown. Page 543 shows some commutators:

$[L^2, \mathbf{L}] = 0 \qquad [\mathbf{p}^2, \mathbf{L}] = 0$

Then 16.30 shows us that we really already know about L^2.

Remember from quantum wave mechanics that $p = (\hbar/i)\nabla$ and $L = r \times p = (\hbar/i)\, r \times \nabla$. These are the representations of p and L in the "x,y,z" world, and you could compute the momentum say of a state by doing this: $\langle\psi|p|\psi\rangle = \langle\psi|x\rangle\langle x|p|x'\rangle\langle x'|\psi\rangle$ and the thing in the middle is $\delta(x-x')(\hbar/i)\nabla$ (I think). Here in E&M Jackson is using the same operators with $\hbar=1$ just because they are convenient for his manipulations which are going to involve the same $Y_{\ell m}$ that appear in spherical coordinates in QM. In this chapter we are not "interpreting" L very much, just using it as a tool. If you write ∇^2 in sphericals, you get 16.30 where L^2 takes care of the angular stuff. So, I am happy with all this stuff on p 542-543.

16.2 Multipole Expansion of the Fields. First off the bat, Jackson writes two 3-equation sets which are claimed to be equivalent to Maxwell's 4 equations. In each case, the fourth is true because div curl = 0. So let's go with a B expansion, since either will do. Since B as a vector solves the wave equation, we get 16.35. BUT, this expansion as written only solves the wave equation member of our 3-equation set. We have to *impose* that $\nabla\bullet B = 0$ on the coefficients. This turns out to be a huge mess, and Jackson takes us through the gory details, and when the dust settles, we replace 16.35 with a sum with some as-yet-not shown coefficients over a set of basic fields $\mathbf{B}_{\ell m}$ given by 16.42. These have the form of any radial solution times not $Y_{\ell m}(\theta,\phi)$ but $\mathbf{L}\, Y_{\ell m}(\theta,\phi)$. In 16.45 Jackson normalizes these things and calls them $\mathbf{X}_{\ell m}(\theta,\phi)$. He selects X as a letter close to Y, and **X** is a vector.

Now here is a major point. Notice that $r \bullet \mathbf{B}_{\ell m} = 0$ because $\mathbf{r}\bullet\mathbf{L} = 0$ in an operator sense from 16.27. (From mechanics, the angular momentum L is perpendicular to both r and p, since L = r x p.) This means that all the solutions $\mathbf{B}_{\ell m}$ in 16.42 are "transverse magnetic" or TM. The B field is exactly perpendicular to **r**. Note that there is in general no reason why this must be true in a general solution, although it is in a plane wave solution of Maxwell's. But this set of solutions does have the TM property. And of course if you start with E instead, you get the $\mathbf{E}_{\ell m}$ shown in 15.44 and these will all be TE solutions. Luckily these two sets of solutions form a complete set! These $\mathbf{E}_{\ell m}$ and $\mathbf{B}_{\ell m}$ are called "the multipole fields".

Now TM is better called "electric" because we shall soon see that the driving source of such fields is electric charge, whereas the "magnetic" TE solutions are driven by magnetic charge. Recall right now from electric dipole radiation that E stays parallel to p no matter where you go, so this cannot be a TE mode. It turns out to be a TM mode and the dipole B field is going to be fully transverse.

So now we have the Grand Finale of this section which is expansion 16.47. Notice that B and E shown have all possible TM and TE components each. The f_ℓ and g_ℓ are not specific radial functions, they are some arbitrary linear combinations of the two radial basis functions. Notice that the same coefficients appear in each equation like a_E because that is the way the solutions are proportioned, just do it! Similarly, whatever the function f_ℓ is, it must be the same function in both equations, and the same for g_ℓ.

This discussion implies that you can and should study the individual modes by themselves. The big advantage here is that we can apply spherical system boundary conditions most conveniently in this form!

16.3 Properties of the Multipole Fields. The first order of business is to study these fields at very small and very large radius r.

First the small r situation. For electric, we get 15.61 which replicates our multipole expansion equation 4.1 from electrostatics, and 16.48 goes with it. The B field here is kr times smaller in size that the E field, another reason we might call this "electric" instead of TM. For the "magnetic" modes, opposite is true, and you can always reverse things with 16.52.

Now for large r with our usual outgoing boundary condition. In this case the radial function has to be the Hankel-1 which has the right large-r form. (Note that Hankel 2 is the complex conjugate and therefore goes as exp(-ikr)/r which is incoming, not outgoing) For the electric we get B as in 16.53 as the large-r limit, and then E is the usual far-field B x n result, so in the far zone B and E are equal in size. Note Well: all multipoles have solutions which go as 1/r and thus can exist in the large-r limit, there is nothing special about dipole.

Now we stay in this large-r limit and ask the questions: how much energy, and how much angular momentum is carted off by a given multipole field? On page 548 Jackson decides to look at the electric multipoles { I suspect the conclusions are the same for the magnetics, but I don't think he comments on this } and he computes the answers to the two questions. He gives the electric multipole its usual coefficient a_E as shown in 16.57. We are going to assume that this is all we have at the moment.

Recall from page 200 that the instantaneous linear momentum density is $\boldsymbol{\mathcal{P}} = \mu\varepsilon\, \mathbf{S}/c^2 = \mathbf{S}/v^2$ where S = (c/4π) E x H.. Also recall from that page that the instantaneous angular momentum density is $\boldsymbol{\mathcal{L}} = r \times \boldsymbol{\mathcal{P}}$. If we use our time average theorem (see Chap 7 notes), then we add 1/2 and put * on the second field, so we then have $\boldsymbol{\mathcal{P}} = (1/8\pi c)\, E \times H^*$ in this sense, and $\boldsymbol{\mathcal{L}} = r \times \boldsymbol{\mathcal{P}} = (1/8\pi c)\, r \times (E \times H^*)$. Instead of using the symbol $\boldsymbol{\mathcal{L}}$, Jackson uses symbol m for this in 16.61. { He does not want to confuse $\boldsymbol{\mathcal{L}}$ with L }

So, how much U and how much $\boldsymbol{\mathcal{L}}$ is contained in a particular multipole field? The time averaged energy dU contained in a shell of thickness dr is given by 16.60. The time averaged $\boldsymbol{\mathcal{L}}$ in the same shell is given in 16.64. In both cases, the angle stuff is exact, and we use large-r (radiation zone) for the radial stuff. Yes, only $\boldsymbol{\mathcal{L}}_3$ does not vanish because of the raising/lowering business. We get the famous result then that $\boldsymbol{\mathcal{L}}_3 / U = m\hbar/\hbar\omega$ and $\boldsymbol{\mathcal{L}}_1 = \boldsymbol{\mathcal{L}}_2 = 0$. We are saying that an electric multipole field ℓm in the radiation zone only has angular momentum about the polar z axis and the amount is "as if" the field were made of "photons" having energy $\hbar\omega$ and having L3 = m $\hbar$. This is some kind of special ℓm photon!

On page 549 Jackson tries to explain why was don't get an $\ell(\ell+1)$ form for |M| the way we do in quantum mechanics, but his discussion evades me, he is really quoting from another source. This is one example of why engineers probably don't like Jackson's book much, because of this "real physics stuff".

Now, let's talk about how this "ℓm photon" is produced by quantum state transitions of matter, perhaps from atoms. Suppose the quantum states are |JM> and |J'M'>. Then the selection rules are these:

J' must be contained within the series $\mathbf{J} \otimes \boldsymbol{\ell}$ = J-ℓ up to J+ℓ
M' =M + m // Jackson has the other sign on m, I don't see why!
for parity conservation, must have
Parity(|J'M'>) = Parity(|JM>) * $(-1)^{\ell}$ for electric, $(-1)^{\ell+1}$ for magnetic

The first two rules are just conservation of angular momentum .

This process of emitting an ℓm photon is called a "multipole transition". I don't have any idea of what is really going on in such a transition. In general it must involve multiple true photons, since each photon only really has j = 1. Do I have a book where this subject is discussed? On the web I see a guy saying that in spherical geometry, you really do have ℓm photons. I have to imagine this as a coherent state of multiple photons -- a multi-photon state that has quantum numbers ℓm. Still, various people do speak of the ℓm thing as "a photon". I will have to put this subject off to another day. I think there are interesting

implications. Although I have many physics books, none of them seem to address this subject, but I have not looked in depth. (Question: in positronium we have s1, s2 and L combining to make J. In a 2-photon state, is there something like L?)

The bottom line for me regarding multipoles is this: probably in a dielectric material like water, selection rules are probably going to cause all transitions other than electric dipole (electric, $\ell = 1$) to be very weak. If a material *can* emit ℓ=5, that is fine, and this will radiate just as far as ℓ=1. But this probably does not happen in normal materials.

16.4 Angular Distribution of the Multipole Fields. Recall that we had the general multipole expansion for B and E showing in Grand Finale 16.47. At that time, we were supposed to think of the f_ℓ type function as an arbitrary lin comb as in 16.43 with two more arbitrary coefficients that depend on ℓ. Since we have another coefficient layer on top of this (the a_E for example), we could decide to normalize these lower level coefficients so the sum of squares is 1. In any event, in the far zone where we have only h1, we now give it a unity lower level coefficient. Doing this, we then arrive at 16.69 for the large-r form of the full fancy expansion! In this limit, the two fields have the simple n x relationship, and now we see the all-important upper level coefficients exposed in this limit.

Getting the power per angle function 16.70 is a trivial step. For a single pole term we get 16.71 in terms of the fancy X harmonics. If you expand that out in terms of the Y functions, you get a mixture or raising, lowering, and non-changing L operators, so we expect to see a mixture of adjacent m values. The result is stated in 16.72 without proof. Jackson's table on page 551 then shows the exact results of the X^2 object for ℓ=1 and ℓ=2. The dipole interpretation is this: m=0 term would come from a z-aligned dipole of size $p = a/k^3$ as I learn by comparing to 9.23. If you had an x-aligned, you would get a mix of $m = \pm 1$ in the usual combination.

We see the plots for the ℓ=1,2 multiple cases (all of them). Jackson's comment is a good one on page 553: If you have some atoms radiating thermally, say, you would expect an incoherent sum of all the m multipoles for a given ℓ, and 16.73 shows that the result is isotropic as it of course has to be. But, if we stimulate atoms with a special field (plane wave, eg), this will not be the case, I suspect. He concludes by integrating the differential power over solid angle to get very simple results.

16.5 Sources of Multipole Radiation: the Moments.

Digression on magnetization M. Look back at page 151 in the magnetics area which I skipped on this reading. Equation 5.77 shows the notion of a magnetization density $M = n \langle m \rangle$ in exactly the way we talked about $P = n \langle p \rangle$ on page 118. As in the p case, you can have already-existing **m** dipoles in a material like iron which just get lined up, or you can induce your own **m** dipoles. Interestingly, Jackson does not have a little section on "models for M" in his book, the way he does for P. Portis does discuss this on page 243, however. Roughly setting $M = \chi B$ and studying a simple model for an electron, he gets result (24) for χ . This is "diamagnetism". Electrons tend to spin around in a B field in circles and make moments m, so in diamagnetism we are creating these moments. In paramagnetism these orbits already exist and we thermally line them up, giving 38. Spin gets into the act as well, so you can see that this is a complicated subject. Now, for our current purposes, the main point to note about M is that its presence in a material makes an effective current density which is $\mathbf{J_M} = c\,\nabla \times \mathbf{M}$ as on page 152. This is analogous to $\rho_P = -\nabla \bullet P$ as on page 112. Recall that you can say that $\nabla \bullet \mathbf{E} = 4\pi(\rho_{free} + \rho_P)$ but $\nabla \bullet \mathbf{D} = 4\pi\rho_{free}$ where D is defined exactly so it only sees free charge. In similar fashion you define H so that $\nabla \times \mathbf{B} = (4\pi/c)(\mathbf{J_{free}} + \mathbf{J_M})$ but $\nabla \times \mathbf{H} = (4\pi/c)\,\mathbf{J_{free}}$.

So, as we start out here, Jackson seems to be setting $\rho = \rho_{free} + \rho_P$, but he has $\mathbf{J} = \mathbf{J}_{free}$ and he breaks out the M stuff separately, calling it script M. Next, Jackson has to define a new field variable **E'** as in 16.79 to incorporate the free J just to simplify the manipulations that are coming.

The first order of business is to now rewrite our "wave equation sets" by adding the sources, and this is done in 16.80 and 81. The result is an incredible mess! You wonder why people don't do the expansion in the vector potential A where the sources are simple, but I guess the reason is that we don't know how to put boundary conditions on A to find a solution. So the B wave equation is driven by ∇ x **J** and a double curl on M which seems pretty clear. The E' equation is symmetrical in that it is driven by ∇ x **M** and a double curl of J. Hey, where did ρ go? Now we see the motivation for E' ! It is defined exactly so that $\nabla \bullet E' = 0$. The J term is added to E' and then $\nabla \bullet J$ is known in terms of ρ from continuity. So the answer is that the wave equations don't have ρ because they come from the Maxwell curls only. Then this E' trick gets rid of ρ as just mentioned. Fine.

Now things are going to get VERY serious! First, the new B and E' equations match the former B and E equations outside the sources, so there we know the Finale 16.47 applies for B and E' (another reason to do E' as done). So let's try something like 16.82 with asymptotic limits 16.83 to make things consistent with our general notation in the far zone. That is, this 16.83 is setting the scale on our functions f and g. Jackson then swirls around to come up with ODE's for f and g which are driven by complex source terms, see first two equations on page 555. The ODE left sides are exactly what we had earlier way back in 16.5 page 539 (the spherical Bessel operator, so to speak). We already computed the Green's function for this operator on page 541 16.21, and now we are going to use it! It is just a product of two Bessel functions, the inner one j, the outer one h. So, creating a shorthand notation K_E for the messy right side of 16.84, we instantly have the Green's solution 16.86 for f. Taking the large r limit, we are then able to identify the a_E coefficient, and this is then stated in 16.88. Same for a_M below that. Obviously, these things are "moments" of the source situation that drive the multipole fields.

The two results for these moments are then "simplified" and presented again on page 556. Each one is now an integral of source stuff against a $Y_{\ell m}$ function, the whole thing volume-integrated over the source region, and we notice that our friend ρ has reappeared in the first equation.

Let's pause to think about this. If we stimulate a water sphere with a plane wave, we can probably decompose the plane wave into multipole components (soon!). We can then think of these as somehow inducing a J, ρ and M in the water droplet -- and that stage of course involves the "physics" of the matter known as water; perhaps there is no significant M. Those induced sources then in turn imply some multipole moments according to the integrals shown on page 556, and these then give radiated multipole fields for B and E' We know what each moment field E' or B looks like for large r, and in theory we can compute the details at any r, but then we have to do the detailed integrals as in 16.86. This is because we need to know the f and g functions which appear in assumed 16.82. The good news is that we have a general solution to the entire problem in terms of some integrations. By the way, to carry out the induction program mentioned here, we have to assume the radiated fields are locally weak compared to the incoming field, something I want to see a proof of here at some point.

Now in the large λ limit (meaning kd >>1 where source has size d), things simplify because we can take limits of the Bessel functions. We end up with four kinds of moments called Q, Q', M and M' as detailed on page 556-7. We recognize Q as our electrostatic moments! We have $a_E \sim Q + Q'$ and this second Q' thing can arise only if you have M, and in that case, $Q' << Q$. So the main idea is that a_E is mainly driven by ρ, and that is why we called the TM modes "electric". We also have $a_M \sim M + M'$, where M is driven by the free current J, and M' comes only if there is M. M and M' are usually the same

size so you have to worry about both of them if M is present. Now we see why he used script M -- to avoid confusion with these final long-λ moments!

So wow, again, we have a complete solution of things in a form geared to distant radiation problems from a localized source that is small relative to λ.

16.6 Radiation from Atoms and Nuclei. Another section very important to me. For both atoms and nuclei, we are going to use the same format "ball park" estimates for the electric and magnetic sources.

Rule 1: for EM radiation by either atoms or nuclei, we are usually in the long λ limit kd >> 1. Certainly for atoms and visible light this is the case, compare 1A to 5000A. In our simple atomic model, our estimate for kd is ka $\sim(Z_{eff}/137)^2$. For nuclear, ka cover too large a range for us to say something about it.

To say that we have a ρ that is going to drive a *particular* multiple electric field is to say 16.99. Then in the integral 16.94 only one $Q_{\ell m}$ is generated. The dimensions require some kind of e/a^3 factor, where e is the charge of the "thing" that is radiating (electron in atom, proton in nucleus), while a is the size of the thing you are interested (atomic or nuclear radius). Jackson throws in a factor of 3 here probably just to make things look nicer down the road, no comment is given. Maybe each atom has 3 radiating electrons? I think it is just to make things simpler in our later comparisons. So I agree that 16.99 is reasonable for ρ.

Now 16.101 is trickier to understand. Here, he combines the M and M' integrands together. We dimly recall how magnetic dipoles are enhanced by g-factors, so we will allow such a dimensionless factor. The basic unit is going to be the Bohr mageton for atoms. In Livesey page 163 we see that this is the classical magnetic dipole moment of a circulating electron $m = e\hbar/2m_e$ usually called μ_B. The electron spin on its own has a similar moment, give or take a factor of 2. For the nucleus, the appropriate mass is m_p and again moments come from both "orbital" and "spin" presence in the nucleus.

So, in 16.101 we have the g factor, we have a $1/a^3$ for our volume dimension factor, and we have an appropriate magneton factor. The 1/r there must be due to our having $\nabla\bullet$M and not M. Note that in spherical coordinates, the divergence of a constant vector $\mathbf{A} = A\hat{\mathbf{r}}$ is $\nabla\bullet A = A\,(2/r)$. All the divergence terms have a 1/r in them due to the r^2 dr volume factor, so I am happy to have a 1/r, and we need it as well just to have the dimensions be right. Finally, we have our usual $Y_{\ell m}$ for this assumed multipole.

Using these crude estimates for the driving factors of our moment integrals, we get the three simple results shown on page 558 for Q, Q' and M+M'. Right off the bat, we get a major result:

Rule 2: For either atoms or nuclei, Q' <<< Q due to relation 16.103, so forget Q' (the "induced electric")

At the same time, we use 16.98 to talk about our transition rates for radiation between two quantum states. That is, we are simply dividing the total power (as computed in 16.97 for a multipole) by the photon energy which has dimensions 1/time. We associated $1/\tau$ with the rate of transition measured in number of photons/sec. Using this general formula and inserting our electric result 16.100 we get 16.104 for our transition rate.

Aside: { In answer to an earlier question I had, you can add $\boldsymbol{\mu}\bullet\mathbf{B}$ into a Hamiltonian and get quantum split levels due to this thing and I presume these would be M1 transitions. }

Now, in 16.105 we are comparing the rate of electric versus magnetic for the same order ℓ, and then in 16.106 we compare order ℓ to order ℓ+1, which comparison is the same for either electric or magnetic.

Atomic Radiation. In this case, the factor in 16.105 is roughly $(Z_{eff}/137)^2$ which is a very small number unless you are talking X-rays from a deep inner shell of a very heavy atom. For water this is going to be on the order of 10^{-2} to 10^{-4} . So we get this rule:

Rule 3: In atomic radiation of order ℓ, magnetic is *weaker* than electric by 10^{-2} to 10^{-4} $[(Z_{eff}/137)^2]$. So you will see the magnetic only if the electric is somehow forbidden by selection rules.

Rule 4: In either atomic of nuclear radiation of either type (M or E) , the transition rate in order ℓ+1 is smaller than in order ℓ by $\sim (kd)^2 \sim (Z_{eff}/137)^2$, as in 16.108. Therefore, within a multipole family (M or E), the lowest allowed ℓ dominates. Usually this is electric dipole for the E family. So, when electric dipole is fully allowed, you don't have to think about higher order Q moments, assuming they are "stimulated" in a similar strength in an induced radiation situation. Thus, even if we find that a plane wave has large higher Q moments beyond $\ell = 1$, they are not going to matter when we scatter light off water drops.

Now some nuclear-only comments, starting bottom of page 559.

From the 16.104 estimate, you can plot lifetime τ versus ω (or $\hbar\omega = E$) as shown, and you get a different line for each order ℓ due to the power law you see sitting in 16.104, which is $\omega^{2\ell+1}$ so on log log plot we get $\log \tau = -(2\ell+1) \log E + \text{constant}(\ell)$. One point to be made here is this: for a fixed energy release E (draw a vertical line in the plot), you find that the higher order ℓ transitions have longer times and lower transition rates. Jackson claims that if you study a boatload of nuclear EM transitions, they fall in bands close to the lines shown in the figure.

Now for nuclear, what about ratio 16.105 for E versus M of the same order ℓ ? In general, electric are stronger by 25-120 times except for ℓ= 1 where the E1 is inhibited by a characteristic of nuclei (perhaps naively you don't see much a p moment in a ball of protons), so E1 and M1 are about the same in this one case.

Now a slightly different reading on ratio 16.106. For a given parity, if the lowest order of M is ℓ, then the lowest order in the E family for the same transition will be $\ell + 1$, so 16.110 is a ratio you really care about. In nuclei, you might find that Q2 is 5% of M1 as being typical. So if the lowest allowed is M, then you might have *some* noticeable E in the next higher order. However, if the lowest allowed is E_ℓ, then 16.111 shows that the next parity allowed transition $M_{\ell+1}$ is completely negligible.

16.7 The very famous center fed antenna. Total antenna length is d. Do NOT assume kd >> 1. Study this case very carefully please. There is no M, so ignore those terms in 16.91 and 16.92. Inside the antenna wires, $\mathbf{r} \times \mathbf{J} = 0$. This kills off the only remaining term in the a_M equation, so a major fast result:

(1) there are no magnetic modes at all, so all modes are going to be electrics such as electric dipole E1

Now for current, Jackson for the moment just assumes some unknown I(r) that vanishes at the ends of the antenna wires. He can then write J as in 16.113 and then ρ as in 16.114, using delta functions. Since J is in the $\hat{\mathbf{r}}$ direction, and since we have $\mathbf{r} \bullet \mathbf{J}$ in the a_E equation, the integrand has no ϕ dependence, which means that:

(2) only the m=0 electric modes survive. We knew this anyway from symmetry.

Now, the integrand has no θ dependence either, apart from the $Y_{\ell m}$ function. Since the even ℓ $Y_{\ell 0}$ functions are odd, they are killed off as well, so

(3) only the ℓ = odd modes survive.

So we now want to compute up the surviving modes which are ℓ odd and m=0 using 16.118, and now we need a model for the current I(r). The wave equation *suggests* exp(ikz) type dependence along the wires, as if we had a plane wave in space along the z direction. If you made a parallel wire transmission line, you would expect to see exp(ikz) along the line, but this is another subject Jackson does not treat in his book (well maybe he does). You could regard the conductors as mere boundary conditions and the wave travels along them at the full speed of light with exp(ikz) in the ideal case. When there are ohmic and radiative losses, you have to modify this result. So Jackson is saying lets try to ignore radiation loss and ignore ohmic loss and put in the idealized current shape which then is 16.119 since it has to vanish at the ends. Boom, you can now get the result as in 16.120! All done! This is the general result (with our assumptions as stated) regardless of the relationship between d and λ.

Jackson next considers the half-wave and full-wave antennas as examples. For the shorter antenna, the table shows that the E1 is most of it, and the E3 is about 5% and the E5 is 0.1%, we are talking amplitude here. For the longer antenna, things converge more slowly. The E3 is now 33% of E1 and the E5 is 3% in amplitude. Things get messier when they are large compared to λ, keep that in mind!

He goes on to compute the power distribution for the sum of E1 + E3 for these two antennas. The math is done on page 565 with some mildly messy numerical results.

Now back in Chapter 9 we solved the arbitrary length center-fed antenna with the same assumptions as here (same current assumed, same ignoring of radiative and ohmic losses), but there we used the full power far-zone result (9.8) so we computed all orders at once! Here we are just doing E1 + E3, so it is interesting to compare the results. Jackson did write down the half and full wave "exact" results on page 279, so we can compare our approximation here to that. The plots on page 566 are for this purpose.

Aside: I recall how my Harvard independent study guy King used to pooh-pooh books like Jackson's for assuming the current is not affected by the radiation. "Real" antenna people like King take this into consideration. Recall also that I have a whole fat Transmission Line Theory book by this same King.

16.8 Expand a plane wave in sphericals. The result we want is 16.127 or 16.128, and Jackson derives these quickly by just taking the large r' limit of the exp(ikR)/R result we already found in 16.22. He gets here to use his "sum rule" found earlier. (It is always nice to get future use out of an obscure result!)

Look at 16.129. For sure all ℓ orders appear here, although only with m=0. The z-axis here can be thought of as the k axis in which case we identify γ = θ. There is lots of energy in lots of orders, so you cannot say a plane wave is a dipole E1 or anything like that! In fact, a plane wave is going to try to induce all $Y_{\ell 0}$ modes into an object that will radiate. We should see this happen in the conducting sphere example to follow. Some day I will try to graphically add the terms to see how a plane wave arises here!

Now Jackson starts over without clearly explaining his motivation, but I am now aware of it. In this section, Jackson says: " Think about 16.128 but now add a unit vector to it so you have a real field instead of just a scalar function. In fact, take the traditional linear combinations of those unit vectors that define the circular polarizations. We like these because they are going to allow us to use $L_{\pm}$ operators in our calculation which will greatly simplify things. Now using this, let's derive result (16.139). We are going to need this result in the next section in the problem of scattering from a conducting sphere! "

Sidetrack and Confusion: I tried to show the these circular polarizations somehow diagonalize $L_z = -i\partial_\phi$ with ± 1 eigenvalues, but concluded that in fact, for a given t and z, a circ pol plane wave has no ϕ dependence, so $L_z = 0$. In fact, if I compute $\mathbf{L}\, e^{i\mathbf{k}\bullet\mathbf{r}}$ I find that $d\mathbf{L} = \mathbf{L}\, e^{i\mathbf{k}\bullet\mathbf{r}}\, dV = k\, e^{i\mathbf{k}\bullet\mathbf{r}} \sin\theta\, \hat{\boldsymbol{\phi}}\, dV$ for a little volume dV at location r,θ,ϕ relative to **k**. If I integrate this around the ϕ ring at r, θ, we get zero. So then the integral over the entire plane wave is also 0. So the total "angular momentum" of a plane wave is 0. BUT, I have here not really computed the L of a field, only of a scalar function $e^{i\mathbf{k}\bullet\mathbf{r}}$, and L can only be applied to a scalar function. Still, one might talk about $\mathbf{L}\, \mathbf{E}$ where we really mean $\mathbf{L}\, E_i$ and then it is true. In a plane wave, we have photons traveling in the k direction, so the ring integral just described really is zero. I don't see how the circ pol changes this situation.

However: if we look in the problems on page 200, we see that Jackson has in fact pondered this question with a medium ε,μ. In the above paragraph, I was thinking of L as the operator $\mathbf{L} = (1/i)\, \mathbf{r} \times \boldsymbol{\nabla}$. However, on page 200 we show that a EM field has an angular momentum density $\boldsymbol{\mathcal{L}} = r \times \boldsymbol{\mathcal{P}}$ where this thing $\boldsymbol{\mathcal{P}}$ is the linear-momentum density of the field, given by $\boldsymbol{\mathcal{P}} = \boldsymbol{S}/v^2$ where **S** is the Poynting vector, and v = velocity = $c/\sqrt{\mu\varepsilon}$, so $\boldsymbol{\mathcal{P}} = \sqrt{\mu\varepsilon}\, \boldsymbol{S}/c^2$. In problems 6.11 and 6.12, which I have now more or less done, we show that a localized plane wave (Gaussian beam like) with circ pol does in fact have a z component of angular momentum that is ±1 when scaled in the usual way $N\hbar/\hbar\omega$. So my confusion was to confuse p with $\boldsymbol{\mathcal{P}}$ and L with $\boldsymbol{\mathcal{L}}$. Also, in the following when we expand a true plane wave in sphericals, we will again find that for each multipole component ℓ of such a plane way, we still get a z component of angular momentum that is ±1. Here the units are m = ±1. There are no components of a plane wave that have m > 1.

Resume: So how does Jackson derive 16.139? He starts by assuming the expansion 16.131 for a plane wave of circular polarization. This is the Grand Finale 16.47 where we have assumed j for both radial functions. The reason for doing this is that anything else blows up at r=0, but plane waves do not blow up there. From this expansion, we can extract the coefficients as in 16.133,4. Now in 16.133 for E we put our simple form 16.130, the plane wave form e^{ikz} . This is the point at which our choice of circ pol gives us the $L_\pm$ operators in 16.135, resulting in 16.136. At this point we shove in our earlier expansion 16.127 for e^{ikz} and out come simple results 16.137,8. Thus, we now know the coefficients in our original expansion 16.131 and the result is 16.139. Notice that the m sum went away because all the coefficients only exist for m =1 or m=-1, depending on which circ pol you are talking about.

If we think of our plane wave as photons, although we still get a sum over all ℓ (recall the ℓm photons discussed earlier), we only get |m|=1, suggesting that the plane wave does have a component of angular momentum about the z axis that is in fact ± 1. This makes us think of a photon with two states of longitudinal polarization, but we know photons only have transverse polarization, so not quite clear how you fit the photon picture in with our plane wave expansion! Defer this to a rainy day.

16.9 The Conducting Sphere Scattering Problem!

This is a big problem, and is probably the closest thing in Jackson to my dielectric water sphere scattering problem, so pay close attention! See details of the solution at the end of this document! We assume separate forms for incoming and outgoing fields. For the incoming fields, we use our plane wave expanded in the very fancy 16.139 that we just pondered in the previous section. For the outgoing we use 16.141 where NOW (unlike in the plane wave expansion in the last section) we put Hankel-1 everywhere because it alone has the right asymptotic behavior e^{ikr}/r.

Now, why do we only have m = ±1 in expansion 16.141? Jackson is uncharacteristically silent on this question. We suspect it has to do with conservation of $\mathcal{L}_3$. I will create my own argument. Suppose we localize the incoming plane wave to a very long but finite packet, by slightly mixing frequencies. Then we can talk about "before" and "after" the scattering process. The sphere might absorb some energy in the process, but maybe it is not going to absorb any angular momentum. If we knew this to be true, then we might argue that the outgoing wave must have the same L_3 as the incoming. But still this seems vague to me.

Let's see if Fizpatrick has anything to say about this. This is a guy on the web who has made lots of Jackson-like notes. Here is the top level for his notes from one class:

http://farside.ph.utexas.edu/~rfitzp/teaching/jk1/lectures/lectures.html

This is the HTML presentation of his notes. He has another web page

http://farside.ph.utexas.edu/~rfitzp/teaching.html

which shows ALL his course, and also leads to PDF versions of his notes. These are much better since the equation numbers are correct. I have downloaded his multipole notes. His notes follow extremely closely to Jackson, same notation, he admits that Jackson is his main source.

He makes this comment: "a spherically symmetric scatterer cannot couple different m components". Well that is not much of a proof! I just asked Jim about this.

I think the real answer is this: you could assume all the m terms in the scattered wave and then you would have coefficients a(ℓ,m) in the outgoing equation. But when you then tried to have the total fields satisfy the boundary conditions at the sphere, you would find that the other m waves have to be zero or you cannot make the match. In other words, what you would get is something like this:

$$\hat{\mathbf{r}} \bullet \mathbf{B} = 0 \Rightarrow \quad \Sigma\, f(\ell,m,r=a)\, \mathbf{X}_{\ell,m} \quad = F(\ell,r=a)\, \mathbf{X}_{\ell,\pm 1}$$

and you would then say that since each $X_{\ell,m}$ has its own characteristic angular dependence, the match requires that all the other m coefficients vanish. This would be true for any boundary conditions that were "spherically symmetric" because the same general argument above applies. So OK.

I now accept 16.141 as the general form of the scattered wave. The boundary conditions are extremely elegant, being 16.142 where $\hat{\mathbf{n}} = \hat{\mathbf{r}}$. The form 16.143 is very helpful in arriving at the boundary conditions 16.144, along with the fact that $\hat{\mathbf{r}} \bullet \mathbf{X}_{\ell m} = 0$. Notice that the $\hat{\mathbf{r}} \bullet \mathbf{B} = 0$ equation determines the α coefficients, and the $\hat{\mathbf{r}} \times \mathbf{E} = 0$ determines the β ones. If you write these coefficients only using hankels, you see that each ratio in 16.145 is just a phasor as shown in 16.146, so this makes for a very simple formula for the coefficients 16.148. The solution to the entire problem is in these little phase shifts called δ_ℓ and δ'_ℓ which are functions of k, a and ℓ. The large and small k limits are shown in 16.149, 150. Notice that α and δ are associated with the *magnetic* (TE) fields in 16.141. This is the coefficient called a_M in 16.47.

So the final result for the scattered wave is stated in 16.151 where the α and β are done as phasors times sines. Keep in mind: this is the exact result for this problem, even very close in to the sphere, so we really DO have this problem solved. In the radiation zone the limit for B is shown in 16.152. Since the phases in a partial wave are not likely to be equal, the polarization in each partial wave of the resulting wave is elliptical.

Jackson goes on to write down the differential cross section as in 16.155 and he remarks that it is "rather complicated". In the dielectric sphere problem, Jim thought you need 40 partial waves before you actually see the rainbow stuff. We can of course do the large and small k limits (relative to sphere radius a). The exact total cross section however can be done (as is often the case, recall unitarity) and you find that each partial wave contributes as $(2\ell+1)$ times the sum of the sines of the phase shifts.

The long λ (small k) limit is dominated by $\ell=1$. Notice, by the way, that $\ell=0$ plays no role in this problem. You see that $\ell=0$ is missing from 16.139, the incident wave. In fact, $X_{00} = \mathbf{L}Y_{00} \sim \mathbf{r} \times \nabla Y_{00} = 0$ because Y_{00} is just a constant. So anyway, the $\ell=1$ dominates and is stated in 16.157 which is restated in 16.160 and is the same for either circ pol incident state. Notice that both the δ and δ' terms contribute to this result, so it is a mix of E1 and M1. The plot shows that most of the scattering is backward in this limit. The front/back asymmetry arises from the $\cos\theta$ term which arises from the cross terms which come from interference between the E1 and M1.

Finally, the total cross section in this limit shows the Rayleigh's Law k^4 dependence. I agree that we have seen this arise in our current problem, but it is not obvious to me that this would be case for any scattering that has mainly $\ell=1$. No doubt you can prove it to be the case, and we know that it is the case in simple dipole radiation models as we have made earlier. For example, on page 271 we know that S the Poynting will be proportional to k^4 in any dipole problem, so I guess that clinches it!

In any event, the work done in this section was first done by Mie and Debye in 1908-1909.

References: Born and Wolf give the *general* sphere solution, allowing it to have dielectric and conducting properties! I might go check this out somewhere. // I just web-ordered 7th edition for $64 hardback, I hope it still has that solution Jackson mentions. (he refs a 1959 Principles of Optics edition).

Problems: The dielectric sphere problem is assigned as problem 16.12, the last one. I don't see a solution to this messy problem on the web, I can imagine why, it is a mess.

I notice that our friend Fitzpatrick entitles this section of his "version of Jackson" as Mie scattering. In doing so, he notes that you can write out everything in terms of the coefficients without ever using the boundary conditions, then apply them at the last minute. He then does the same conductor as Jackson, so he does not attempt the general problem!

Appendix A. Details of Jackson's Boundary Conditions for the Conducting Sphere

The first thing to understand is this: the three vectors ($\hat{\mathbf{r}}$, $\mathbf{X}_{\ell m}$, $\hat{\mathbf{r}} \times \mathbf{X}_{\ell m}$) are mutually perpendicular. How do we know this? Well, $\mathbf{X}_{\ell m} = \alpha_\ell (\mathbf{L}Y_{\ell m})$ where $\alpha_\ell = (1/\sqrt{\ell(\ell+1)})$. We know for sure that $\hat{\mathbf{r}} \bullet \mathbf{X}_{\ell m} = 0$ because we know $\hat{\mathbf{r}} \bullet \mathbf{L} = 0$. This says that $\mathbf{X}_{\ell m}$ lies in a plane perpendicular to the unit vector $\hat{\mathbf{r}}$. The third vector here gives 0 when dotted with $\hat{\mathbf{r}}$ or $\mathbf{X}_{\ell m}$, since $A \bullet (A \times B) = 0$, so it also lies in that same plane with $\mathbf{X}_{\ell m}$ but is perp to $\mathbf{X}_{\ell m}$. Of these three vectors, we are only claiming that $\hat{\mathbf{r}}$ has unit length. The vector $\mathbf{X}_{\ell m}$ is a vector of functions of θ and ϕ so is likely not to be of unit length for all such angles. You can and should think of $\mathbf{X}_{\ell m}$ and $\hat{\mathbf{r}} \times \mathbf{X}_{\ell m}$ as the two perpendicular "transverse" vectors, as if you were thinking about polarization of a plane wave in the $\hat{\mathbf{r}}$ direction.

The second thing to understand is Jackson's 16.143. I have proven this by hand for an arbitrary radial function f(r), the function doesn't have to be spherical Bessel's the way Jackson might imply. The

important thing to realize is that this equation is showing that the quantity ∇ x [f(r) $\mathbf{X}_{\ell m}$] can be decomposed into a radial piece (the first term), and a transverse piece that is in the $\hat{\mathbf{r}}$ **x** $\mathbf{X}_{\ell m}$ direction.

Now about the boundary conditions. The two shown we understand: there can be no tangential E field, and there can be no normal B field. What about the other two possible boundary conditions here? What about normal E? The answer is (pill box straddles surface) that surface charge will arrange itself however it needs to in order to make normal E be what our solution makes it come out being! The same thing is true for tangential B, think of a loop and B = 0 inside the conductor. Surface currents K will flow however they must to make tangential B come as it comes out. So this is why we do not try to "impose" two more boundary conditions.

Here then are the fields as Jackson arranges them:

$$\mathbf{E}_{inc} = (1/2)\Sigma\, i^{\ell}\sqrt{4\pi(2\ell+1)}\,[\;2j_{\ell}(k_1 r)\,\mathbf{X}_{\ell,\pm1} \pm 2/k_1\,\nabla \times \{\, j_{\ell}(k_1 r)\,\mathbf{X}_{\ell,\pm1}\}\;]$$

$$\mathbf{B'}_{inc} = (1/2)\Sigma\, i^{\ell}\sqrt{4\pi(2\ell+1)}\,[\mp 2i\, j_{\ell}(k_1 r)\,\mathbf{X}_{\ell,\pm1} - 2i/k_1\,\nabla \times \{\, j_{\ell}(k_1 r)\,\mathbf{X}_{\ell,\pm1}\}\;]$$

$$\mathbf{E}_{sc} = (1/2)\,\Sigma\, i^{\ell}\sqrt{4\pi(2\ell+1)}\,[\,\alpha_{\pm}(\ell)\,h^{(1)}_{\ell}(k_1 r)\,\mathbf{X}_{\ell,\pm1} \pm \beta_{\pm}(\ell)/k_1\,\nabla \times \{\, h^{(1)}_{\ell}(k_1 r)\,\mathbf{X}_{\ell,\pm1}\}\;]$$

$$\mathbf{B'}_{sc} = (1/2)\,\Sigma\, i^{\ell}\sqrt{4\pi(2\ell+1)}\,[\mp i\,\beta_{\pm}(\ell)\,h^{(1)}_{\ell}(k_1 r)\,\mathbf{X}_{\ell,\pm1} - i\alpha_{\pm}(\ell)/k_1\,\nabla \times \{\, h^{(1)}_{\ell}(k_1 r)\,\mathbf{X}_{\ell,\pm1}\}\;]$$

Our boundary conditions are these:

$$\hat{\mathbf{r}} \bullet (\mathbf{B'}_{inc} + \mathbf{B'}_{sc}) = 0 \qquad \#1$$

$$\hat{\mathbf{r}} \times (\mathbf{E'}_{inc} + \mathbf{E'}_{sc}) = 0 \qquad \#2$$

For condition #1, notice that since $\hat{\mathbf{r}} \bullet \mathbf{X}_{\ell m} = 0$, we only pick off the second term in each B field expansion. Since the second term itself is then expanded as shown in 16.143, and since $\hat{\mathbf{r}} \bullet (\hat{\mathbf{r}} \times \mathbf{X}_{\ell m}) = 0$, all that survives is the first term in 16.143 for each field. If we ignore the overall constant factors and the $Y_{\ell m}$ then we get this:

$$-2i/k_1\, j_{\ell}(k_1 r) - i\alpha_{\pm}(\ell)/k_1\, h^{(1)}_{\ell}(k_1 r) = 0 \qquad \#1$$

and this is just what Jackson shows in 16.144.

Now consider condition #2; more terms survive. Out of each field we will now have an $\hat{\mathbf{r}} \times \mathbf{X}_{\ell m}$ term *and* an $\mathbf{X}_{\ell m}$ term. The first term in each field expansion obviously contributes to the $\hat{\mathbf{r}} \times \mathbf{X}_{\ell m}$ amount, but what about the second term. If we expand the second term using 16.143, the first term yields *nothing* since $\hat{\mathbf{r}} \times \hat{\mathbf{r}} = 0$. We are left with a contribution from the second term which looks like $\hat{\mathbf{r}} \times \hat{\mathbf{r}} \times \mathbf{X}_{\ell m}$. . But this is easily shown to be - $\mathbf{X}_{\ell m}$. Now, since the two terms are along different (and mutually perp) transverse direction vectors, the coefficients of the two terms must *separately* meet the boundary condition! Thus, we arrive at two conditions from condition #2:

$\hat{\mathbf{r}} \times \mathbf{X}_{\ell m}$ terms: $\quad 2j_{\ell}(k_1 r) + \alpha_{\pm}(\ell)\, h^{(1)}_{\ell}(k_1 r) = 0$

$-\mathbf{X}_{\ell m}/r$ terms: $\quad \pm 2/k_1\, \partial_r[r\, j_{\ell}(k_1 r)] \pm \beta_{\pm}(\ell)/k_1\, \partial_r[r\, h^{(1)}_{\ell}(k_1 r)] = 0$

Now it is lucky that the first condition here replicates the condition we already got from BC #1. The second condition is then as Jackson states in 16.144b.

Note that we have 2 equations in 2 unknowns (ignoring ± etc).

Appendix B. Changes in the presence of μ and ε

In order to even be able to attack the dielectric sphere problem (which I do in a separate document), you have to know how to "interpret" all of the Chapter 16 formulas in the case that you have a uniform ε and μ. I think I have found a good way to do it.

Section 16.2 on deriving the Grand Finale formula. You might think you could just replace B with H and E with D and have the με stuff cleared out of Maxwell's equations, but that is not the case. Looking at page 178, the problem is that the Faraday's law will then say that curl D = -1/c ∂_t H * με. So we need another plan.

Go back to the starting point 16.31 and add με as shown based on 7.1 page 202 which shows the four Maxwell's in this case, notice there is a single insertion point. Go ahead and do the time derivatives and get 16.32. We are going to end up rescaling the B field, so the div equations won't change. We care about the curl equations. These are:

$\nabla xE = ikB$ $\qquad k = \frac{\omega}{c} = \frac{\omega}{v} * \frac{v}{c} = k'\beta$ $\qquad \beta = 1/\sqrt{\mu\varepsilon}$

$\nabla xB = -ik\mu\varepsilon E = -ik\beta^{-2} E$ $\qquad \sqrt{\mu\varepsilon} = \beta^{-1}$

Now define the usual modified k (call it k') as shown above right. Then we get

$\nabla xE = ik'\beta B$	$\nabla xE = ik'(\beta B)$	$\nabla xE = ik'B'$
$\nabla xB = -ik'\beta^{-1}E$	$\nabla x(\beta B) = -ik'E$	$\nabla xB' = -ik'E$

Therefore, we have shown that we can rewrite 16.32, our starting point, by making two replacements:

1) replace k with k'	$k' = k / \beta$	$\beta = 1/\sqrt{\mu\varepsilon}$
2) replace B with B'	$B' = \beta B$	

Once we have done this, all four equations look exactly the same. This is the main point!!!

<u>The triplet equation groups page 543.</u> These are as shown, except make the two replacements shown above.

<u>The multipole fields page 545</u>**.** These are as shown, except make the two replacements shown above. Notice that all Bessel functions will now have (k'r) as argument.

<u>Grand Finale 16.47 page 546</u>**.** Put in k' for k everywhere, and the first equation is for B'. These are just the same changes made in the steps above.

Section 16.3 equations. Equations like 16.48 and 16.51 are just limits with constants not shown, so we are not bothered by not seeing $1/\varepsilon$ in 16.51. The swap rule 16.52 of course should have our B' in place of B. I would put a B' in 16.53 and therefore also in 16.56. Same in 16.57 along with replacing k with k' there.

Now in 16.58 we have to make a different change! The true formula for u has ED and BH. So we would then write this as ε EE + $(1/\mu)$ BB. If we now rescale to our B = B'/β, we get εEE+ $(1/\mu)$ β^{-2} B' B' = ε[EE + B'B']. In the radiation limit being done here, we have |E| = |B'| from modified 16.56. Then we can write the energy $u = 2\varepsilon (B')^2$. Therefore, 16.60 will have an extra ε out front.

Then in 16.61 we really should have H there and H = B/μ. But then put in B = B'/β, and then 16.61 will have $1/(\mu\beta)$ out front. Using 16.57 with its actual k' and B' then gives 16.62 with both B' inside, and with $(1/\mu)$ out front. Thus, the result 16.65 will have this $1/\mu$ out front. The ratio in 16.66 will then say that M/U(modified) = M/U (shown) * $(1/\mu)$ * $(1/\varepsilon)$. So the ratio is then no longer as shown and I do not know how to interpret the modified result.

Section 16.4 on distribution. The true time averaged power will be ExH = $(1/\mu)$ E x B = $(1/\mu\beta)$ E x B' = $\sqrt{\frac{\varepsilon}{\mu}}$ E x B'. Thus, we should have this square root factor out in front of 16.70.

Step 1: Let's go through the general multipole expansion and add the presence of uniform μ and ε. Start with 7.1 page 202 which shows the four Maxwell's in this case, notice there is a single insertion point. So, I have marked up how the two 3-equation sets on page 543 are altered.

One alteration is that you have to think of k as ω/v instead of ω/c, but only in places that got their k from the wave equation! This means inside any Bessel function like $j_\ell(kr)$, the multiplying k in there must be the $k = \omega/v$ version of k. However, the external factors of k are unchanged, namely, the 1/k that you see in the multipole fields on 545 except you have to add one $\mu\varepsilon$ factor as shown for E in the electric mode case only.

Looking then at the general expansion 16.47, we have to add same $\mu\varepsilon$ factor that we added into 16.42. Note that this is NOT just a rescaling of the a_E coefficient because the two expansions are coupled.

On page 548 I added new factors as shown.

Next, how is the huge 16.139 going to be altered?

Section 16.5 on moments. Looking at 16.77, Jackson is starting with the full equations including H, and he has left the polarization charge grouped in with ρ. So I have no modifications to make to this entire section.

Section 16.7 on center-fed antenna. Everything goes exactly as stated. However, when we get to the radiated power formula 16.121, we should add the same $\sqrt{\frac{\varepsilon}{\mu}}$ factor out front as on page 550 described above, and in all other power formulas in this section.

Section 16.8 on Plane Wave Expansions. First consider 16.130. As usual, k = k' in the exponent. The second equation there should be modified according to 7.13 on page 204 which says $B_0 = \beta^{-1} E_0$. Thus the second equation is correct if we just replace B by B'. So we just make the usual 2 changes in 16.130.

As for 16.131, it is just a case of the Grand Finale 16.47, and we know to think of B' on the left of the second, and all k are k'. Luckily for us, all the solution for the coefficients is unchanged and we end up with 16.139 with just our usual 2 changes.

Section 16.9 on the Conducting Sphere. As usual, replace B with B' everywhere and k with k'. We do this in the assumed form of the outgoing waves as well as in 16.139. The boundary condition can be taken as n•B' = 0 so nothing changes throughout. The final results 16.151 and 16.152 are the same except interpret k and B as k' and B'.

Now we come to the power formulas. There should be a $\sqrt{\frac{\varepsilon}{\mu}}$ factor out front in 16.155 for the same reason as discussed above (really it is ExH = (1/μ) ExB = (1/βμ) ExB').