Chapter 6: Faraday, Maxwell, Potentials, Gauges, Green's Solutions, Conservation Laws

This is a grab-bag chapter that constructs the known important results for "time-varying fields". Earlier in the book we did things in electrostatics and magnetostatics, here those things are updated.

6.1 Faraday's Law. In this section, Jackson reviews the 1831 discovery of Faraday that if you change the magnetic flux surface integral on a current loop, a voltage is induced around that loop which causes a current to flow in that loop according to Ohm's Law. This voltage is usually written $\mathcal{E}$ and is historically called the "electromotive force". It is, however, the same kind of potential (voltage) we had in electrostatics, and it is as if a battery were in series with the loop, although the battery cannot be localized anywhere in the loop. We now know this voltage is due to the Maxwell equation $\nabla \times \mathbf{E} = -\dot{\mathbf{B}}/c$. Jackson notes that the term Lenz's Law is used to describe the sign of the induced voltage -- it opposes the change in flux.

In this little section, Jackson first assumes an unknown constant k in Faraday's law, as if $\nabla \times \mathbf{E} = -k\,\dot{\mathbf{B}}$. He then considers a loop in motion and uses Galilean invariance to show that in the moving loop, charges are seeing a field $E' = E + k(v \times B)$. But from Biot-Savart (force on a current in a B field) we conclude that $k = 1/c$. Recall that Biot-Savart was developed in Chapter 5, as the force on a test current similar to Coulomb's law of the force on a test charge. [I have a separate note written up on this section.] By the way, this force on a particle is valid all the way up to the speed of light for v.

So, the end result here is $\nabla \times \mathbf{E} = -\dot{\mathbf{B}}/c$. Jackson does not claim to "derive" this equation. He just found what the constant must be to be consistent with Biot-Savart.

6.2 Energy in a B field. I have spent about 3 hours trying to clean up Jackson's logic in this section, but to no avail. I tried to draw a reasonable tiling picture and so on. My conclusion is that this is a bogus presentation, something Jackson normally does not do! He starts with the idea that if you have a wire with a current flowing in it, and you increase the flux through the wire by making a change δB in the magnetic field in the vicinity, then some work $\delta W = I\,\delta V = (I/c)d(\text{flux})$ must be done by the current in the wire to make this new increased B field. Jackson then tries to tile his wire circuit somehow with a mesh of loops, and then he tries to apply this δW idea to one of the little tiling loops. It is very unclear whether the tiling loops are supposed to replace the wire, or whether there is supposed to be current density J in all space inside the starting wire, or what! He assumes that the tiling wires have a cross sectional area $\Delta\sigma$ and that $\mathbf{J}\,dV = I\,d\boldsymbol{\ell}$ where $dV = \Delta\sigma\,d\ell$. So the fact that he has J being along the tiling loop suggests that he is trying to *replace* the original wire with a mesh of wires. But does this mesh then form a thin open surface with some small thickness that spans the original wire loop? If so, then the integral in 6.12 would be just over the volume of this mesh, not over the volume of all space. Then the conclusion would be that the total work done is 6.15 integrated over the volume of this mesh. But this seems wrong, because we know it should be over the volume of all space where we have fields from our wire. The whole presentation is very, very ugly and I am sure that Jackson regretted it and has replaced it with some better argument in his 2nd and 3rd editions.

Therefore, I am abandoning this entire section of Jackson's book, including the trailing comments on work done in a magnetization etc. The result is of course well known.

6.3 Maxwell's Contribution. Jackson claims that in 1865, Maxwell was staring at the equations 6.22 that seem to explain all electrical phenomena to that date. But Maxwell saw something wrong. Since divcurl=0, the second equation implied that divJ = 0, in violation of continuity. He knew this should be $d\rho/dt$. He got this fixed up by adding $1/d\ \partial D/\partial t$ to the RHS, and since $divD = 4\pi\rho$, this was exactly what was needed. Besides making the equations consistent with continuity, this little addition allowed one to show that either field satisfies the wave equation, and this allowed waves as solutions to the equations, and this led to Maxwell's idea that probably light was these waves, traveling through the EM ether. Maxell noticed that the velocity implied by his equations was pretty close to the measured speed of light. This was a huge breakthrough and that is why he gets his name on these equations, although he only really added the displacement current term.

6.4 Vector and Scalar Potentials.

(1) Conjecture that any **B** can be written as $\mathbf{B} = \nabla \times \mathbf{A}$ where **A** is some "vector potential". At least we know that this form guarantees that $\nabla\bullet\mathbf{B} = 0$ since in general div curl = 0 for anything.

(2) In statics we knew that $\nabla \times \mathbf{E} = 0$ so we conjectured that $\mathbf{E} = -\nabla\phi$ since curl grad = 0 for anything. We were able to also satisfy $\nabla\bullet\mathbf{E} = 0$ if we insisted that $\nabla^2\phi = 0$.

(3) With time dependence, we have $\nabla \times \mathbf{E} = -(1/c)\ \dot{\mathbf{B}}$ instead of 0. With $\mathbf{B} = \nabla \times \mathbf{A}$ this means we have that $\nabla \times (\mathbf{E} + \dot{\mathbf{A}}/c) = 0$ so now we should choose $\mathbf{E} + \dot{\mathbf{A}}/c = -\nabla\phi$ since that form satisfies this equation since curl grad = 0. So we end up with:

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \text{and}$$

$$\mathbf{E} = -\ \dot{\mathbf{A}}/c\ - \nabla\phi$$

We have now replaced 6 quantities with 4 quantities, perhaps an improvement. At this point, we have satisfied two of Maxwell's equations: $\nabla\bullet B = 0$ and Faraday. The other two Maxwell's equations are shown top of page 180. The above equations can both be written as $F^{\mu\upsilon} = \partial^{\mu}A^{\upsilon} - \partial^{\upsilon}A^{\mu}$, so the E and B fields are really part of a tensor. See page 379.

6.5A The Lorentz Gauge

(4) If we assume the condition $\partial_{\mu}A^{\mu} = 0$ (known as the Lorentz Condition) on the potential (where $\partial_0 = \partial_t/c$ $\partial^0 = -\partial_t/c$ and for proper scaling, and $x^0 = ct$), then the two equations decouple into the two shown on page bottom, which we can combine as $\Box^2 A^{\mu} = 4\pi J^{\mu}$. So at this point we have "three" equations, those two on the bottom, and the Lorentz condition.

(5) Consider the gauge transformation $\mathbf{A'} = \mathbf{A} + \nabla\Lambda$ along with $\phi' = \phi - \dot{\Lambda}/c$, or $A'^{\mu} = A^{\mu} + \partial^{\mu}\Lambda$. Notice that such a transformation *does not alter* the E and B fields shown in 6.29 and 6.31. Thus the potentials A'^{μ} and A^{μ} are equivalent solutions to a problem that give the same observable field results. There appears to be a whole "family" of solutions A^{μ} that are equivalent. In the tensor notation from above we

had that $F^{\mu\upsilon} = \partial^{\mu}A^{\upsilon} - \partial^{\upsilon}A^{\mu}$ so when you do $A'^{\mu} = A^{\mu} + \partial^{\mu}\Lambda$, you get $F'^{\mu\upsilon} = F^{\mu\upsilon}$ because the two second derivative terms obviously cancel each other.

(6) Consider now our "three equations" as noted in (4) above. Suppose our initial pick for A^{μ} gives $\partial_{\mu}A^{\mu} = f \neq 0$. Then $\partial_{\mu}A'^{\mu} = \partial_{\mu}A^{\mu} + \partial_{\mu}\partial^{\mu}\Lambda = f + \Box^2\Lambda$. Then all we need do is pick Λ such that $\Box^2\Lambda = -f$ [a sort of Poisson equation in 4D] and we have achieved $\partial_{\mu}A'^{\mu} = 0$. So there is a subset of our initial "family" of solutions that give the same physical fields , which subset meets the Lorentz Condition. This smaller family of solutions is called **the Lorentz Gauge**.

(6A) If we are in the Lorentz Gauge, then

$$\partial_{\upsilon}F^{\mu\upsilon} = \partial_{\upsilon}\{\partial^{\mu}A^{\upsilon} - \partial^{\upsilon}A^{\mu}\} = \partial^{\mu}(\partial_{\upsilon}A^{\upsilon}) - \Box^2 A^{\mu} = -\Box^2 A^{\mu} = -\partial_{\mu}F^{\mu\upsilon}$$

so that we can write $\partial_{\mu}F^{\mu\upsilon} = \Box^2 A^{\mu} = (4\pi/c)J^{\mu}$.

(7) Once we are "in the Lorentz gauge", we can still do further transformations $A''^{\mu} = A'^{\mu} + \partial^{\mu}\Lambda'$ where $\Box^2\Lambda' = 0$ [a sort of Laplace equation in 4D], and we "remain" in the Lorentz Gauge. These further transformations move us within the smaller family of solutions.

6.5B The Coulomb Gauge (aka the Transverse Gauge)

(8) Instead of requiring the condition $\partial_{\mu}A^{\mu} = 0$, we here require that $\nabla\bullet\mathbf{A} = 0$. The first thing you see on the top of page 180 is that electrostatics as we studied it just continues! We solve for ϕ in terms of ρ just as we always did, which is what 6.45 says. But now the "other" equation 6.33 has a junk term in it as shown 6.46.

It turns out that you can partition J as shown in 6.48. To prove this (I did it in problem 6.6), you first show that the expressions shown in 6.49 and 6.50 really do add up to J, this is a non-trivial piece of work requiring two vector identities. Secondly, you note that the required curl and div conditions on the two pieces are met because curl grad = 0 and div curl = 0. Again, this is a highly non-trivial decomposition.

At this point the rest is easy. That "junk term" I mentioned is just $4\pi J_{\ell}/c$ and you arrive at the very simple result 6.52 which says that, in this gauge, A is driven only by J_t. Of course figuring out what J_t actually is requires doing that messy integration 6.50, but it can be done. So in this gauge, you have again obtained decoupling of the A and ϕ equations. Specifically, ϕ is as in electrostatics with 6.45, driven instantaneously by the charge. Then you have only 6.52 to solve with 6.50 as the source. This too is an instantaneous integral.

Application of this gauge: no sources (no ρ and no J) implies that $\phi = 0$ and also $J_t = 0$, so your entire problem is the homo wave equation for **A** with no sources and 6.53.

6.6 The Green's function for the wave equation.

I remember doing this in detail, and there are several results to note. The original equation is 6.54 with a driving term shown and a -4π sitting there too -- a scalar time-dependent wave equation. The G definition also has the -4π as in 6.55, and then 6.56 says how we would use this G if we knew what it was to then solve the full problem. The solution for G in momentum space is 6.59, and when we 4-Fourier that back into real 4-space we get the famous result which is 6.64 which has the time-retarding delta function times

1/R. So if we look at how we use G to get a full solution, we have 6.65, and you see the famous 1/R appearing there. If you do the time integral, you get the even more famous solution which is 6.66. So: the solution of the time-dependent wave equation driven by function -4πf is the space integral of f at retarded time, divided by R.

6.7 Solving the time-dependent scalar wave equation with and without sources f using the Green's Function Method: the Poisson initial-time solution, and the Kirchhoff Huygens integral.

Back on page 18 in electrostatics, we had showed that 1/R was the Green's Function of the Laplace, and we commented there that you could then add to your formal 1/R Green's function any solution F(x,x') of the homogeneous Laplace equation, and that this extra freedom allows us to match boundary conditions, such as making G = 0 on some closed surface. This concept was simply adding homo solutions to the "particular" solution.

Here we have a similar situation. We have found that δ(time)/R is the formal Green's function, and we ought to be able to add to it another function F which solves the homogeneous wave equation. We now have a hyperbolic ODE so the only way to have a win is specifying the solution ψ and ψ' on a finite open surface. In our 4D equation, so to speak, an "open surface" is usually taken to be all of 3D space plus the boundary $t = t_0$, some initial time. So the idea is to specify ψ and ψ' (normal derivative) on a closed 3D surface at this initial time. Remember that this was "too much" in the electrostatic case, but it is what we need here.

But Jackson does not "add something to the Green's Function" the way he did back in that electrostatics chapter. Instead, he sort of starts from scratch. Using Green's Theorem, and an infinite volume with bounding surface S at infinity, we get the result 6.70 where we can junk that last surface integral term because we just assume ψ drops off there I guess. There is obviously some condition to think about, but Jackson has his hands full with other details. At time $t_0 = 0$, suppose we have ψ and ψ' described by some functions F and D as show in 6.71. Then if you park yourself at the origin by setting **x** = 0, you can get result 6.73. The first term is our familiar particular solution from 6.66. Added to this, we see a surface term at radius ct. We have found a solution to the problem that meets the boundary conditions 6.71 and it is good at any time t. It's only weakness is that it is only at x = 0, but I imagine the references show the general result. Notice that you have to integrate the boundary value stuff over the entire closed 2D surface, all 4π steradians. This fancy 6.73 solution is called the **Poisson's solution**. It seems that the surface integral samples a shell of both boundary value functions. The more time passes t, the farther out is that shell being sampled. I am not sure exactly how one would use this, I need to see some sample problems, but I think it is beyond this book but we have some references.

Again, the boundary conditions F and D have to be known ONLY at time t=0, that is the "open surface" idea. After that, it takes off on its own according to 6.73. Notice that at a later time, it is sampling the surface functions *as they were* at t=0, but of course it has to go out to distance r' = ct to find those values.

Going back to 6.70, suppose we can drop the initial value parts terms, and suppose there are no sources f, then we have *only* the surface term which is the last term in 6.70 which we dropped earlier. So we have then 6.74. By manipulation, Jackson converts this to 6.76 which gives the field ψ inside a volume in terms of a surface area integral around the volume of ψ and various derivatives of same (space and time). This is all being done in the time domain, so we have the "retarded" label on the guts. This is the **Kirchhoff diffraction-type integral formula**, but it looks quite foreign to me because I am used to seeing in with sine time dependence only, where we have just two terms and not three, and where we have exp(ikR)/R and not just 1/R. I know that when you undo the retardation into a time integral with the

delta showing and put in the sine time dependence, that is how you pick up the exp(ikR) factor. How three terms become two terms is pretty hazy, I would have to do the conversion to k space in detail. // Well if we sneak a peak ahead at page 281, we have a repeat of our result 6.76, and Jackson takes this right to k space and now I am reminded that there really are 3 terms, though we usually drop one of them. This then serves as Jackson's derivation of the Kirchhoff scalar formula. When he does his vector Kirchhoff, he starts from scratch again.

6.8 Poynting's Theorem (1884): Conservation of Energy

This is a very good and quick derivation. How can we "do work" on a system of E and B fields?

Think of a point charge in motion v. The force that an E,B system of fields exerts on this charge is $\mathbf{F} = q\mathbf{E} + (1/c)\,\mathbf{v} \times \mathbf{B}$. The work done by the fields on the charge is $dW = \mathbf{F}\bullet\mathbf{dx} = \{ q\mathbf{E} + (1/c)\,\mathbf{v} \times \mathbf{B}\} \bullet \mathbf{dx}$. The work per unit time is then given as $dW/dt = \{ q\mathbf{E} + (1/c)\,\mathbf{v} \times \mathbf{B}\} \bullet \mathbf{v} = q\mathbf{E} \bullet \mathbf{v}$. The important point is that the magnetic field can never do work on a particle because the magnetic force is always at right angles to the velocity vector of the particle! Now write $(dW/dt)\,dV = \mathbf{E} \bullet \mathbf{v}\, q\, \delta(\mathbf{r} - \mathbf{a})dV = \mathbf{E} \bullet \mathbf{J}\, dV$. This is the work done per second in a small volume dV by the fields on a current. So this then takes us to Jackson's starting point:

power applied by fields to sources = $\int \mathbf{E} \bullet \mathbf{J}\, dV$.

Jackson then goofs around with Maxwell's equations and manages to rewrite the RHS above as two terms, shown in the RHS of 6.81. This 6.81 is the conservation of energy. It says that, in time interval dt,

decrease in stored energy =
energy lost by fields to sources + energy lost through bounding surface

In differential form we have 6.82. In either case we interpret the quantity $(c/4\pi)\,\mathbf{E} \times \mathbf{H}$ as the rate at which power flows out per unit area, and this is called **S**, Poynting's vector. It is energy flow per unit area. We have not done any time averages here, this is all instantaneous. You could add an arbitrary curl of something to S and not change anything, but no one ever does that. Jackson has assumed that the media are linear, otherwise you have extra energy storage and loss mechanisms such as hysteresis.

6.9 Conservation of Linear Momentum

We take a similar starting point with $\mathbf{F} = q\mathbf{E} + (1/c)\,\mathbf{v} \times \mathbf{B}$ as the force of fields on a charge. But this is $d\mathbf{p}/dt$, the momentum change put onto the charge. Integrate this as in 6.89 over dV to get the total momentum transferred from fields to charges. Think of this as an increase in $\mathbf{P}_{mech}$. Now the game is to fiddle with the RHS of 6.89 using the two Maxwells shown in 6.90. After purely mathematical shufflings, we end up with 6.100 as a restatement of our equation. Now one reason for the extra complication we encounter here is that the thing being conserved, linear momentum, is a vector, whereas in the previous section it was the scalar: energy. The resulting equation like 6.100 will be a vector equation. In order to maintain an elegant notation, the dyadic notation is used, but you can really just regard 6.100 as the three

component equations and for each component i we have the divergence of $\mathbf{T}\bullet\mathbf{e}_i = T_{ji}$ which we may regard as a vector in the index j. But the result really is tensor in nature, no getting around that.

So what does 6.100 say? The first issue is the identification of the linear momentum $\mathbf{P}_{field}$ of the E and B fields according to 6.94. Jackson has not mentioned photons, but I will, in order to confirm this result. Write:

energy passing thru wall piece of area dA in time dt = $(c/4\pi)$ ExB dA dt = S dA dt

photons passing thru wall piece of area dA in time dt = $(c/4\pi)$ ExB dA dt / $\hbar\omega$

momentum per photon = $\hbar\omega/c$

total mom passing thru wall piece of area dA in time dt = { $(c/4\pi)$ ExB dA dt / $\hbar\omega$ } * $(\hbar\omega/c) = \Delta P$

volume density of momentum $\mathcal{P} = \Delta P/dV = \Delta P/[dA(cdt)]$
= { $(c/4\pi)$ ExB dA dt / $\hbar\omega$ } * $(\hbar\omega/c)$ / [dA(cdt)] = { $(c/4\pi)$ ExB } * $(1/c^2) = (1/4\pi c)$ ExB = S/c^2

This confirms the factor of $1/c^2$ that he gets. One factor comes from momentum per photon, the other comes from the thickness of the volume.

So looking at the LHS of 6.93 or 6.100, we see that the left hand side represents the rate of increase in mechanical momentum of the charges plus the rate of increase in momentum stored in the field within our volume. Momentum is thus "coming in from somewhere" and adding to both these terms. Thus, in 6.100, we have to interpret the *vector* quantity n•T as a momentum flux on the surface that is feeding momentum in through the boundary! Just as we made a distinction between stored energy and energy flux through the boundary, here we make a distinction between stored momentum and momentum flux through the boundary. What symbols are we going to use :

u = energy density	$\mathbf{S}$ = energy flux
$\mathcal{P}$ = momentum density	$\overleftrightarrow{\mathbf{T}}$ = momentum flux (the Maxwell stress tensor)

Now imagine that at our wall we have a complete absorber on the "right side". Then in time dt we know that $\hat{\mathbf{n}} \bullet \overleftrightarrow{\mathbf{T}}$ dA momentum sinks into the absorber, so the pressure on the surface will be

pressure on surface = $\hat{\mathbf{n}} \bullet \overleftrightarrow{\mathbf{T}}$

For example, if the surface is in the z=constant plane, we have

Pressure = $\hat{\mathbf{z}} \bullet \overleftrightarrow{\mathbf{T}} = \mathbf{T}_{3j}$ = a vector

$\text{Pressure}_3 = T_{33} = (1/4\pi)[E_3E_3 + B_3B_3 - (1/2)(E^2 + B^2)]$

This would be the normal pressure on the wall, pushing it back -- we call this radiation pressure. However, for general fields, there are going to also be tangential pressure components. Simple case is a

plane wave hitting an absorber at an angle. We expect tangential pressure pushing the wall sideways, in addition to the normal pressure component.

Application: put a black ball on a spindle and shine a laser at it at an grazing angle. We expect the laser to keep the ball rotating against the loss of friction, and we could compute the force on the ball using this T thing! [Hey guess what: I think that is what a radiometer is! One in vacuum should rotate away from the white or mirrored sides. But ones with gas go the other direction due to a thermal effect.]

Question: what are the absolute intensities of E fields in light of various kinds encountered every day? Think of sunlight as a kilowatt per square meter, integrated from IR through UV I suppose. Assume all at an average λ of 5000A. Conversion says we have then $S = 10^6$ erg/sec /cm^2 . This tells us that $E^2 = 10^{-3}$ erg/cm^3, so that $E = 10^{-6}$ dyne/esu. Thus, field storage is on the order of milli-ergs/cc and radiation pressure is milli-dynes/cm^2. Suppose a radiometer vane weighs 1 gram and is 1 cm^2 in area. The linear acceleration on this could be 10^{-3} cm/sec^2 so in 1 second it is not going very fast. Too bad, web sites say that radiation pressure is not enough and that heat convection does it! The toy radiometer spins the wrong way from a radiation pressure point of view.

6.10 Macroscopic Equations

Jackson has already come up with expressions for u and **S** and $\mathcal{P}$ in terms of fields E,D,B,H. In this section, he thinks again about the macro fields being now spatial *and* temporal averages of micro fields (including this time ϕ and A). One important result is 6.112 which reminds us exactly about the effective charge due to the presence of a dielectric (usually on a boundary), as well as the effective current arising from magnetization M and also from P. This dP/dt term seems new to me. Naturally it was not present in the static equations! Jackson's main point in this section is that he is doing everything from scratch, using only Maxwell's equations. [So does this mean I can use J = dP/dt to compute a dielectric radiation situation? I was always sort of stumped on how to start in the calculation of A when you could not do the little parts trick].

So, if we start with the energy conservation law in terms of the D and H fields and start stripping out the wrappings on these fields, the law appears as 6.117. The point is that we have now identified the extra terms due to M and P, and these are showing up as adders to the current as in 6.112. The point is that the field not only does work on free current J in the form J•E, but it also does work (at least reactive work) on M and P exactly as shown in this equation! You basically have inductance and capacitive work staring you in the face here. Since these terms are reactive, and represent temporary storage and not power loss, Jackson argues that it is natural to incorporate them in the D and H field presentation.

Problems. Interestingly, the ones I have done or looked at below are the same ones (apart from 6) that Jackson had us do when I took the class.

Problem 6.6. See hand-written solution, shows that you can separate currents as claimed for the Coulomb Gauge.

Problem 6.8: statement of the main chapter conserved quantities

We have already derived the expressions for u. Keep in mind the asymmetrical way in which D and H are defined:

$\mathbf{D} = \varepsilon\mathbf{E}$ $\qquad\qquad$ $\nabla\bullet\mathbf{D} = 4\pi\rho_{free}$

$\mathbf{H} = (1/\mu)\,\mathbf{B}$ $\qquad\qquad$ $\nabla\ \mathbf{x}\ \mathbf{H} = (4\pi/c)\,\mathbf{J}_{\mathbf{free}}$

So the first energy formula here for u has ED + BH inside.

The S we are happy with.

The thing he is calling g here is our momentum density $\mathcal{P}$, and seems that the result here is D x B which is a slightly unexpected combination. If you trace things through on page 192, this is in fact the result.

We could go trace out the derivation of the tensor (I have some pencil marks in the book from when I once did this), and the result is that you always have DE and HB at each quadratic location. So I am not really doing this problem, I am just reviewing the results that it states.

Problem 6.9: Angular Momentum of the Fields and its Conservation Law

I am not doing this problem, just studying its results. If we look at angular momentum instead of linear momentum, here are the things we find:

$\mathcal{L} = \mathbf{r}\ \mathbf{x}\ \mathcal{P} = \mathbf{r}\ \mathrm{x}\ [\ (1/4\pi c)\ \mathbf{D}\ \mathbf{x}\ \mathbf{B}\ ]$ $\qquad$ since $\qquad \mathcal{P} = (1/4\pi c)\ \mathbf{D}\ \mathbf{x}\ \mathbf{B}$

For angular momentum (a vector!) we again have to ask what is the tensor that describes the describes the flow of angular momentum through a boundary and the answer is $\overleftrightarrow{\mathbf{M}} = \overleftrightarrow{\mathbf{T}}\ \mathbf{x}\ \mathbf{r}$.

Now we know that in quantum mechanics we have

$\mathbf{L} = \mathbf{r}\ \mathrm{x}\ \mathbf{p} = (\hbar/\mathrm{i})\ \mathbf{r}\ \mathrm{x}\ \nabla$ $\qquad\qquad$ $\mathbf{p} = (\hbar/\mathrm{i})\ \nabla$

These are differential operators. One should not confuse these operators with the $\mathcal{L}$ and $\mathcal{P}$ density functions for an EM field. You cannot take L_3 and apply it to something like E or B or A or ϕ and expect to magically get the same thing multiplied by $\pm$ 1 and claim that you have located a photon and this is its spin relative to the z axis. On the other hand, you should find this to be the case when you apply L_3 to a multipole field $Y_{\ell m}$, that is, you are diagonal with eigenvalues m. See next couple of problems.

Problem 6.11. Classical model for a photon group

The E field shown has some kind of unspecified slow (relative to λ) transverse cutoff function (could use a Gaussian, as in a Gaussian Beam), and has circular polarization, so you might try to think of this as some kind of semi-localized photon. When you set $\nabla\bullet E = 0$, you get exactly 0 from the function shown. The ∂_z picks up ik from the exponent, and the z factors then cancel the ∂_x and ∂_y factors. The form for B comes from the ∇xE Maxwell equation where we agree to drop second derivatives of E_0 .

What does this set of fields look like? The B is always perp to E due to the 90 degree rotation of the i factor. The wave is going to the right. In one, the B leads by 90 degrees, and in the other it lags by same. There is a z component due to the transverse confinement of the beam, due to the divergence requirement.

Problem 6.12. Continuation of 6.11.

For L3, I guess we might compute u and $\mathcal{L}$ for an infinite cross section disk across the beam, of finite thickness. We would count on the E_0 function to converge this integration. We would do this for both u and $\mathcal{L}$ and then take the ratio. There are some things that puzzle me here, however. What is the meaning of the object **E•B h**ere? We know the transverse part is going to be 0, but this is not obvious with the -- well I guess it just works, yes, just do it, fine. Clearly the z direction cannot converge so don't try to integrate that way. Now let's assume we get the result shown, I have not done this calculation. If we interpret this plane wave as some number N of photons all in the same state, then $u/N = \hbar\omega$ and $\mathcal{L}/N = \pm\hbar$ and the ratio is then exactly as shown, so that would be our interpretation. As for the final question, if a field has cyl symmetry, then no transverse direction in space is picked out so you cannot have any transverse vector quantity be non-zero (is this right...?)