Proof of (6.5) Jackson PhL 1.22.03

Imagine a contour that is moving in a direction v. Imagine we are interested in the area integral over a surface bounded by this contour of some vector field A,

$$I = \int_{C(t)} A(r) \bullet dA$$

If we want the total derivative of I, we get one contribution from dA/dt, but we get another one because the contour moves. The first term is obvious. The second term can be written like this:

$$\left[\int_{C(t+dt)} A(r) \bullet dA - \int_{C(t)} A(r) \bullet dA \right] / dt$$

If you draw a picture of the contour moving in direction v, you see that the values of A that are picked up at time t+dt were the values that A had at t but at a distance shifted by v dt. This is just because the entire contour and its attached surface has shifted by v dt. Thus we can rewrite the above as:

$$\left[\int_{C(t)} A(r+vdt) \bullet dA - \int_{C(t)} A(r) \bullet dA \right] / dt = \int_{C(t)} (v \bullet \nabla) A(r) \bullet dA$$

where this δA first term expansion used here is obvious if you just write out the components.

$$A(r+a) = A(r) + (a \bullet \nabla) A(r) + \ldots$$

Now we have this little vector identity,

$$\nabla \times A \times v = (v \bullet \nabla) A - v(\nabla \bullet A)$$

where v is a constant so all gradients on it vanish, killing two of the usual terms in this identity. Now, if A happens to be B, the magnetic field, we know that $\nabla \bullet B = 0$ so we ignore the second term. So we rewrite our term as follows:

$$= \int_{C(t)} \nabla \times (A \times v) \bullet dA$$

But Stokes lets us relate an area integral like this with a curl to line integral

$$= \int_{C(t)} (A \times v) \bullet dl$$

and this then is the rightmost term in Jackson's equation (6.5) on page 172.