Some Gradient Facts PhL 1.31.03

Lemma: $\nabla(r) = \hat{\mathbf{r}}$ Proof: $(\partial r/\partial x_j) = 1/2 * 1/r * 2\, x_j = x_j/r = \hat{r}_j$

Theorem: $\nabla(\, f(r)\,) = f'(r)\ \hat{\mathbf{r}}$ Proof: $\partial f/\partial x_j = (\,\partial f/\partial r)\,(\partial r/\partial x_j) = f'(r)\ \hat{r}_j$

Applications:

f=f	$\nabla(\, f(r)\,) = f'(r)\ \hat{\mathbf{r}}$
f=r	$\nabla(r) = \hat{\mathbf{r}}$
f=1/r	$\nabla(1/r) = -\,\hat{\mathbf{r}}\,/r^2$

Lemma: $\nabla \times \mathbf{r} = 0$ Proof: $\varepsilon_{ijk}\partial_j r_k = \varepsilon_{ijk}\delta_{jk} = \varepsilon_{ijj} = 0.$

Theorem: $\nabla \times [\, f(r)\,\mathbf{r}\,] = 0$

Proof: $\nabla \times [\, f(r)\,\mathbf{r}\,] = \nabla(\, f(r))\ \times \mathbf{r}\ +\ f(r)\,\nabla \times \mathbf{r}\ = f'(r)\ \hat{\mathbf{r}} \times \mathbf{r} +\ f(r)\,\nabla \times \mathbf{r}\ = 0 + 0 = 0$

Applications: $\nabla \times \hat{\mathbf{r}}\ = 0$

Lemma: $\nabla \bullet\ \mathbf{r} = 3$ $\partial_j r_j\ = 1+1+1 = 3$

Theorem: $\nabla \bullet\ [\, f(r)\,\mathbf{r}\,] = 3\, f(r) + r\, f'(r)$

Proof: $\nabla \bullet\ (\, f(r)\,\mathbf{r}) = \mathbf{r} \bullet \nabla(\, f(r)\,) + f\,\nabla \bullet \mathbf{r} = \mathbf{r} \bullet f'(r)\ \hat{\mathbf{r}} + 3\, f = 3\, f + r\, f'(r)$

Applications:

f = f	$\nabla \bullet\ [\, f\,\mathbf{r}\,] = 3\, f + r\, f'$
f = 1	$\nabla \bullet\ \mathbf{r}\ = 3$
f = 1/r	$\nabla \bullet\ \hat{\mathbf{r}} = 2/r$

Theorem: $(\mathbf{k} \bullet \nabla)\ [\, f(r)\,\mathbf{r}\,] = f(r)\,\mathbf{k}\ + r\, f'(r)\ (\mathbf{k} \bullet \hat{\mathbf{r}}\,)\ \hat{\mathbf{r}}$

Proof: $(\mathbf{k} \bullet \nabla)\ [\, f(r)\,\mathbf{r}\,] = \Sigma\, k_i \partial_i\ [\, f(r)\, r_j\,] = \Sigma\, k_i\, [\, f(r)\, \delta_{ij} + r_j\, \partial_i\, f(r)] = f(r)\,\mathbf{k} + [\mathbf{k} \bullet \nabla(f(r))\,]\,\mathbf{r}$
$= f(r)\,\mathbf{k} + [\mathbf{k} \bullet\ f'(r)\ \hat{\mathbf{r}}\ ]\,\mathbf{r}$

Applications:

f = f	$(\mathbf{k} \bullet \nabla)\ [\, f\,\mathbf{r}\,] \quad = f\,\mathbf{k}\ + r\, f'\ (\mathbf{k} \bullet \hat{\mathbf{r}}\,)\ \hat{\mathbf{r}}$
f = 1:	$(\mathbf{k} \bullet \nabla)\ \mathbf{r}\ = \mathbf{k}$
f = 1/r:	$(\mathbf{k} \bullet \nabla)\ \hat{\mathbf{r}}\ = (1/r)\ [\ \mathbf{k}\ -\ (\mathbf{k} \bullet \hat{\mathbf{r}}\,)\ \hat{\mathbf{r}}\ ]$

Theorem: $\nabla \times [f(r)\,\mathbf{r}\,] \times \mathbf{k} = -2\,\mathbf{k}\, f(r) + r\, f'(r)\,\{ -\mathbf{k} + (\mathbf{k}\bullet\hat{\mathbf{r}})\,\hat{\mathbf{r}}\,\}$

Proof: $\nabla \times [f(r)\,\mathbf{r}\,] \times \mathbf{k} = -\mathbf{k}\,(\nabla\bullet[f(r)\,\mathbf{r}]) + (\mathbf{k}\bullet\nabla)\,[f(r)\,\mathbf{r}]$

$= -\mathbf{k}\,[\,3\, f(r) + r\, f'(r)\,] + [\, f(r)\,\mathbf{k} + (\mathbf{k}\bullet\hat{\mathbf{r}})\,[r\, f'(r)]\,\hat{\mathbf{r}}\,] = -2\,\mathbf{k}\, f(r) + r\, f'(r)\,\{ -\mathbf{k} + (\mathbf{k}\bullet\hat{\mathbf{r}})\,\hat{\mathbf{r}}\,\}$

Applications:		
	$f = f$	$\nabla \times [f\,\mathbf{r}\,] \times \mathbf{k} = -2\,\mathbf{k}\, f + r\, f'\,\{ -\mathbf{k} + (\mathbf{k}\bullet\hat{\mathbf{r}})\,\hat{\mathbf{r}}\,\}\partial$
	$f = 1$	$\nabla \times \mathbf{r} \times \mathbf{k} = -2\,\mathbf{k}$
	$f = 1/r$	$\nabla \times \hat{\mathbf{r}} \times \mathbf{k} = -(1/r)\,[\,\mathbf{k} + (\mathbf{k}\bullet\hat{\mathbf{r}})\,\hat{\mathbf{r}}\,]$
