Point charges as current: using delta functions PhL 1.28.03

Consider a point charge q moving at some slow constant velocity **v**. What is **J** and what is ρ?

$\rho(\mathbf{r}) = q\,\delta(\mathbf{r} - \mathbf{a})$ //point charge located at position **a**

We know this is right because we integrate over a surrounding volume to get q. Next, let's guess for **J**:

$\mathbf{J}(\mathbf{r}) = q\,\mathbf{v}\,\delta(\mathbf{r} - \mathbf{a})$

To verify this conjecture, consider (**v** is a constant vector)

$\nabla\bullet\,\mathbf{J}(\mathbf{r}) = q\,(v\bullet\nabla)\delta(\mathbf{r} - \mathbf{a})$

On the other hand, we can see that

$\partial_t\rho(\mathbf{r}) = q\,\partial_t\,\delta(\mathbf{r} - \mathbf{a}) = q\,(\partial_t\,\mathbf{a})\bullet\nabla_{\mathbf{a}}\,\delta(\mathbf{r} - \mathbf{a}) = q\,\mathbf{v}\bullet(-\nabla_{\mathbf{r}})\,\delta(\mathbf{r} - \mathbf{a}) = -\mathbf{q}\,(v\bullet\nabla)\delta(\mathbf{r} - \mathbf{a})$

Since this satisfies $\nabla\bullet\,\mathbf{J} = -\,\partial_t\rho(\mathbf{r})$, we know it must be the right form for **J**. Remember that continuity just says that if there is a net outflow of current from a volume, the enclosed charge must decrease.

Example in the z direction

Let $\mathbf{v} = v\,\hat{\mathbf{z}}$. Then $\mathbf{J}(\mathbf{r}) = q\,v\,\delta(\mathbf{r} - \mathbf{a})\,\hat{\mathbf{z}}$. Then $\nabla\bullet\,\mathbf{J}(\mathbf{r}) = \partial_z J_z = q\,v\,\partial_z\,\delta(\mathbf{r} - \mathbf{a})$, where ∂_z acts only on the z factor of the delta function. Now imagine an aligned box containing the charge. Integrate this divergence over the box and we get $qv\int_0^h dz\,\partial_z\,\delta(z - a_z)$ = only the parts = $qv\ [\,\delta(h - a_z) - \delta(a_z)\,]$. This is only non-zero when the charge is passing through the upper or lower face of the box. On the other hand, if we integrate the current density over the box, we get $Q(a_z) = q\,\theta(a_z)\,\theta(h - a_z)$. That is, it is non-zero only if a_z is in the box. If we apply a time derivative to this Q, we use $\partial_x\,\theta(x) = \delta(x)$ to get

$\partial_t\,Q(a_z) = q\,\partial_t[\theta(a_z)\,\theta(h - a_z)] = q\,\partial a_z/\partial t\ \partial_{az}\,[\theta(a_z)\,\theta(h - a_z)]$ // chain rule
$= q\,v\ \partial_{az}\,[\theta(a_z)\,\theta(h - a_z)]$ // recognize v
$= q\,v\,\{\,\theta(a_z)[-\delta(h - a_z)] + \delta(a_z)\,\theta(h - a_z)\,\}$ // product rule
$= q\,v\,\{\,\theta(h)[-\delta(h - a_z)] + \delta(a_z)\,\theta(h - 0)\,\}$ // use deltas
$= q\,v\,\{\,[-\delta(h - a_z)] + \delta(a_z)\,\}$
$= -\,q\,v\,[\,\delta(h - a_z) - \delta(a_z)\,]$

So thing do work out. It just seems a little odd to say that $\nabla\bullet\,\mathbf{J}(\mathbf{r}) = qv\,\partial_z\,\delta(\mathbf{r} - \mathbf{a})$ in this case, and generally

$\rho(\mathbf{r}) = q\,\delta(\mathbf{r} - \mathbf{a})$ $\mathbf{J}(\mathbf{r}) = q\,\mathbf{v}\,\delta(\mathbf{r} - \mathbf{a})$ $\nabla\bullet\,\mathbf{J}(\mathbf{r}) = -\,\partial\rho/\partial t = qv\,\nabla\,\delta(\mathbf{r} - \mathbf{a})$

Application to a mechanical dipole radiator

Imagine charges q and -q separated by distance a cos(ωt) **.** In other words, +q is located at

q is at: $\mathbf{a}(t) = (d/2)\exp(-i\omega t)\,\hat{\mathbf{z}}$ -q is at: $-\mathbf{a}(t) = -(d/2)\exp(-i\omega t)\,\hat{\mathbf{z}}$

$\rho(\mathbf{r}) = (+q)\,\delta(\mathbf{r} - \mathbf{a}) + (-q)\,\delta(\mathbf{r} + \mathbf{a}) = q\,[\delta(\mathbf{r} - \mathbf{a}) - \delta(\mathbf{r} + \mathbf{a})]$

The velocity of the +q charge is $\partial_t\,\mathbf{a}(t) = -i\omega\,\mathbf{a}(t)$, while that of -q is the opposite. So we have

The current density is

$$\mathbf{J}(\mathbf{r}) = (+q)(-i\omega\,\mathbf{a}(t))\delta(\mathbf{r} - \mathbf{a}) + (-q)(+i\omega\,\mathbf{a}(t))\delta(\mathbf{r} + \mathbf{a}) = -i\omega q\,\mathbf{a}(t)\,[\delta(\mathbf{r} - \mathbf{a}) + \delta(\mathbf{r} + \mathbf{a})]$$

Now use Jackson (9.13) to compute the n=0 term of the vector potential! Note that the two currents are in the same direction! We get:

$$\mathbf{A}(r,t) = \frac{e^{ikr}}{cr}\int \mathbf{J}\,d^3r = -ik\,\frac{e^{ikr}}{r}\,[2\,q\,\mathbf{a}(t)] = -ik\,\frac{e^{ikr}}{r}\,[2\,q\,(d/2)\exp(-i\omega t)\,\hat{\mathbf{z}}] = -ik\,\frac{e^{ikr}}{r}\,\{qd\hat{\mathbf{z}}\}\exp(-i\omega t)$$

The thing $\{qd\hat{\mathbf{z}}\}$ is usually called **p**, the **dipole moment**,

$$\mathbf{p} = q\,d\,\hat{\mathbf{z}} \qquad \mathbf{A}(r) = -ik\,\frac{e^{ikr}}{r}\,\mathbf{p} \qquad \text{// agreeing with (9.16)}$$

We could also compute **p** by integrating :

$$\mathbf{p} = \int \rho\,\mathbf{r}\,d^3r = \int q\,[\delta(\mathbf{r} - \mathbf{a}) - \delta(\mathbf{r} + \mathbf{a})]\,\mathbf{r}\,d^3r = q\,\mathbf{a} - (q\,(-\mathbf{a})) = 2\,q\,\mathbf{a} = q\,d\,\hat{\mathbf{z}}$$

Note that things are defined with and without the time phase in various places.