Some Questions PhL 1.30.03

1. What is the energy density in a plane wave?

In statics, page 21 Jackson shows that $|\mathbf{E}|^2/8\pi$ is the energy density of an E field. The || is there only because E is a vector, not because something is complex-valued. The understanding here is that **E** is a real-valued vector. When we do plane waves, we usually say $\mathbf{E} = E_0 \exp(ikz)\,\hat{\mathbf{z}}$. We really mean the real part of this thing which is $E_0 \cos(kz)\,\hat{\mathbf{z}}$. The peak energy density of such a plane wave stored only in the E field would then be $E_0^2/8\pi$. The average however is the average of cosine squared which is 1/2, so $\langle U_E \rangle = E_0^2/16\pi$. But in a plane wave, the B field has the same energy density, we get $\mathbf{\langle U \rangle = E_0^2/8\pi}$ as the total average energy density.

Notice that the average energy density of a plane wave is independent of frequency ω. Higher frequency plane waves have a fewer number of photons that each have a higher energy quantum.

2. What is the momentum of a plane wave?

(a) Photon has energy hυ and momentum p such that $E^2 - p^2 = p_\mu p^\mu = mc^2 = 0$ so $p = h\upsilon/c = h\beta$. This would then be $p = h\omega/2\pi c = \hbar k$. That sounds familiar. But E&M classical theory does not know about $\hbar$, so question remains unanswered.

(b) Use the fact that, for photons, p = E/c where E is energy. Then average energy density U in a volume is $E_0^2/8\pi$ in free space, so: dEnergy = $(E_0^2/8\pi)$ dV and dp = $(E_0^2/8\pi c)$ dV. So the average momentum density is P = $(E_0^2/8\pi c)$. For a unit amplitude plane wave, we have P = $(1/8\pi c)$. The number of photons per unit volume is $n = (E/dV)/E_1 = (E_0^2/8\pi) / \hbar\omega = (E_0^2/8\pi)/\hbar\omega = n$. The number of photons in a volume dz dA is ndzdA, they all bounce off our mirror in time Δt, so force $F = \Delta p/\Delta t$. We have $\Delta p = 2np\ dA dz$, and Δt = dz/c (the time for this volume to hit the wall completely), so F = 2npc dA or Pressure = 2npc = $(E_0^2/4\pi)$. So we are in the end able to show these things about momentum without reference to a photon, except we need the idea that p = E/c and then P = U/c. I am afraid this can only come from quantum theory. Notice that the momentum of an individual photon $\hbar k$ seems to be irrelevant so our momentum calculation. The momentum of a plane wave is not proportional to k. In a plane wave, as we increase k, we increase the energy of a photon, but at the same time we decrease the number of photons, so the momentum p or P for a given value of E_0 remains unchanged.

Summary for a plane wave of the form $\mathbf{E} = E_0 \exp(ikz)\,\hat{\mathbf{z}}$ **:**

average **energy density** = $(E_0^2/8\pi) = U$

average **momentum density** = $(E_0^2/8\pi c) = P = U/c$

average energy flow per unit area per unit time = $\Delta U/\Delta t = (E_0^2/8\pi)\ dz / dz/c = (E_0^2/8\pi)*c = U*c$
This is also called "**energy flow**" or "**power/area**" and symbol S of Poynting is usually used.
Some people refer to power/area as **intensity** and use symbol I.

average momentum flow per unit are per unit time = $\Delta P/\Delta t = (E_0^2/8\pi c)dz / dz/c = (E_0^2/8\pi) = U = P*c$
This is called "**momentum flow**" but there is no term like "power" to describe this thing.

average **pressure** on a mirror = $F/dA = (\Delta p/\Delta t)/dA = 2*PdV/(dA\ dz/c) = 2*(P*c) = 2*U = (E_0^2/4\pi)$

Comment: If you have an absorber instead of a mirror, radiation pressure is half the above.

3. Problematic bogus calculation of Pressure

Average energy density in a plane wave is $E_0^2/8\pi$, see above. Imagine plane wave bouncing off a mirror. Push the mirror dz into the field, stored field energy decreases by dz $E_0^2/8\pi$ per unit area just because there is less field present. And this reduction occurs in both the incoming and the outgoing beams? Or should you first compute the total coherent E and B fields to get the true stored energy in the beam space? If I do this, I find that $E = 2E_0 \sin(kz)\sin(\omega t) = B$. Then I get $\langle E^2\rangle = (2E_0)^2 * 1/2 = 2\ E_0^2$ and the same for $\langle B^2\rangle$. Then I would claim that the average energy density in the volume to the left of the mirror is the sum of these which would be $4\ E_0^2$, then we add the 8π to get $\langle u\rangle = 4\ E_0^2/\ 8\pi = E_0^2/\ 2\pi$. If this is right, then we would claim that after the mirror has moved dz, the field has lost stored energy $(E_0^2/\ 2\pi)$ dz per unit area. Now, the pushing hand does work on the mirror which does work on the field, so you would think the field energy would *increase*, not decrease! This simple argument gives twice the right answer for pressure with the wrong sign, since pressure = F/dA = dU/(dzdA)! Something is not kosher with this argument.

Idea: while we are pushing on the mirror for time dt and it is moving at speed v, the reflected beam is Doppler up-shifted. The frequency of the reflected beam is upshifted in frequency by amount $1/(1 - 2v/c) \sim 1+2v/c$ during the time of movement. This ought to have something to do with the problem!

After spending about 2 full hours on this paradox, I have no solution to it. So I don't know how to compute radiation pressure using a change-in-energy-stored argument. I sent an email to Jim, I bet he will have the obvious answer that I am unable to see today.

4. What about Poynting's Vector?

Start off with **S** = k **E** x **B** where k is to be determined. This is for real fields E and B. If we write S = k E x B in esu units, then $|S| = k\ |E|^2$ instantaneous. Then $|S|_{peak} = k\ E_0^2$ and $\langle|S|\rangle = k\ E_0^2/2$. But from the section above this must be this must be $U*c = (E_0^2/8\pi)*c = (E_0^2/2)*(c/4\pi)$, so $k = c/4\pi$. Therefore

S = $(c/4\pi)$ **E** x **B** instantaneous energy flow => $|S|_{peak} = (c/4\pi)\ E_0^2 = (E_0^2/4\pi)*c$

$\langle$**S**$\rangle = (c/8\pi)$ **E** x **B** = $(E_0^2/8\pi)*c$

How do we adapt this for complex E and B? Try **S'** = j **E** x **B*** where j is to be determined. For plane wave in esu units we have $|S'| = j\ E_0^2$ because the phasors cancel. Clearly S' so defined cannot refer to an instantaneous value at some z because there is no function of z left. Consider:

$|S'| = j\, E_0^2 = (j/k)\, k\, E_0^2 = (j/k)\, |S|_{peak} = (2j/k) \langle |S| \rangle$

If we select j = k, then S' refers to a peak value, and if we pick j = k/2, then S' is an average value. I think Jackson on page 205 wants to be thinking average flow, so he would say

$\mathbf{S'} = (c/8\pi)\, \mathbf{E} \times \mathbf{B^*}$ average energy flow // as we see in 7.14

5. How do we come to associated p with ∇, and in what context(s) ?

In wave theory, consider a plane wave exp(ikx). The operator $(\hbar/i)\nabla$ yields $\hbar k$ which we know is the momentum of a photon in the plane wave (see 2(a) above) . So we would associate $\mathbf{p} = (\hbar/i)\nabla$. So this association, which is the same in the Saxon QM book, requires quantum theory.

On page 542 of Jackson, we seem him use $\mathbf{p} = (1/i)\nabla$ as a shorthand, so he can us $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ as another shorthand. For him, these are operators closely related to the quantum operators, but they are have the $\hbar$ scaled out. If we apply Jackson's **p** operator to exp(ikx), we get just k, the momentum of a photon in units of $\hbar$. I should think of these things as mere shorthands in Jackson. But if we talk about individual photons, they have that interpretation.

6. What is the potential of a plane wave, both ϕ and A ?

Let $\hat{\boldsymbol{\varepsilon}}_t$ be an arbitrary unit vector transverse to **k** of the plane wave. Then try this form:

$\mathbf{A} = (1/ik) \exp(i\mathbf{k}\bullet\mathbf{r})\, \hat{\boldsymbol{\varepsilon}}_t$

Then

$\mathbf{B} = \nabla \times \mathbf{A} = (1/ik) \{ \nabla[\exp(i\mathbf{k}\bullet\mathbf{r})] \times \hat{\boldsymbol{\varepsilon}}_t \} + \exp(i\mathbf{k}\bullet\mathbf{r})\, \nabla \times \hat{\boldsymbol{\varepsilon}}_t .$

The second term is 0 since $\hat{\boldsymbol{\varepsilon}}_t$ is a fixed vector. Then $\nabla[\exp(i\mathbf{k}\bullet\mathbf{r})] = i\mathbf{k} \exp(i\mathbf{k}\bullet\mathbf{r})$ so we have

$\mathbf{B} = \nabla \times \mathbf{A} = (1/ik) \{ i\mathbf{k} \exp(i\mathbf{k}\bullet\mathbf{r}) \times \hat{\boldsymbol{\varepsilon}}_t \} = \exp(i\mathbf{k}\bullet\mathbf{r})\, \hat{\mathbf{k}} \times \hat{\boldsymbol{\varepsilon}}_t$

So this looks like a reasonable B field for a plane wave in the k direction. Given this B and this k, we then know that

$\mathbf{E} = \exp(i\mathbf{k}\bullet\mathbf{r})\, \hat{\boldsymbol{\varepsilon}}_t$ $\qquad\qquad$ $\hat{\boldsymbol{\varepsilon}}_t \times (\hat{\mathbf{k}} \times \hat{\boldsymbol{\varepsilon}}_t) = \hat{\mathbf{k}}$

because then ExB is in the k direction. Now we know that

$\mathbf{E} = -\nabla\phi - (1/c)\partial_t \mathbf{A} = -\nabla\phi - (1/c)(-i\omega)(1/ik) \exp(i\mathbf{k}\bullet\mathbf{r})\, \hat{\boldsymbol{\varepsilon}}_t = -\nabla\phi + \exp(i\mathbf{k}\bullet\mathbf{r})\, \hat{\boldsymbol{\varepsilon}}_t = -\nabla\phi + \mathbf{E}$

Thus, if we take ϕ = constant in space, we get the right answer.

Summary: If we start with these potentials:

$$(\mathbf{A}, \phi) = (e^{i\mathbf{k}\bullet\mathbf{r}} \hat{\boldsymbol{\varepsilon}}_t / ik , \phi(t))$$

we obtain these plane-wave fields:

$$\mathbf{E} = e^{i\mathbf{k}\bullet\mathbf{r}} \hat{\boldsymbol{\varepsilon}}_t \qquad \mathbf{B} = e^{i\mathbf{k}\bullet\mathbf{r}} \hat{\mathbf{k}} \times \hat{\boldsymbol{\varepsilon}}_t$$

where E is polarized in the $\hat{\boldsymbol{\varepsilon}}_t$ direction. Notice that

$$\nabla\bullet A = (1/ik)\, \hat{\boldsymbol{\varepsilon}}_t \bullet \nabla (e^{i\mathbf{k}\bullet\mathbf{r}}) = e^{i\mathbf{k}\bullet\mathbf{r}} (\hat{\mathbf{k}} \bullet \hat{\boldsymbol{\varepsilon}}_t)$$

$$1/c\, \partial\phi/\partial t = f'(t)/c$$

Since these don't add up to 0, we are not in a Lorentz gauge here (see 6.36). Nor are we in the Coulomb gauge. This is one of an infinite possible sets of potentials that give these fields.

7. In non-relativistic <u>classical</u> mechanics, how do E and B fields interact with a particle?

(a) The E and B fields put a force on a particle which is F = qE + (v/c) x B. The first term is normal in electrostatics. The second term represents an E field that appears in a moving frame of reference. I think this force law comes entirely from Maxwell's four equations.

(b) A static charged particle makes an E field which is E = $-\nabla\phi$ where ϕ is the potential made by the particle, and we find ϕ by solving the Poisson equation.

(c) If the particle makes a current J because it is moving, then we get an A that is a certain integral of J, and then B = ∇xA and E then comes from a Maxwell equation.

(d) In general, we have J^μ driving an ODE for A^μ which encapsulates (b) and (c) above.

However, we know that the fields affect the charges and the charges affect the fields, so not obvious what general solutions exist. We often talk about a particle in some "external fields" which are imposed from the outside. We also talk about particles moving and making fields, for example, some "fixed currents" in an antenna. We also talk about particles without fields and fields without particles.

8. In non-relativistic <u>quantum</u> mechanics, how do E and B fields interact with a particle?

I think this is going to have something to do with **A•J** and maybe ϕ ρ.

For electric field, we should be able to add a Hamiltonian term -eEz for field in z direction. Maybe add something like -evxB for magnetic field. I will find some books where this is done.

9. In relativistic <u>classical</u> mechanics, how do E and B fields interact with a particle?

Jackson talks about some of this in his book. At least how fast moving particles generate fields. I forget how the force equation changes (if it does at all) for a fast-moving particle

10. In relativistic <u>quantum</u> mechanics, how do E and B fields interact with a particle?

This is perhaps Dirac electron theory with fields not yet quantized.

11. In relativistic <u>quantum</u> mechanics and <u>quantum</u> field theory, how do E and B fields interact with a particle?

The general idea is you write a Lagrangian and add the interaction term e $A^{\mu} J_{\mu}$ which couples the photon field to the electric current, represented by the electron Dirac spinor business.