

Engineering Education: Presentation of Maxwell's Equations in Historical Perspective and the Likely Desirable Outcomes

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Abstract

It is suggested that where possible fundamental concepts and theories be presented from a historical perspective in undergraduate engineering education. Apart from being fairer to the contributions of the several researchers whose works often significantly contribute to a later and much-acclaimed theory, such an approach can also be employed to intellectually challenge the young students, and to encourage them to pursue life-long learning. Against this background, a possible method of presenting Maxwell's equations in a historical context is presented.

Keywords: History; education; electromagnetic fields; electromagnetic waves; Maxwell's equations; Maxwell

1. Introduction

Maxwell's theory rightfully represents a very significant component in undergraduate electrical engineering curricula of most universities. A particularly salient feature of this (as also almost all other) work of Maxwell is the blending of "the imaginative and the analytical faculties to produce results partaking of both natures" [1, p. 242]. "Through Maxwell's theory," James Jeans pointed out, "electricity first became a mathematically exact science and the same might be said of other larger parts of physics" [2, p. 30]. Thus, apart from forming the foundation of electromagnetic theory, it is the development of Maxwell's equations with the unification of electricity and magnetism that has "forced on us" the special theory of relativity, which "serves as a touchstone in modern physics for the possible forms of interaction between fundamental particles" [3, p. 514]. It has been recently shown that Maxwell's equations can be used to prove some of the relativistic effects [4]. Thus, the importance of Maxwell's theory extends well into fundamental physics.

Physics has four hundred years of history behind it. For an interesting account of the history of electricity and magnetism, which is "in large measure the history of science itself" [3, p. 22], the reader can refer to [1, 5]. For example, a very brief sketch can be found in [6]. In fact, Maxwell's electromagnetic theory also represents the cumulative efforts of several great mathematicians and experimenters. A very readable sketch of the evolution of electromagnetics leading up to Maxwell's equations and beyond can be found in [2]. However, there is a "lack of historical perspec-

tives on the genesis (of these equations) in current engineering and mathematics curricula" [7, p. 4]. Of course, one of the main reasons for this is a time constraint. It is the intent of this article to propose that the presentation of Maxwell's equations from a historical perspective is both necessary and advantageous in undergraduate education.

The article proceeds as follows. First, an illustrative comparison of some of Maxwell's original equations is made with those currently used, so that the student may develop an appreciation as to how they have developed from an apparently complex form into a more elegant form. In so doing, the significance of the contributions made by Maxwell himself in the development of modern vector notation is highlighted. Next, the four equations are discussed from historical perspectives. Finally, the significant outcomes of the discussion are presented, and the article is concluded.

2. Maxwell's Equations

Maxwell presented his equations in 1864, in a memoir entitled "A Dynamical Theory of the Electromagnetic Field," presented to the Royal Society. In 1873, he presented the more extensive *Treatise* [8]. In this work, he used the Cartesian coordinate system of Des Cartes, as that method was "still the most familiar to students of science" [8, p. 9]. Some of the original Maxwell's equations and their modern version are presented in Table 1 [8, 11] for illustration, so as to facilitate an appreciation of how subsequent

Table 1. Three of the nine of Maxwell's original equations and their equivalent vector forms.

Original x, y, z Component (Modern Vector) Description	Original Equation	Equivalent Vector Equation
Total current p', q', r' ($\mathbf{J}_T$) due to <i>true current</i> p, q, r (conduction current, $\mathbf{J}$) and <i>electric displacement</i> f, g, h (displacement current, $\mathbf{D}$)	$p' = p + \frac{df}{dt}$ $q' = q + \frac{dg}{dt}$ $r' = r + \frac{dh}{dt}$	$\mathbf{J}_T = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$
Relationship between magnetic force α, β, γ (magnetic field intensity, $\mathbf{H}$), magnetic permeability, μ , and electromagnetic momentum F, G, H (magnetic vector potential, $\mathbf{A}$)	$\mu\alpha = \frac{dH}{dy} - \frac{dG}{dz}$ $\mu\beta = \frac{dF}{dz} - \frac{dH}{dx}$ $\mu\gamma = \frac{dG}{dx} - \frac{dF}{dy}$	$\mu\mathbf{H} = \nabla \times \mathbf{A}$
Equations of electromotive force P, Q, R (electric field intensity, $\mathbf{E}$); ψ is electric potential (electric scalar potential), $\mathbf{v}$ is the velocity of a conductor moving in an isotropic medium	$P = \mu \left(\gamma \frac{dy}{dt} - \beta \frac{dz}{dt} \right) - \frac{dF}{dt} - \frac{d\psi}{dx}$ $Q = \mu \left(\alpha \frac{dz}{dt} - \gamma \frac{dx}{dt} \right) - \frac{dG}{dt} - \frac{d\psi}{dy}$ $R = \mu \left(\beta \frac{dx}{dt} - \alpha \frac{dz}{dt} \right) - \frac{dH}{dt} - \frac{d\psi}{dz}$	$\mathbf{E} = (\mathbf{v} \times \mathbf{B}) - \frac{\partial \mathbf{A}}{\partial t}$

work helped express the equations more elegantly. For a complete listing, the reader can refer to [2, p. 185]. It may be noted that in the first column of the table, the vector quantities in bold face are preceded by their corresponding rectangular components. For example, p, q , and r represent the rectangular components of the conduction current vector, $\mathbf{J}$.

Despite making use of the Cartesian component representation, Maxwell advocated the use of Hamilton's "Calculus of Quaternions," as the relationship between physical quantities in electrodynamics could be expressed "far more simply" by many fewer of Hamilton's equations. According to Maxwell, "one of the most important features of Hamilton's method is the division of quantities into Scalars and Vectors....[In this method] the position of a point in space is defined by the vector drawn from a fixed point, called the origin, to that point.... Any physical quantity (which may be either a scalar or a vector), whose value depends on the position of the point, is treated as a function of the vector" [8, p. 10-11]. It can thus be considered that quaternions have a vector component in them. When we consider the quaternion expression given by Maxwell to describe the fact that the line integral of electric field intensity, $\mathbf{E}$ (the term "electromotive intensity" was used by him), is independent of the path of integration [8, p. 143],

$$\mathbf{E} = -\nabla\phi, \quad (1)$$

where ϕ is the scalar potential, it is clear that this equation is the same as that written in modern vector notation. Similarly, when Maxwell presented the electromagnetic equations in quaternion form [12, pp. 257-259], he evidently used $S \cdot \nabla \mathbf{D}$ to represent $\nabla \cdot \mathbf{D}$, and $V \cdot \nabla \mathbf{H}$ to represent $\nabla \times \mathbf{H}$. Thus, in a certain sense, it can be deduced that in [6, 12], Maxwell essentially introduced and made use of vector notation [13]. Moreover, he defined and introduced the operators ∇ and ∇^2 , although he did not then use the terms "del" and "Laplacian: for these operators [7, p. 90]. Thus,

although the modern mathematical formulation of Maxwell's equations as known today is generally attributed to Hertz (who obtained the scalar forms of the equations in 1884) and Oliver Heaviside (who obtained the vector forms in 1886) [2, p. 223], Maxwell's own contribution to the development of modern vector notation was also significant. Table 2 presents the modern equations with their physical interpretation in differential and integral forms.

It is apparent from Table 2 that the change to vector notation resulted in an elegant and compact representation of Maxwell's equations. More generally, this change also "produced a symmetric mathematical representation that reinforced the perception of physical symmetries between the various fields" and greatly inspired later developments in physics [14].

In what follows, we present the historical development of these four equations. It may be noted again that a complete historical development is beyond the scope of this paper, and we will only include what can be considered to be the more significant of the contributions. The basic idea is to inform the students about the involvement of many more thinkers than are normally mentioned in class.

3. Historical Perspective

3.1 Gauss' Law for Electrostatics

In 1767, Joseph Priestley, taking a clue from an experimental observation originally made by Benjamin Franklin, deduced that the charge of a uniformly charged spherical shell would reside on its surface, and that the electric force between two charges would vary inversely as the square of the distance between them. From 1771 to 1781, Henry Cavendish was actively engaged in valuable

Table 2. Maxwell's equations in current form.

Name	Differential Form/ Integral Form	Physical Interpretation
Gauss' law for electrostatics	$\nabla \cdot \mathbf{D} = \rho$ $\oint_S \mathbf{D} \cdot d\mathbf{s} = \int_V \rho dv$ $\mathbf{D}$: electric flux density/ electric displacement density ρ : volume charge density	The net outward flux through a closed surface is equal to the net charge enclosed by the surface
Gauss' law for magnetism	$\nabla \cdot \mathbf{B} = 0$ $\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$ $\mathbf{B}$: magnetic flux density	The net outward magnetic flux through a closed surface is equal to zero
Faraday's law	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ $\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s}$ $\mathbf{E}$: electric field intensity	A change with time in magnetic field causes electric field
Ampere-Maxwell law	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$ $\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{s} + \frac{d}{dt} \int_S \mathbf{D} \cdot d\mathbf{s}$ $\mathbf{H}$: magnetic field intensity $\mathbf{J}$: conduction current density	A change with time in electric field causes magnetic field

electrical research. However, he chose not to publish his papers. The reason for this could have been his preference for aloofness. According to Feather [5], Cavendish "assiduously avoided contact with his fellow men throughout his adult life....Occasionally he would present the results of his own researches for publication...but much more frequently he did not." (His papers were rescued by Maxwell nearly a century later.) In fact, Cavendish had reexamined Priestley's inverse-square law by more careful measurements. However, the inverse-square law for electrostatics is traditionally attributed to Charles Coulomb, who, in 1785, carried out a systematic experimental verification of the law using a torsion balance [3].

In 1812, Karl Friedrich Gauss rediscovered the divergence theorem (earlier discovered by Joseph Lagrange for gravitation in 1764) that applies in situations where the inverse-square law is valid [2, p. 6]. Starting from this theorem, he derived Gauss' law for electrostatics; however, he did not publish this work until 1867 [15].

In the year 1837, Faraday conducted systematic experiments on dielectric-capacitance dependence. In this experiment, he constructed two concentric spherical shells, with the outer one such that it could be dismantled exactly into two halves. The space between the two spheres could be filled with insulators. He used this capacitor configuration with the outer sphere grounded and the inner sphere isolated. Based on the experiments, he showed that the capacitance of a capacitor with solid or liquid dielectric was more than that of the capacitor with air in between the conductors [5]. It is sometimes considered that Gauss' law is just a mathematical statement of this experiment of Faraday [16]. Some other authors suggested that Gauss' law is based on Coulomb's force law [7, 17]. However, considering the following historical accounts, it appears that Gauss' law needs to be viewed essentially as an original mathematical discovery, rather than as an experimental result:

1. Gauss' theorem is due originally to Gauss. Gauss' law for electrostatics was discovered in 1812 [2], while Faraday's experiment cited above was done in 1837 [5]. Even assuming that there might be some ambiguity in historical accounts, the following further points can be considered.
2. Maxwell, in developing the idea of electric displacement leading to the statement of the law that "the displacement outwards through any spherical surface concentric with the sphere is equal to the charge on the sphere" [8, pp. 54-57], used a reasoning based on Faraday's experimental investigations on the "influence of the medium on the capacitance of condensers" [5, p. 116]. However, this has to be understood in conjunction with Maxwell's statement in the Preface to his *Treatise* that he expressed the "mathematical discoveries" (italics added by the author) of Gauss (and others) in terms of ideas derived from Faraday as it is "much better" this way. He also talked about the "electromagnetic speculation" (italics added by author) originated by Gauss. When introducing the theorem of Gauss in Article 59, Maxwell described it as a general theorem that indicated the "conditions of solution of problems in the distribution of electricity" [8, p. 62].
3. While Whittaker [1, p. 240] said that Gauss was one of those natural philosophers who speculated much "on the manner in which electric and magnetic influences are transmitted through space," Feather [5, p. 21] described Gauss' theorem as "original."

Of course, it is possible that the much earlier Coulomb's law had its *influence* on the discovery by Gauss. It may be noted here that Coulomb's law is derivable from Gauss' law, and hence is not counted as one of Maxwell's equations [17].

3.2 Gauss' Law for Magnetism

In 1695, Sir Matthew Hale, in his posthumously published work, wrote that "every smallest Particle of this Magnet, every little Dust thereof should have the same Conformation that the entire Magnet had, every little Particle having his Poles...perfectly Analogal to the great Magnet, whose Dust it is" [5, p. 62]. In 1750, John Michell rediscovered that the simplest magnet "possesses two poles of equal strength but opposite character," and also that "the law of mutual attraction or repulsion between magnetic poles is an inverse-square law" [5, p. 61]. Subsequent attempts to obtain magnetic poles also failed, and it was recognized that if a magnet was divided in two, it only resulted in two dipolar magnets, and so on. In 1785, Coulomb made use of long and thin magnets and "came very near to establishing the law of magnetic action," and pointed out that magnetic polarity needed an atomic theory for a complete understanding. In 1824, S. D. Poisson developed a thorough mathematical theory of magnetization, where for the first time he introduced the concept of a *magnetic dipole* moment. This can be considered as a formal recognition then of the fact that magnetic monopoles had not been found to exist. Later, in 1832, Gauss carried out a careful experimental validation of the inverse-square law [5]. Since Gauss' theorem applies to the situations where the inverse-square law is valid, it can be directly applied to the case of a magnetic field, leading to the "law of nature" that the total outward magnetic flux over a closed surface is zero, as magnetic poles have until now not been obtained. This law can rightfully be called Gauss' law for magnetism, for example, as in [18]. It may be noted that some authors preferred not to assign a name to this law.

Unlike the other Maxwell's equations, Gauss' law for magnetism is contingent upon no magnetic monopole being observed in nature. This question of a magnetic monopole is not a closed question (as, in fact, is any scientific question, in principle), and its existence is viewed as an interesting speculation. One of the important reasons for this interest is charge quantization, one of the most profound mysteries defying an understanding until now, and "Dirac's argument [in 1931]...that the mere existence of one magnetic monopole in the universe would offer an explanation of the discrete nature of electric charge" [3, p. 273].

3.3 Faraday's Law

Faraday's first original contribution to experimental electromagnetics was made in 1821, when he demonstrated the influence of a magnet on a current. In 1822, he resolved to "convert magnetism into electricity." After continued investigations for about ten years, in 1831 he presented his qualitative results that "an induced current flows in a closed circuit whenever the current in a neighboring circuit changes in magnitude," and "whenever another circuit in which a steady current is flowing is made to approach or recede from the circuit in question." In 1832, he quantitatively showed that the "magnitude of the induced current was inversely proportional to the resistance of the wire." He said that this phenomenon, whereby electric current was produced from magnetism, represented electromagnetic induction. In 1850, he showed

that "the induced electromotive force is directly proportional to the rate of change of the number of lines of force threading the circuit." (In the intervening period, he was studying the flow of current through liquids [5].) This is Faraday's law.

Thus, the essence of the law of electromagnetic induction was presented in 1831. An observation relating to an important extension of the induction of currents was made and reported by Joseph Henry in 1832, and later by William Jenkin, in 1834. It is acknowledged that Faraday was "undoubtedly entitled to the full honour" for the discovery of electromagnetic induction [1, p. 173].

3.4 Ampere-Maxwell Law

In 1819, Hans Oersted opened up a new field of inquiry when he found that a current-carrying wire deflected a nearby magnetic compass. (It was this work that motivated Faraday to work on the reverse possibility of producing current from a magnetic field.) A lithograph showing Oersted's original demonstration, and also a model of the apparatus used by him, were given in [19]. Oersted qualitatively deduced that an electric current passing through a long straight wire produces a magnetic field in the surrounding space, with the lines of force forming circular loops about the wire as a common axis. He also showed "that the deflection of his compass needle was increased when the long straight current-carrying wire was looped back on itself, so that it passed both above and below the needle." An account of Oersted's research was published in July 1820. Later in the same year, Jean Biot and Felix Savart provided a quantitative and empirical description of Oersted's finding, in the process presenting what is now called the Biot-Savart law [5]. This law relates the magnetic field at any point in space to the current in the wire that generates it.

The results of Oersted's experiments were confirmed by Ampere, who carried out systematic experiments on the effects of electric currents on one another. He discovered that if in two parallel wires the current flow was in the same direction, there would be attraction between them, and that if it was in opposite directions, there would be repulsion. A drawing of the original table used by Ampere in his experimental studies appeared in [20], for example. His studies eventually led to the refinement of the galvanometer as a current-measuring device, originally invented by Schweigger [20]. A model of Oersted's version of Schweigger's galvanometer can be viewed in [21].

Significantly, while considering Oersted's experiments, Ampere from the beginning referred to "the *mutual* action of a current-carrying wire and a magnet" rather than the "action on a magnetic needle by a current-carrying wire." During the next five years, he carried out theoretical and experimental investigations towards formulating a combined theory of electricity and magnetism. Ampere's most significant work on electricity and magnetism was published in 1826. It was called *Memoir on the Mathematical Theory of Electrodynamical Phenomena, Uniquely Deduced from Experience*. It contained a mathematical derivation of the electrodynamic force law, and it describes four experiments. This theory became "fundamental for 19th century developments in electricity and magnetism" [19]. Ampere's law was contained in this memoir:

$$\nabla \times \mathbf{H} = \mathbf{J} \quad (2)$$

Maxwell was full of appreciation for Ampere's work: "[Ampere's theory is] one of the most brilliant achievements in

science...It is perfect in form and unassailable in accuracy; and it is summed up in a formula from which all the phenomena may be deduced, and which must always remain the cardinal formula of electrodynamics" [5].

In 1861, Maxwell published a "mechanical model" of the electromagnetic field, in which he presented a modified Ampere's law [5]:

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \quad (3)$$

where the additional term represents what Maxwell called the displacement current. Several justifications were advocated in the literature for introducing this term [22]. One of them was the observation that Equation (1) is inconsistent with the principle of charge conservation [23]. By purely theoretical arguments, Maxwell was able to remove this defect by the addition of displacement current. Apart from curing the inconsistency defect, this addition also brought about a pleasing symmetry, by suggesting that a changing electric field induces a magnetic field, just as a changing magnetic field induces an electric field (Faraday's law). This speculative discovery represented the most brilliant and radical concept introduced by Maxwell, and formed the basis of his electromagnetic theory.

4. Discussion

Maxwell's theory, contained in the above four equations, discussed the most basic implications of electrodynamics, including the nature of electric charge, displacement current, and generation and detection of electromagnetic waves. However, it was "inevitable that a theory so novel and so capacious as that of Maxwell should involve conceptions which his contemporaries understood with difficulty and accepted with reluctance" [1, p. 254]. His theory and ideas "were expanded, modified, and made understandable after his death" in 1879 by Hertz and "The Maxwellians" (George Fitzgerald, Oliver Lodge, and Oliver Heaviside) [11]. In 1888, Hertz, by his brilliant experiments, succeeded in generating electromagnetic waves, and thus confirmed Maxwell's theory some 22 years after it had been proposed. Needless to say, all these efforts, including those of Maxwell's contemporaries and predecessors, have had profound technological implications.

Several other distinct qualities stand out when Maxwell's theory is examined from a historical perspective, as above. An important thing was his modesty: he did not proclaim his theory loudly, but rather just introduced it as "another way of treating the subject" [8]. This quality of his was also emphasized in his biography [24]. Maxwell's generous acknowledgment and appreciation of the contributions of others was also an attribute that invites attention, particularly these days, with increasing instances of plagiarism. It is appropriate to reproduce here what Maxwell had to say about two such significant contributors – Gauss and Faraday – in the preface of his *Treatise* [8]:

Gauss...brought his powerful intellect to bear on the theory of magnetism, and on the methods of observing it, and he not only added greatly to our knowledge of the theory of attractions, but reconstructed the whole of magnetic science as regards the instruments used, the methods of observation, and the calculation of the results, so that his [approach] may be taken as models of physical research....

...before I began the study of electricity I resolved to read no mathematics on the subject till I had first read through Faraday's *Experimental Researches in Electricity*....If by anything I have here written I may assist any student in understanding Faraday's modes of thought and expression, I shall regard it as the accomplishment of one of my principal aims – to communicate to others the same delight which I have myself found myself in reading Faraday's *Researches*. [Thus, Maxwell acknowledged that Faraday's experiments represented a great influence on him.]

In this connection in general, it can be thought that even teaching a theory can be made more ethical by including where possible the works of others that contributed to that theory.

Finally, the importance Maxwell attached to keeping scientific spirit (and hence, openness) alive was evident, for example, by his admission that he had not been able to account by mechanical considerations for the stresses in the dielectric material [25]. Even while trying to extend his theory to also understand the cause of gravitation, he said he was "unable to understand [as to how to] go any further" [1, p. 258]. In his *Treatise*, he provided "a step by step path through" Faraday's experiments, as he believed "reading Faraday was a way to cultivate a scientific spirit" [25].

In addition to the above, in the course of presenting displacement current and the possible presence (or absence) of magnetic poles, the level of current understanding and the need for furthering the work can be discussed to motivate the students to appreciate that fundamental electromagnetics is still an open area for research.

At the University of Nottingham Malaysia Campus, the author incorporates substantial historical material from this paper and from the references cited herein into the teaching of certain electrical-engineering modules. In order to understand whether the students found this incorporation to be beneficial, a survey was conducted in the Spring 2007 semester. An analysis of the responses led to the result that the students, in an overall sense, appeared to exhibit a positive perception. The details of this survey and its results will be the topic of a future paper [see below].

5. Conclusion

The importance and benefits of teaching engineering and scientific concepts from a historical perspective in undergraduate education were considered. Such an approach is likely to open up the students' interest in the subject and beyond. Emphasizing this point was the essential intent of this paper. In this light, Maxwell's equations were discussed from a historical perspective.

During the discussion, the significance of Maxwell's contribution to the development of modern vector notation was highlighted. It was suggested that Gauss' law was essentially a mathematical discovery, and that the "no-magnetic-monopole" law can rightfully be called Gauss' law for magnetism.

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7. References