Units used in E&M PhL 1.22.03
Review 10.27.22

1. About Units in General

Define versus Use. Some equations *define* a new unit, other equations just *use* the unit in some manner in the solution of a problem. For example, $E=mv^2/2$ and $E=mgh$ both use an energy unit on the left, but neither is the defining equation for an erg. Perhaps E = FxD is the defining equation here. Although $E=mv^2/2$ is true for what it means, since we are not defining the erg by this equation, we would never say 1 erg = 1/2 gm-cm/sec^2 the way we might say 1 foot = 12 inches. But we *would* use the general notion that erg = gm-cm/sec^2 in terms of "dimensions" to check work we do, see next item. A third type of equation is used to *rescale* a unit, such as L(in) = 12 L(ft) and for such an equation, it is always correct to say that 1 ft = 12 in. See below. [ok]

Two ways of handling units in an equation. Consider:

F(nt) = m(kg) a(m/sec^2)	F = 36 kg *10 m/sec^2 = 360 nt
	F = m kg * a m/sec^2 = m*a nt

In the equation on the left, m(kg) means "mass measured in kilograms". In this form, m(kg) is just a number, it has no units! We have already handled the units with the parentheses. The two numbers on the right when multiplied give the number on the left. When we solve a problem, however, we often write things as done on the top right. The method shown on the bottom right seems confusing to me -- writing units like this seems better suited to doing numerical calculations and making sure you get no errors. With numbers, you will cancel and combine units as required and that acts as a check on your work. But from a theoretical point of view, I find the notation on the top left to be much clearer and unambiguous. Sometimes equations have a different form if one uses different units, so F = ma would not be "enough" if that were the case for that equation, which it is not. But for the Coulomb force law, it is the case! [ok]

Defining a Unit by Rescaling. Consider:

1 foot = 12 inches	L(inch) = 12 L(foot)

The thing on the left is the *definition* of a foot, if the inch is already defined. Notice how the labels are reversed in the right equation. [ok]

Converting Equations to Different Units. The above is an example of a rescaling of some unit. Here is my general notation for doing this:

L(inch) = L(foot) * { foot/inch } // { " right over left" }

or

1 foot = 1 inch * { foot/inch } // { " left over right" }

The symbol { foot/inch } is meant as a dimensionless fraction. [ok] You put foot = 12 inch on the top, cancel the inch, and end up with {...} = the number 12. {foot/inch} does **not** mean "feet per inch",

because if it did, that would give 1/12 which is not what is meant! When a dimension is on the bottom, you have to reverse the rule. For example:

v(m/sec) = v(mile/hour) * {miles/m} * {sec/hour}

= v(mile/hour) * {mile/ft}* {ft/in}* {in/cm}* {cm/m} * {sec/hour}

= v(mile/hour) * {5280} * {12}* {2.54}* {1/100} * {1/3600} = .45 v(m/hour)

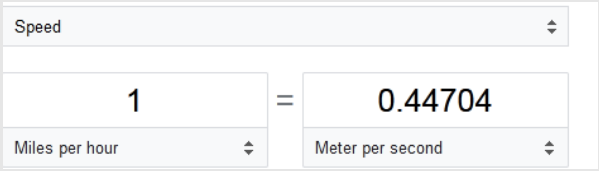
Try to verify the first line (only) above: It says this:

v(m/sec) = v(mile/hour) * {miles/m} * {sec/hour}
= v(mile/hour) * 1609.34* (1/3600)
= v(mile/hour) * 0.444706 [ok]

I think my inversion rule would then say

1 mile/hour = 0.444706 m/sec

And here is verification



This provides a 100% error-free clean way to convert any quantity from any units to any other units! We will use this method many times in what follows.

2. Units used in Electricity and Magnetism; comments applicable to all Systems

The Two Basic Equations for Defining Charge and Related Quantities. The first of these is associated with Mr. Coulomb's work done around 1788, now known as his law. The second is associated with Mr. Ampere's work done around 1827 (and other work of Biot and Savart and Oersted around the same time). Following Jackson's appendix, we install constants k1 and k2 into these equations:

$F = k1*qQ/r^2$ (1) / force between two charges

$dF/dx = k2*2iI/D$ (2) / force between two parallel wires

The second equation is really an integration of a more primitive equation which is this:

$\mathbf{dF} = k2* i\mathbf{dx} \times (Id\mathbf{X} \times \mathbf{r})/r^3$ (3)

so the factor of 2 appearing in (2) just comes from the integration to get that result.

Sometimes (1) is used to define a unit of charge, and sometimes (2) is used. Different choices of k1 and k2 yield the different "systems of units" used in E&M.

How are k1 and k2 related to Maxwell's Equations? The first order of business is to say how we are going to define the electric field. Every system does this in the same way *with no constant* as follows:

$F = qE$ from which we conclude that: $E = k1*Q/r^2$

That is to say, there is no constant between F and qE. The associated Maxwell equation must have this form as we show in the words that follow:

$\text{div}\mathbf{E} = 4\pi k1\ \rho$ $\int E \bullet dA = \int \text{div}E dV = 4\pi k_1 \int \rho dV$

When we integrate the divE equation (Gauss's Theorem) over a sphere around a point charge, doing int(E.dA) makes 4π from the angle integral, so there has to be a 4π put into the divE equation RHS to cancel it. The main point is that the k1 appearing in (1) also appears in the divE equation.

In similar fashion we can rewrite (3) as

$\mathbf{dF} = (1/\alpha)\ i\mathbf{dx} \times d\mathbf{B}$ $d\mathbf{B} = \alpha\ k2^*\ (Id\mathbf{X} \times \mathbf{r})/r^3$ // 1st is Biot-Savart (4)

We should think of the first equation in the manner of F = qE. For electrics, we use "q" as a test charge to sense the E field. For magnetics, we use "idx" as a test current to sense the B field. When equation (3) is split into two parts this way, allowing a definition of the B field, we have the chance to add a new constant which we might have called k4, but Jackson calls it α so we will call it α as well. Since it cancels when the first two equations are glued back together to give (3), it can be anything. [In the Gaussian cgs system it is set to c, in mks to 1, for example.]

Of course when we later learn that E and B are part of the same tensor $F^{\mu\upsilon}$, we must choose k1, k2 and α so E and B had the same dimensions, and this is one major benefit of the cgs-Gaussian system. This is of little concern in the practical units.

So the right equation defines the B field produced by a practical current, and the left shows the force of that field on a test current. But the left equation now stated this way *also* applies to the force on a test current in the presence of a fixed-magnet field B. We now know this is due to the same current idea, but it was not known in 1810 (say), so this first formula correlated fixed-magnet B with electro-magnet B and implicitly said the two B's were the same thing. The internal currents idea of M then quickly followed.

As noted, the first equation is like F = qE, but because it is more complicated in nature and correlated previously separated ideas, it gets a name associated with it, in honor of Biot and Savart who, in 1820, discovered the law as the force on a wire in the presence of a fixed-magnet. Oersted in 1819 had found that a current deflected a compass, but did not measure or analyze the nature of that force.

The associated Maxwell equation is this (Ampere's Law)

$\text{curl}\ \mathbf{B} = 4\pi\ \alpha\ k2\ \mathbf{J}$ $\int B \bullet dl = \int \text{curl}B dA = 4\pi\alpha k_2 \int J \bullet dA$

To see the connection here, the simplest case is the straight wire. Integrate dB from (4)-right over the wire and you get B = α k2 2 I/r. Compute this same B from Stokes theorem applied to the curl equation shown above, you get 2πrB on the left from the line integral, and you get 4πα k2 I on the right, so B = α k2 I /r again. This shows why we had to put the 4π and k2 into the curl B equation.

So, in this section we have related our constants to the two Maxwell equations that have sources (ρ and J). The divB equation is uninteresting since it is just 0. We still have to worry about the 4th Maxwell equation about curl E, and we have to worry about the "other term" in the curl B equation, known as the "displacement current". We do these things right now.

A simple problem regarding the term curl B = k4 ∂E/∂t.

Let's assume a constant k4 in this equation in current-free space and see if we can connect it with k1 or k2. To do this, let's study a simple problem. Imagine a tiny ring (radius a) moving toward a point charge at speed v, all flat-on and symmetric. If I apply the above bolded curl rule to this problem,

$$\int B \bullet dl = \int curlB \bullet dA = +k4 d/dt(\int E \bullet dA)$$

Using this formula, we can compute the B field in the tiny ring to be:

$$B = k1\ k4\ Q\ a\ v / r^3$$

In this problem, the ring is so small, that E is constant on the ring, so electric flux is just πa^2 E = πa^2 k1 Q/r^2. Putting k4 d/dt on this makes - k4 2 v/r^3, while the left side gives 2πaB. The 2πa cancels on both sides, giving the above result.

Now I treat the same problem by sitting on the ring and watching Q approach at speed v, I can identify Qv with Idx in the Biot-Savart B formula above, and I get that

$$B = \alpha\ k2\ Q\ a\ v/r^3$$

where sinθ from the cross product is a/r. Comparing these two results, we find that,

k4 = α k2/k1.

Notice that this is just what Jackson has put in his (A.8).

A simple problem regarding curl E = -k3 ∂B/∂t (Faraday's Law)

The simple problem needed here is to put a square loop at the edge of a region of constant B field, say between the poles of a magnet, and have the loop move to the right, say, at speed v. The equation of interest here is this

$$\int E \bullet dl = \int curlE \bullet dA = -k_3 d/dt(\int B \bullet dA)$$

In our simple problem, the change of flux on the right is Byv where y is the height of the loop, so the RHS is then k3 Byv in magnitude. The electrons in the moving loop wire feel a force **dF** = (1/α) i**dx** x d**B** as noted above. Putting J = qvδ(r-a) and integrating in all 3D over a tiny "wire", i dx = qv, and we then get that the force is F = (1/α) qvB. We interpret this as being due to an electric field E' = F/q = (1/α) vB in the rest frame of the wire, and the line integral then gives (1/α) yvB. Thus, **k3 = 1/α**.

Jackson on page 172 deals with this issue in a more general way, and here is a summary of his method. You write the usual Stokes equation with the above curl rule with constant k3 (he calls it k there). You then consider the situation of a current loop moving in the presence of a lab B field of some sort. The electric field in equation 6.4 is denoted E' to remind us that it is the field in a frame moving with the wire, because a contour integration is done in that frame. That is a key point. We are interested in the field that pushes a current in the wire, and that field must be the field that is in the rest frame of the wire. So I think (6.4) is evaluated in the lab, but you use the field E' as just stated. Then in (6.5) he computes the flux change in the lab and comes up with the famous two terms -- I proved the second term on a separate sheet. This lets you write (6.6) and then interpret E as the field in the lab. We have then proven the field transform law (6.8) for non-relativistic velocity v. We still have our k3 in this formula. Now comes the magic trick. In the lab, an electron in the wire is in fact a current of the form q v. We know from Biot-Savart that such a current feels a force from the B field, and that force is (1/α) q v x B. So *Jackson's* conclusion is that k3 = 1/α. (In this section he uses k for k3, but uses k3 in his appendix.)

Maxwell's Equations and the Connection to Speed of Light. So far then, based on the discussions above, we have these Maxwell's Equations:

div**E** = 4πk1 ρ curl **E** = - (1/α)∂**B**/∂t (Faraday) k1 = c^2 • k2

div **B** = 0 curl **B** = 4π α k2 **J** + (α k2/k1) ∂_t **E**

It took some doing just to get to this point! Now if you do ∇x∇xE and get the free-space wave equation, you conclude that **k1/k2 = c^2** in your system of units whatever it is -- we have not picked one yet. This is because plane waves must travel the speed of light. If you start with ∇x∇xB, you reach the same conclusion. Either way you pick up both curl equations and you therefore conclude that **k1/k2 = c^2.**

Do the constants k1 and k2 and α have dimensions?

If we look back at our original two defining equations in which k1 and k2 first appear, and if we imagine that each of our constants have some kind of dimensions, then the dimensions in these two equations are only consistent IF this is true: (k1/k2) must have the dimensions of m^2/sec^2 (say in mksa). We know that in fact k1/k2 = c^2 so this all makes perfect sense. This does NOT tell us what we have to do for k1 and k2 separately, however. Similarly, constant α may or may not have dimensions, since we can make it be anything we want. Since α is a rescaling of the B-field, giving it dimensions will simply change the dimensions of the B field.

There are really only two reasonable choices to make about dimensioning k1 and k2. One choice is to give each of these units its "natural" dimensions such that the dimensions cancel on both sides of the two basic equations.

F(force) = k1(force-L^2/$charge^2$) *q(charge)Q(charge)/r$(L)^2$ (1)

$$dF(\text{force})/dx(L) = k2(\text{force-time}^2/\text{charge}^2)*2i(\text{charge/time})I(\text{charge/time})/D(L) \qquad (2)$$

In this case, we cannot make any connection between the charge unit and the other units. This is a good approach to take if neither k1 nor k2 is going to be unity.

The other reasonable choice is to make k1 = 1 with no dimensions, or to make k2 = 1 with no dimensions. If this is done with k1, then the equation is regarded as making a connection between the unit of charge and the base units of force and length. You get force = charge^2/L^2 . If this is done instead with k2, then you get force = charge^2/time^2 which is different!

The mksa system takes the first approach noted above. The esu system sets k1 = 1 and this defines the esu unit of charge such that dyne = esu^2/cm^2. The emu systems set k2=1 so dyne = emu^2/sec^2

What about D and H?

As Jackson points out on page 617, this allows 6 more constants to appear in your world:

$$D \equiv \varepsilon_0 E + \lambda P = \varepsilon' E \qquad H \equiv (1/\mu_0) B - \lambda' M = (1/\mu')B$$

I have used ε' and μ' because, although Jackson says everyone uses μ and ε, I see that Bleaney on page 19 have D = εε$_0$E, so I would then say ε' = εε$_0$ for Bleaney, whereas Portis on p 112 uses just ε' = ε.

Since we are defining D and H by these equations, we are really free to set all 4 constants arbitrarily. Of course P and M are zero in free space. In this limit, ε becomes ε$_0$ and μ becomes μ$_0$ and this is the *motivation* for these notations, as if these numbers were properties of free space. Jackson says that λ and λ' are always both 1 or 4π.

It is desirable to have the divD and the curlH equations "look nice".

From above we have div**E** = 4πk1 ρ, so in free space we would then have div D = (4πε$_0$ * k1) ρ. The choice of k1 and ε$_0$ is always coordinated so that the factor in parens is either 4π or 1, for "unrationalized" or "rationalized". In all systems shown on page 618, these are the only two possibilities for divD.

The other interesting equation is curl **B** = (4π α k2) **J** + (α k2/k1) ∂_t **E.** In free space we would have this result: curl **H** = (4π α k2/μ$_0$) **J** + (α k2/ (k1 μ$_0$ε$_0$)) ∂_t **D.** Again, k2 and μ$_0$ are chosen so the first factor (in front of the J) is either α or 4πα. In Jackson's page 618 list, there are two choices for α (1 and 1/c), so we end up with <u>four</u> different forms for the curlH equation in different systems!

Jackson does not comment on the χ type factors one sees in P = χE and M = χH. There is some variation in how these things are done as well. Bleaney says P = χε$_0$E but M = χH, so for each author you have to go look at exactly how they do things! I suspect that geophysicists are interested in H and μ and use units different form transformer designers.

Why batteries are about a volt

We are jumping ahead here a bit, but we want to show what the "volt" is a useful unit size. Using the Bohr radius and the electric charge, we have

$|e| = 1.6 \times 10^{-19}$ C $\qquad r_B = 5.3 \times 10^{-11}$ m

$|e|/ r_B = 3 \times 10^{-9}$ C/m

$V(volts) = k_1^{mks\text{-}C}\ Q(C)/r(m) = 9 \times 10^9 * 3 \times 10^{-9} = 27$ volts as a rough scale

Hydrogen levels are $E = -e^2/(2r_B n^2)$ so that $V = -13.5\ volts/n^2$. In a larger atom there is shielding so the general form is the same, but the radius is larger, so set n = 4 say and we get outer electron of something like Cu being bound by on the order of 1 eV, meaning the valence electron is at a potential of - 1 volt. In a battery, you might pull an electron off Cu to get Cu++, and you might put it onto Zn++ to get Zn. Each of these actions will involve a different energy on the order of 1 eV, so the net result will be on the order of an eV, so the battery will put out something on the order of 1 volt. The ionization and affinity values are affected by the fact that the ions which result are not free, but are in a solution, but this does not change the scale of things. In fact, the energy obtained by putting Cu++ in solution probably offsets its ionization potential. (see chemistry books for more detail)

The potential unit determines the current unit.

In any system, we know that power P = IV. Thus, the product of the current and potential units must be the energy/sec unit in whatever system you are in. In mks we get watts = amperes * volts. In the other systems we get erg/sec = abamperes*abvolts = statamperes*statvolts.

3. The mksa System (= SI system = rationalized-mks).

We do this one first because most lookup information is in these units, which are sometimes called the "practical" units. Also, the world has adopted SI now and SI is "taking over". One writes equation (2) above as:

$$dF(nt)/dx(m) = k2(nt\text{-}sec^2/C^2) * 2i(C/sec)I(C/sec)/D(m) \qquad k2(nt\text{-}sec^2/C^2) = 10^{-7} = \mu_0/4\pi$$

and one then *defines* the size of quantity C (the **Coulomb** unit of charge) by setting $k2(nt\text{-}sec^2/C^2) = 10^{-7}$. In this unit, an experimenter will find that |e| = 1.6e-19 C. WHY was k2 selected to be this strange power? Because then a C/sec = ampere will be a reasonable "practical unit" of current, and will have a simple connection (1/10) with the formerly used abampere unit of the emu system. More on this historical subject later. In fact the **ampere** is really defined as per above, then Coulomb = ampere-sec.

The constant k2 is usually written as $\mu_0/4\pi$. The 4π removes 4π from the Maxwell equation, and the μ_0 is so called because it ends up (via a long path) in the $H = \mu_0 B$ relation in free space. In the mksa system, the constant k2 is assigned its natural dimensions:

$$k2(nt\text{-}sec^2/C^2) = 10^{-7} = \mu_0/4\pi \qquad k2 = 10^{-7}\ nt\text{-}sec^2/C^2 \quad (= 10^{-7}\ henry/meter)$$

The appearance of **"henry"** just comes from $V = L\partial_t I$, and is the way $\mu_0/4\pi$ is usually expressed. Now, because k2 is assigned these "natural" units, we cannot make any connection between the Coulomb and any of our base units. The units just cancel out in the equation, as was noted earlier.

Notice that even with an "amp" of current, the force between two "practical wires" a meter apart is tiny, about 2/10,000,000 of 1 nt, and think of 1 lb = 4.45 nt, so nt really is an everyday unit. So in the *magnetic* world, a Coulomb/sec is not a very "big" thing. If you put the wires 1 cm apart and put 100 amps in each wire, you gain a factor of 10^6, so force is then 2/10 = 1/5 nt ~ 1 lb. This is easily observable by any dolt.

So in this system, charge is *defined* by the *magnetic* equation. The *electric* equation will then read,

$F(nt) = k1(nt\text{-}m^2/C^2) * q(C)Q(C)/r(m)^2 \qquad = 9 \times 10^9 * q(C)Q(C)/r(m)^2$

since we know from the above work that $k1(nt\text{-}m^2/C^2) = k2 * c\ (m/sec)^2 = 1e\text{-}7*(3e8)^2 = 9 \times 10^9$. This constant k1 is usually expressed as $k1 = 1/(4\pi\varepsilon_0)$ so we can have $D = \varepsilon_0 E$ in free space. As with k2, the constant k1 is given its "natural" dimensions:

$k1(nt\text{-}m^2 / C^2) = 9 \times 10^9 = 1/(4\pi\varepsilon_0) \qquad k1 = 9 \times 10^9\ nt\text{-}m^2/C^2 \quad (= 9 \times 10^9$ meter/farad)

where the last comes from Q=CV defining the **farad**. Notice that $k1/k2 = (3*10^8\ m/sec)^2$.

The electrostatic potential is a derived concept, not a definition. We have dW = Fdx so dW/q = F/q dx which is then E dx. This work per charge is called the potential and is this dV = E dx = (F/q)dx. Thus, our definition of the volt is this:

1 volt ≡ 1 nt-m/C

Now for the particular situation of a point charge, we can integrate E from r to infinity to get

$V(volts) = 9 \times 10^9\ Q(C)/r(m) \qquad / = k1\ Q/r$

where we have used the point charge for our definition. Now E = nt/C from its definition, =volts/m here.

In *electrostatics*, a Coulomb is a very large charge! If you put two of them 1 meter apart, the force between them would be 9 billion newtons, enough to explode any physical holding object to shreds in a real hurry! Another way to say this (see next paragraph) is that an aluminum sphere of radius 1 meter holding 1 Coulomb of charge would have a potential of 9 billion volts!

Side question: why don't the electrons just fly off a charged sphere into the vacuum? According to Purcell page 50, an average electron at the surface "feels" half the field. Perhaps we have 10,000 volts on a practical sphere, so the field just outside is 10,000 volts/meter. But this is a mere 10^{-6} volts/Angstrom. If you pull one extreme surface electron 10 A above the surface, you "gain" about $*10^{-5}$ eV of energy. But you would expect the electron to be bound in the metal by something on the order of 1 eV (see battery discussion earlier). Thus, you would need perhaps 1 billion volts on your sphere (~ Coulomb) before electrons would fly off spontaneously. If there is a conductor outside the sphere, then the "work function" to remove an electron goes away, and you get resistive flow. This same thing happens if the medium outside the sphere ionizes into a plasma due to the E field (spark, lightening bolt, etc).

This system is called "rationalized mks" only because we write $k1 = 9 \times 10^9$ as $1/(4\pi\varepsilon_0)$. Since $4\pi k1$ appears in the Maxwell equation, this gets rid of the 4π floating around.

What about α? Jackson says $\alpha=1$. The place to look for this is in Faraday's law. Bleaney page 256 shows clearly that $\alpha=1$ for that book, and the same for Portis on p 382. Bleaney claims to use the **"rationalized mks"** where Portis uses the **"SI units system"** which *incorporates* rationalized mks. Another term used is **mksa** where a stands for ampere.

The magnetic field formula above is $\mathbf{dF} = (1/\alpha)\ i\mathbf{dx} \times \mathbf{B}$ which in mksa is $\mathbf{dF} = i\mathbf{dx} \times \mathbf{B}$. Thus, B has units which are derived from other quantities

B(nt/amp-m) similar to E(nt/C)

(Side note: if you know about antisymmetric $F^{\mu\upsilon}$ whose 6 non-zero elements are B and E, it sure is uncomfortable to have these fields have different units!)

The volt was defined as above, and we usually use E(volt/m) in common parlance. It was decided by someone that we should have a unit of magnetic flux (not field) called the Weber:

$d\phi$(**Weber**) = B(nt/amp-m)•dA(m^2) so that 1 Weber ≡ 1 nt-m/amp

Then we can think of B(Weber/m^2) in common speech.

The current SI unit system (see http://iserver.asa.edu.py/physicsweb/hisotry_of_units.htm) was adopted in a conference in 1971 . At this time, the Tesla was introduced to give the B field its own private unit. Webers were not thrown out. Since Weber was already defined as above, we can regard the Tesla as a derived unit, or we could formally define it this way

B(**Tesla**) 1 Tesla ≡ 1 Weber/m^2 = 1 nt/amp-m

My Bleaney book is 1965 so does not mention Tesla, but Portis is 1978 and does. Jackson is 1962-7 and also does not mention Tesla. [by the way, Hz was adopted by SI in 1960. My 1956 and 1964 handbooks don't use it. I think common use of Hz phased in around 1966 in the US, was cycles per second.]

So, one more comment. At this point we have defined volts and amperes, so one more obvious unit remains:

R(ohms) = V(volts)/I(amps) 1 **ohm** ≡ 1 volt/amp

There are some other SI units that are a bit new to me:

pascal = 1 nt/m^2 for pressure
siemens = amp/volt to replace mhos for conductance

Magnetization M is interesting. Bleaney page 131 is talking magnetostatics and mentions the idea of a magnetic dipole moment m = I(current)S(area of loop), similar to the electric dipole p = qL. If you talk about a volume density of moments, then you have M = N m/V = > M(current/length) like **Ampere/m**. Since M is related to H, we sometimes see H with units of current/length.

4. The cgs-esu-Gaussian system of units

In this system, the starting point is to set k1 = 1 so that F = qQ/r^2 and this of course makes electrostatics very nice and simple. We regard this equation as defining the esu unit of charge also called a statcoulomb (Purcell calls this an esu and does not mention statcoulomb).

F(dynes) = q(esu)Q(esu)/r(cm)^2 k1(dyne-cm^2/esu^2) = 1 but no explicit dimensions!

Since k1 has no dimensions, we regard this as the formal definition of the esu in terms of base units:

1 dyne = 1 esu^2 / cm^2

Notice that E = F/q is in dynes/esu or below it can be statvolts/cm.

Since we know that $k1(nt\text{-}m^2/C^2) = 9x10^9$ from the mksa system, we consider:

$$9x10^9 = k1(nt\text{-}m^2/C^2) = k1(dyne\text{-}cm^2/esu^2) * \{dyne/nt\}*\{cm/m\}^2*\{C/esu\}^2$$
$$= 1 * e\text{-}5 * e\text{-}4 * \{C/esu\}^2$$

Thus: Coulomb = $3x10^9$ **esu** (statcoulomb)

So the statcoulomb = esu is a much smaller thing than a Coulomb. That 1 meter sphere with an esu on it would only have a potential of about 3 volts, much more manageable!

The potential is going to be dV = Edx = F/q dx. The unit will be the **statvolt** and we have:

dV(statvolts) = F(dynes)/q(esu) * dx(cm) 1 statvolt ≡ 1 dyne-cm/esu

The conversion is therefore:

1 statvolt = 1 dyne-cm/esu = 1 nt-m/C * {dyne/nt}{cm/m}{C/esu}
= 1 volt * e-5 * e-2 * 3e9 = **300 volts**

Certainly we would say that a **statamp** = esu/sec, the unit of current. Then **statohm** = statvolt/statamp. The E field is measured in statvolts/cm.

Now moving to the magnetic equation, we have

$dF/dx = k2*2iI/D$ $k2 = k2(dyne\text{-}sec^2/esu^2) = k1/c^2 = 1/c^2 = 1/(9e20)\ sec^2/cm^2$

Remember that the force between currents was already pretty small even with amperes. Here with a current unit that is 3 billion times smaller, our force is really really tiny! Usually in equations one just shows the c factors as "c", and of course here it is in cgs units. Thus we have

$dF(dyne)/dx(cm) = (1/c(cm/sec))^2 *2i(esu/sec)I(esu/sec)/D(cm)$ // force between wires

As noted in general above, in terms of units this equation also says 1 dyne = 1 esu^2 / cm^2.

What about B and Biot-Savart? Recall that

dF = $(1/\alpha)$ i**dx** x d**B** d**B** = α k2* (Id**X** x **r)**/r^3

Jackson in his Gaussian system (Purcell also) selects α = c so these become

dF = (1/c) i**dx** x d**B** d**B** = (1/c)* (Id**X** x **r)**/r^3

This is nice in that we associate a 1/c factor with each current when we do the split here. The factor of c is taken to have its units with it in the Gaussian system. Thus we get

dF(dynes)= (1/c(cm/sec)) i(esu/sec)**dx**(cm) x d**B(**dynes/esu)

The point here was to show that we have **B(**dynes/esu) in the Gauss system. In fact, this unit is called a "Gauss":

B(**gauss**) = B(dynes/esu) 1 gauss ≡ 1 dyne/esu

Notice that E and B both have the same units in this system, dynes/esu, although they are usually called "gauss" for the B field. The term "oersted" is used for the H field, but it is the same unit as the gauss.

How are gauss and Tesla's related? I see no trivial way to compute this, just have to do brute force:

dF(nt) = i(C/sec)**dx(**m**)** x **B(**tesla)
dF(dynes)= (1/c(cm/sec)) i(esu/sec)**dx**(cm) x **B(**gauss)

Divide these two equations to get:

$$\frac{\mathbf{dF(}\text{nt)}}{\mathbf{dF}\text{(dynes)}} = \frac{\mathbf{i(}\text{C/sec)}}{\mathbf{i}\text{(esu/sec)}} * \frac{\mathbf{dx(}\text{m)}}{\mathbf{dx}\text{(cm)}} * \text{c(cm/sec)} * \frac{\mathbf{B(}\text{tesla)}}{\mathbf{B(}\text{gauss)}} =$$

$$\{\text{dyne/nt}\} = \{\text{esu/C}\} * \{\text{cm/m}\} * \text{c(cm/sec)} * \frac{\mathbf{B(}\text{tesla)}}{\mathbf{B(}\text{gauss)}}$$

$$\text{1e-5} = \text{1/3e9} * \text{1e-2} * \text{3e10} * \frac{\mathbf{B(}\text{tesla)}}{\mathbf{B(}\text{gauss)}}$$

So

$$\frac{\mathbf{B(}\text{tesla)}}{\mathbf{B(}\text{gauss)}} = \text{1e-4} \Rightarrow \text{B(gauss)} = \text{10,000 B(Tesla)} \Rightarrow \mathbf{1\ Tesla = 10{,}000\ gauss}$$

We could also relate the E field units by comparing volts/meter to statvolts/cm. So

E(volts/m) = E(statvolt/cm)* {statvolt/volt} * {m/cm} = E(statvolt/cm)* 300 * 100
= 30,000 E(statvolt/cm)

=> **1 statvolt/cm = 30,000 volts/m**

So in mksa the magnetic field unit is bigger, but the electric field unit is smaller.

What about magnetic flux. I think people used this

1 maxwell = 1 gauss-cm^2 compare to: 1 Weber = Tesla/m^2

Divide and use T = 1e4 gauss to conclude that **1 Weber = 10^8 maxwell.**

There are really no other units that people use in this system. The inside back cover of Purcell gives a good summary of things.

5. The cgs-emu system of units

Now, on to the emu system. Sometimes units here are called "absolute" units, hence the terms like abvolt. We take as our starting point,

dF(dyne)/dx(cm) = k2*2i(abC/sec)I(abC/sec)/D(cm) k2(dyne-sec^2/abC^2) = 1 (dimensionless)

dF(nt)/dx(m) = k2(nt-sec^2 / C^2) * 2i(C/sec)I(C/sec)/D(m) k2(nt-sec^2 / C^2) = 10^{-7} = $\mu_0/4\pi$

We write the second line only so we can do a division to find how abC are related to C:

1e5 = 1e7 * {C/abC}^{2} => **abcoulomb = 10 coulomb**

This factor entirely arises from the nt/dyne shift being offset by the k2 shift. Of course the **abamp** will be an abcoulomb/sec.

Now k1 = k2*c^2 = c^2 = (3e10 cm/sec)^2. This we are going to have in the electrostatic side of things,

F(dynes) = c^2 (cm^2/sec^2) q(abC)Q(abC)/r(cm)^2 and
E(dynes/abC) = c^2 (cm^2/sec^2) Q(abC)/r(cm)^2

The potential will be V = Edx so that

1 **abvolt** = 1 dyne-cm/abC
= 1 nt-m/C * {dyne/nt}*{cm/m}*{C/abC} = 1 volt * 10^{-5} * 10^{-2} * 10^{-1}
= **10^{-8} volt**

Next we come to Biot-Savart and we have to face the α question: (k2=1)

dF = (1/α) i**dx** x d**B** d**B** = α (Id**X** x **r**)/r^3 // 1st is Biot-Savart (4)

If we set α = 1 (as it is in mksa), then we could conclude that B(Tesla) = 10^{-3} B(emu). A factor of 10 comes from abcoulomb to coulomb, and a factor of 100 comes from 1/distance. However, I think no one uses an emu unit like this, I think they use the Gauss from the cgs-esu system instead! This could be fixed by making α = 1/10. Then we would make no distinction between the two cgs systems for B. However, we would than require that **dF** = 10 i**dx** x d**B.** This has to be the case of B = Gauss and i(abC/sec), no way around it! If B is the same in emu as esu, then so is B-flux, the maxwell.

Obviously if we have abamps and abvolts, we are going to have **abohms**.

6. Some History

The emu system is the counterpoint to the esu system because here we are going to take k2 = 1 with no dimensions, whereas in esu we take k1 = 1. What does this system use for α? Was it the historical system used by Ampere et al? Do I know any sources that use this system? Geophysics may have historically used this system. The note below suggests that abvolts were used prior to 1881, so maybe the emu system with cgs was the standard for everything prior to that time!

My guess is that before 1881, everyone doing electric currents work was using these "ab" units and cgs base units, but was just calling them "electromagnetic units". Surely the "ab" for absolute is a retroactive addition from the modern day to distinguish current amps from historical amps which are today's abamps. The claim made below is that **in 1881 abvolt was replaced with volt that was 10^8 times larger and more practical.**

Why was the ampere defined as 1/10 of an abamp (with the result that the Coulomb became 1/10 of the emu unit (abcoulomb))? The reason just comes from P = IV as noted earlier,

1 amp = 1 joule/volt = 10^7 erg / (10^{-8} abvolt) = (1/10) abamp

So the potential unit (volt) determines the current unit (ampere) and thus the charge unit (coulomb). Also:

abohm = abvolt/abamp = 10^{-8} volt/(10 amp)= 10^{-7} ohm

Some quotes of interest from the web:

" First to adopt a customary metric system was France in 1799. Progress towards its adoption elsewhere has been distinctly less rapid now that those opposed to it are no longer decapitated. Anyway, the CGS system enjoyed wide acceptance until 1881 when the then new electricity industry decided that the e.m.u. abvolt was too small for practical use and succeeded in introducing a value 10^8 times larger: our present value for the volt. The ohm and the amp were also re-scaled. "

" The First International Conference of Electricians (Paris, 1881) adopted the British Association definition of the ohm and added definitions for the volt, ampere, coulomb, and farad. Thus was born the "absolute practical system of electrical units":

- absolute, because the units were defined solely in terms of mechanical units (length, mass and time)
- practical, because the sizes were much more convenient than the cgs units.

The ampere was a derived unit, defined as the current produced in a conductor with a 1-ohm resistance when there was a potential difference of 1 volt between its ends"

" In 1948 it was decided that the centimetre, the gram and the erg (the CGS unit of energy) should be replaced by the larger metre, kilogram and joule. This was called the MKSA system and it restored coherence "

More on the order in which things were done (1780 - 1881)

1785 Coulomb's law for static charges
1800 Volta makes batteries, allowing people to have wires with currents in them for the first time
1820 Oersted sees current in wire deflect a compass needle
1820 Arago sees current in wire attract iron filings.
1820 Biot, Savart and Ampere get force on current from B and between two currents
1825 Ampere puts out a summarizing paper
1827 Ohm's law in Die Galvanische Kette, Mathematisch Bearbeitet paper

1874 BAAS adopts cgs
1881 conference also adopts practical units including amp, ohm, volt

During all this time, no one knew about electrons and protons. Charge Q was regarded as some kind of invisible fluid that came out of matter, maybe two fluids. The fluid could flow through a wire, and could be produced with a battery. The idea that the fluid was made of individual particles was unknown until

1897. The connection between light waves and this charge fluid through the Maxwell equations must have seemed very strange. That did not stop electric motors from being built in 1831.

Dalton summarized basic ideas about *atoms* around 1803, they were round solid balls with interesting properties. In 1897 Thomson showed that cathode rays were individual charged "corpuscles" much lighter than atoms, later called electrons, and they were assumed to be pieces of atoms. People made atomic models like the **Plum Pudding Model** with a mixture of + and - charges swimming around in the ball. Only in 1911 did **Rutherford** infer the presence of charged positive atomic nucleus. The fact that light was quantized in hυ was found separately from the photoelectric effect (**Einstein**), and this with the nucleus idea because the wave model for hydrogen of **Bohr** in 1913, and so started the probability idea and quantum mechanics. We now know that E&M is just quantum mechanics for a massless particle. Relativity (again Einstein) came along in 1905 and soon got mixed with QM.

Notes added 9/21/08

The Lorentz Force

What does this look like in terms of our constants like k_1 ? For the electric part, we know that

$$\mathbf{F} = q\mathbf{E}$$

in all unit systems, as was noted above. Our Biot-Savart from above says $\mathbf{dF} = (1/\alpha)\, i\mathbf{dx} \times d\mathbf{B}$. For a point charge q moving at velocity **v** we would say that $i\, d\mathbf{x} = q\,\mathbf{v}$ (consider q spread out in a little box of thickness dx, where charge moves in x direction, and dA area). Notice that in i dx = q v there are no questions about units, both sides are charge-L/T. So then $\mathbf{dF} = (1/\alpha)\, q\,\mathbf{v} \times d\mathbf{B}$. So this tells us that

$$\mathbf{F} = q[\,\mathbf{E} + (1/\alpha)\,\mathbf{v} \times \mathbf{B}\,] \qquad \text{// the Lorentz force}$$

According to Jackson page 616, in the 5 unit systems he lists the only values that appear for α are 1 or c.

Relativistic Units

In relativistic work, we like to use A^μ as the 4-potential so we would certainly like $V = \varphi = A^0$ and **A** to have the same units! We also want to have $\mathbf{B} = \nabla \times \mathbf{A}$ and $E = -\nabla V$. What conditions does this impose on our various unit constants like k_1 ? First, we see from the above that if V and A have the same units,

$$\dim(E) = \dim(V)/L$$
$$\dim(B) = \dim(V)/L$$

so we are certainly happy with this trivial outcome. It is consistent with $\mathbf{F} = q\,(\mathbf{E} + (\mathbf{v}/c) \times \mathbf{B})$, so we certainly want to select α = c and not α = 1. This single choice of α makes us happy with regard to the units of E and B being the same, and thus the units of V and **A** will be the same. We still then have complete freedom to choose k_1 and k_2 but of course we still must have $k_1/k_2 = c^2$.

Fine Structure Constant

Wiki comments on the various ways to interpret this thing, always called α and not to be confused with the α of our discussion above. I like this one: it is the ratio of two energies. One energy is that potential energy of two electrons separated by distance s, and the other is the energy of a photon which has a wavelength of 2πs. The 2π causes $\hbar$ instead of h to appear as shown below.

(energy of two electrons separated by distance s) = E_1 = Ve = k_1e^2/s .

energy of photon of wavelength $\lambda = 2\pi s$: First, $c = \lambda/T = 2\pi s\, \nu \Rightarrow \nu = c/(2\pi s)$

$\Rightarrow E_2 = h\nu = hc/(2\pi s) = \hbar c/s$

Then we have

$\alpha = E_1/E_2 = k_1e^2/s * s/(\hbar c) = k_1e^2/(\hbar c)$ = dimensionless = 1/137

Notice from $F = k_1 * qQ/r^2$ that the combination $(k_1 * qQ)$ has units of force x L^2 and is therefore independent of how you define charge.

Units used in Bjorken and Drell volume 1 page 100 on Coulomb Scattering

Since they are setting $\hbar = c = 1$, it is a little hard to tell what they are doing, but I think we can figure it out. Looking at p 100 equation A, it appears that they are using $k_1 = 1/(4\pi)$. This implies that $k_2 = 1/(4\pi c^2)$. As noted above, as long as α = c (our alpha), units of E and B are same, and units of V and A are same, so we still have the k_1 freedom. With this choice of k_1 and α, we have Maxwell's equations as follows (from above),

div**E** = ρ curl **E** = - (1/c) ∂**B**/∂t (Faraday) $k1 = c^2 \bullet k2$

div **B** = 0 curl **B** = (1/c) **J** + (1/c) ∂**E**/∂t

where we used these facts:

$4\pi\, \alpha\, k_2 = 4\pi\, c\, /(4\pi c^2) = 1/c$

$\alpha\, k2/k1 = c\, /c^2 = 1/c$

According to Jackson page 616, this system of units is called the Heaviside-Lorentz System. If we had chosen $k_1 = 1$, our charge unit would be the "esu" = statcoulomb, where from above

Coulomb = $3x10^9$ **esu** (statcoulomb)

If we compare two systems which have the same non-EM units, then we know that k_1q^2 = constant, so we can then say

$q_{HL}^2/4\pi = q_{esu}^2$ or $q_{HL} = \sqrt{4\pi}\ q_{esu}$

Thus, a HL charge unit is about 3.5 times larger than an esu. So, when reading BD, we need to realize that the electron charge "e" is in this EH units. Also, they would have

$$\alpha = k_1 e^2/(\hbar c) = e^2/(4\pi\hbar c) = e^2/(4\pi) = e_{HL}^2/(4\pi)$$

where in the last instance we assume that $\hbar = c = 1$.

About $\hbar = c = 1$

Recall that $\hbar$ has units of angular momentum which is $\ell^2 m/t$ while c of course is just ℓ/t. So we have three units to play with: ℓ, m and t. Suppose we choose t = 1 nsec = 1 tick. Then we would have ℓ = 3 meters in order to get c to be correct, approximately, so call this 1 door. The speed of light is then 1 door/tick.

Meanwhile, we know that $\hbar$ = 1.055 Joule-sec = 1.055 m^2 kg / sec. We want to define a mass unit to make $\hbar$ come out being one. Call this unit the speck. Then

$$\hbar(\text{door}^2 \text{ joe/tick}) = \hbar(\text{m}^2 \text{ kg / sec}) \{ \text{m /door} \}^2 \{\text{kg / speck}\} \{\text{tick/sec}\}$$

$$= 1.055 \{ \text{m / 3m} \}^2 \{\text{kg / speck}\} \{10^{-9} \text{ sec/sec}\} = 1.055 \times 10^{-9} / 3^2 * \{\text{kg / speck}\}$$

Therefore, to make this q, we need

$$\{ \text{speck /kg}\} = 1.055 \times 10^{-9} / 3^2 \quad \text{meaning}$$

$$1 \text{ speck} = 1.055 \times 10^{-9} / 3^2 \text{ kg} = \sim 0.117 \times 10^{-9} \text{ kg} = \sim 0.117 \times 10^{-9} \ (10^3 \text{ g}) = .117 \ \mu\text{g}$$

So this would be something like a pollen grain or dust particle, but not like an electron or proton!