Mie Scattering and the M and N functions PhL 2.14.03

These started out being notes on my little Mie Theory 4-page downloaded paper which happens to state all the coefficients in formulas that look like mine.

Review. We have seen how the functions **X**, **Y** and **Z** span the ℓm manifold. We have also seen how one solves an E&M problem separately in the two "modes" known as TM and TE. Each mode has its own private set of equations and operates independently. In either Jackson or Carleton's notes we have this basic idea:

TM: $\mathbf{B}_{\ell m} = a_E(\ell m)\ f_\ell(r)\ \mathbf{X}_{\ell m}(\theta,\phi)$
$\mathbf{E}_{\ell m} = (i/k)\ \nabla \times \mathbf{B}_{\ell m}$

TE: $\mathbf{E}_{\ell m} = a_M(\ell m)\ g_\ell(r)\ \mathbf{X}_{\ell m}(\theta,\phi)$
$\mathbf{B}_{\ell m} = (-i/k)\ \nabla \times \mathbf{E}_{\ell m}$

That is to say, for each ℓm, we can have the two independent solutions noted above.

The quantities $f_\ell(r)$ and $g_\ell(r)$ each represent a linear combination of spherical Bessel functions. The two coefficients of this linear combination are dependent on ℓ and must be determined from boundary conditions. Usually they are normalized so the sum of their squares is 1, but this is not necessary, since the $a_E(\ell m)$ type coefficient can pick up the slack. The asymptotic behaviors of $j_\ell(kr)$ and $h^{(1)}_\ell(kr)$ are especially useful in meeting these boundary conditions, the first being finite at r=0, the second having outgoing spherical wave behavior.

The normalization of the function $\mathbf{X}_{\ell m}(\theta,\phi)$ varies with author, but always has this form

$$\mathbf{X}_{\ell m}(\theta,\phi) = C(\ell,m) * \mathbf{r} \times \nabla Y_{\ell m}(\theta,\phi)$$

$C(\ell,m) =$	$+1$	Carleton
	-1	Mie
	$\frac{-i}{\sqrt{\ell(\ell+1)}}$	Jackson

When we write **Z** and **Y**, we always mean Carleton's normalization for these two basis functions. It would seem that the **X** vector basis function, one of the two transverse ones, is somehow more "fundamental" than the other two, since it provides the "main field" in each of the modes. The "other" field that goes with each mode is found from the $\nabla \times$ operation as shown, and we can write out:

$$\nabla \times [f_\ell(r)\ \mathbf{X}_{\ell m}(\theta,\phi)] = -C(\ell,m)\ \{\ (f'_\ell(r) + 1/r)\ \mathbf{Z}_{\ell m}(\theta,\phi) + \ell(\ell+1)\ f_\ell(r)/r\ \mathbf{Y}_{\ell m}(\theta,\phi)\ \}$$

Thus, as we well know, the "other field" in a given (TM or TE, transverse) mode has components in both the **Z** (other transverse) and the **Y** (radial) direction. We can rewrite the above vector expansion as follows:

$$\nabla \times [f_\ell(r)\ \mathbf{X}_{\ell m}(\theta,\phi)] = f'_\ell(r)\ \hat{\mathbf{r}} \times \mathbf{X}_{\ell m}(\theta,\phi) - C(\ell,m)\ \ell(\ell+1)\ f_\ell(r)/r\ \hat{\mathbf{r}}\ Y_{\ell m}(\theta,\phi)$$

where we have kept it general so that anybody's **X** can be used here. This appears as 16.143 in Jackson where his C(ℓ,m) is as shown above. To do this, we used the fact (from our appendix in other paper) that

$$\mathbf{Z} = -\,\hat{\mathbf{r}} \times \mathbf{X}_{\text{Carleton}} = -\,(1/\,C(\ell,m))\,\hat{\mathbf{r}} \times \mathbf{X}$$

Now when we write the full expansion over all the modes, we get this result:

$$\mathbf{B} = \Sigma_{\ell m}\,[\;\; a_E(\ell m)\;\; f_\ell(r)\;\mathbf{X}_{\ell\mathbf{m}}(\theta,\phi) \;+\; (-\,i/k)\,\nabla \times \{\, a_M(\ell m)\;\; g_\ell(r)\;\mathbf{X}_{\ell\mathbf{m}}(\theta,\phi)\}\;\;]$$

$$\mathbf{E} = \Sigma_{\ell m}\,[\;\; (i/k)\,\nabla \times \{\, a_E(\ell m)\;\; f_\ell(r)\;\mathbf{X}_{\ell\mathbf{m}}(\theta,\phi)\} \;+\; a_M(\ell m)\;\; g_\ell(r)\;\mathbf{X}_{\ell\mathbf{m}}(\theta,\phi)\;\;]$$

and this appears as the "grand finale" 16.47 in Jackson, which I now see is not all that grand. It is just the statement of what we are saying all along in the VSH expansion. The game is to determine the two coefficients $a_E(\ell m)$ and $a_M(\ell m)$ as well as the spherical Bessel implied linear combination coefficients for each ℓm. Notice that in the form above, there is no immediate need to expand the secondary terms in either way shown above, but we know we can do it.

The Mie basis functions M and N. I call them Mie because I suspect they appeared in his 1908 paper in this form. In my little 4-page downloaded Mie Theory paper we see the following definition:

$$\mathbf{M}_{\ell\mathbf{m}} = \nabla \times [f_\ell(r)\,Y_{\ell m}(\theta,\phi)\,\mathbf{r}\,] = \nabla \times [\,(f_\ell(r)/r\,)\;\mathbf{Y}_{\ell\mathbf{m}}(\theta,\phi)\,]$$

Warning: The $Y_{\ell m}(\theta,\phi)$ which appear in this Mie paper may have different normalization than we are used to, but we will absorb any difference into our linear combination of spherical Bessel coefficients, which are already implied when we write the generic $f_\ell(r)$

From the first form, we can quickly simplify as follows:

$$\nabla \times [\,f_\ell(r)\,Y_{\ell m}(\theta,\phi)\,\mathbf{r}\,] = \nabla\,[\,f_\ell(r)\,Y_{\ell m}(\theta,\phi)] \times \mathbf{r}$$

because the second term has $\nabla \times r$ which is 0. In the next step, can see that the

$$\nabla\,[\,f_\ell(r)\,Y_{\ell m}(\theta,\phi)] = f_\ell(r)\,\nabla\,Y_{\ell m}(\theta,\phi) + f'_\ell(r)\,\hat{\mathbf{r}}\;\;Y_{\ell m}(\theta,\phi)$$

so the second term gives nothing since $\hat{\mathbf{r}} \times \mathbf{r} = 0$. So we end up with

$$\begin{aligned}\mathbf{M}_{\ell\mathbf{m}}(r,\theta,\phi) = \nabla \times [f_\ell(r)\,Y_{\ell m}(\theta,\phi)\,\mathbf{r}\,] &= -\,f_\ell(r)\;\;\mathbf{r} \times \nabla\,Y_{\ell m}(\theta,\phi)\\ &= -\,f_\ell(r)\,\mathbf{X}_{\ell\mathbf{m}}(\theta,\phi)/C(\ell,m) \quad \text{// our old friend}\end{aligned}$$

Thus, we can write our exact same multipole fields as

TM: $\mathbf{B}_{\ell\mathbf{m}} = -\,a_E(\ell m)/C(\ell,m)\;\;\mathbf{M}_{\ell\mathbf{m}}(r,\theta,\phi)$
$\mathbf{E}_{\ell\mathbf{m}} = (i/k)\,\nabla \times \mathbf{B}$

TE: $\mathbf{E}_{\ell\mathbf{m}} = -\,a_M(\ell m)/C(\ell,m)\;\;\mathbf{M}_{\ell\mathbf{m}}(r,\theta,\phi)$

$$\mathbf{B}_{\ell m} = (-i/k)\, \nabla \times \mathbf{B}$$

At this point, why not define

$$\mathbf{N}_{\ell m}(r,\theta,\phi) = (1/k)\, \nabla \times \mathbf{M}_{\ell m}(r,\theta,\phi) = (1/k)\, \nabla \times \nabla \times [f_\ell(r)\, Y_{\ell m}(\theta,\phi)\, \mathbf{r}\,]$$

Then we can be even more compact and say

TM:
$$\mathbf{B}_{\ell m} = [\, - a_E(\ell m)/C(\ell,m)]\ \mathbf{M}_{\ell mE}(r,\theta,\phi)$$
$$\mathbf{E}_{\ell m} = i\,[\, - a_E(\ell m)/C(\ell,m)]\ \mathbf{N}_{\ell mE}(r,\theta,\phi)$$

TE:
$$\mathbf{E}_{\ell m} = [\, - a_M(\ell m)/C(\ell,m)]\ \mathbf{M}_{\ell mM}(r,\theta,\phi)$$
$$\mathbf{B}_{\ell m} = -i\,[\, - a_M(\ell m)/C(\ell,m)]\ \mathbf{N}_{\ell mM}(r,\theta,\phi)$$

Here I have added E (electric) and M (magnetic) labels on the M and N functions to make us remember that each one can have a different spherical Bessel function. That is, we might write

$$\mathbf{M}_{\ell mE}(r,\theta,\phi) = \nabla \times [f_\ell(r)\, Y_{\ell m}(\theta,\phi)\, \mathbf{r}\,]$$
$$\mathbf{M}_{\ell mM}(r,\theta,\phi) = \nabla \times [g_\ell(r)\, Y_{\ell m}(\theta,\phi)\, \mathbf{r}\,]$$

Now let's rescale our coefficients in the obvious manner $c_E(\ell m) = - a_E(\ell m)/C(\ell,m)$ to get this final form

TM:
$$\mathbf{B}_{\ell m} = c_E(\ell m)\ \mathbf{M}_{\ell mE}(r,\theta,\phi) = c_E\, \mathbf{M}_{\ell mE}$$
$$\mathbf{E}_{\ell m} = i\, c_E(\ell m)\ \mathbf{N}_{\ell mE}(r,\theta,\phi) = i\, c_E\, \mathbf{N}_{\ell mE}$$

TE:
$$\mathbf{E}_{\ell m} = c_M(\ell m)\ \mathbf{M}_{\ell mM}(r,\theta,\phi) = c_M\, \mathbf{M}_{\ell mM}$$
$$\mathbf{B}_{\ell m} = -i\, c_M(\ell m)\ \mathbf{N}_{\ell mM}(r,\theta,\phi) = -i\, c_M\, \mathbf{N}_{\ell mM}$$

where we are trying to achieve a goal of very compact notation. At this point we can write our full expansion as

$$\mathbf{B} = \Sigma_{\ell m}\, [\ c_E\, \mathbf{M}_{\ell mE} - i\, c_E\, \mathbf{N}_{\ell mE}\ ]$$

$$\mathbf{E} = \Sigma_{\ell m}\, [\ i\, c_E\, \mathbf{N}_{\ell mE} + c_M\, \mathbf{M}_{\ell mM}\ ]$$

which I have to admit is pretty compact.

Now the paper I have has a different notation for things:

them		me
even e	=>	a special meaning not discussed yet! (electric)
odd o	=>	a special meaning not discussed yet! (magnetic, see below)
n	=	ℓ , the main index on the spherical harmonics
ℓ	=	m, the minor index
m	=	n, the index of refraction (they call this the "optical constant")
ka	=	x, known as "the size parameter"

The superscript on the N and M functions relates to the choice of Bessel function. The functions have label 3 (which would be a Hankel 1) and field has subscript s. The "internal" wave (inside the dielectric) has label 1 (which would be j_ℓ) and subscript t (transmitted), while the plane wave expression also has the internal 1 label and subscript i. The three labels are correct. You can see that the coefficients in (1.4) and (1.5) are defined in a peculiar manner, perhaps as Mie did things.

Status: So, at this point I am happy about the expansions (1.4) for the scattered wave, and (1.5) for the transmitted or internal wave. My big problem is (1.6) for the plane wave incident. I know from Jackson that the simplest forms for the plane wave are with circular polarizations, and we have

$$\mathbf{E}_{\pm} = \Sigma_\ell \ (i)^\ell \sqrt{4\pi(2\ell+1)} \quad [\quad j_\ell(kr)\, \mathbf{X}_{\ell,\pm 1} \quad \pm \quad (1/k)\, \nabla \times \{ j_\ell(kr)\, \mathbf{X}_{\ell,\pm 1} \} \ ]$$
$$\mathbf{B}_{\pm} = \Sigma_\ell \ (i)^\ell \sqrt{4\pi(2\ell+1)} \quad [\mp \ i\, j_\ell(kr)\, \mathbf{X}_{\ell,\pm 1} \quad - \quad (i/k)\, \nabla \times \{ j_\ell(kr)\, \mathbf{X}_{\ell,\pm 1} \} \ ]$$

For each partial wave ℓ, we know that the m = ±1 are involved. The formula (1.6) does not show this detail. What is most annoying is that the author simple makes no comment whatsoever about polarization, as if it did not even exist.

Aside: What would happen if we averaged over the two polarizations for the intensity? We could think about the Poynting vector time averaged which would be

$$\langle S_{\pm} \rangle = c/8\pi \ \mathbf{E}_{\pm} \times \mathbf{B}_{\pm}^{*}$$

Because of the *, there will be non-vanishing cross terms. Rewrite as

$$\mathbf{E}_{\pm} = \Sigma_\ell \ (i)^\ell \sqrt{4\pi(2\ell+1)} \quad [\quad j_\ell(kr)\, \mathbf{X}_{\ell,\pm 1}$$
$$\pm \ (1/k) \ \{ - C(\ell,\pm 1) \ \{ \quad (f'_\ell(r) + 1/r)\, \mathbf{Z}_{\ell \pm 1}(\theta,\phi) \ + \ \ell(\ell+1)\, f_\ell(r)/r\, \mathbf{Y}_{\ell \pm 1}(\theta,\phi) \quad \} \ \} \ ]$$

$$\mathbf{B}^{*}_{\pm} = \Sigma_\ell \ (i)^\ell \sqrt{4\pi(2\ell+1)} \quad [\mp \ i\, j_\ell(kr)\, \mathbf{X}^{*}_{\ell,\pm 1}$$
$$- \quad (i/k) \ \{ - C(\ell,\pm 1) \ \{ \quad (f'_\ell(r) + 1/r)\, \mathbf{Z}^{*}_{\ell \pm 1}(\theta,\phi) \ + \ \ell(\ell+1)\, f_\ell(r)/r\, \mathbf{Y}^{*}_{\ell \pm 1}(\theta,\phi) \quad \} \ \} \ ]$$

It is the X-Z cross terms that survive as in Carleton (7), along with the diagonal terms. I see no trivial argument that says you can somehow replace $\mathbf{X}_{\ell,\pm 1}$ with $\mathbf{X}_{\ell,0}$ in any kind of average sense.

So I have now (momentarily)" lost the faith" in this paper due to the unsupported expansion (1.6). This makes me not understand the quoted Mie coefficients, although they do look very much like the ones I got. I did find that the coefficients in fact did not depend on the polarization state, something that seemed strange at the time. So for the moment, I will not regard this as a confirmation source for my coefficients, but will instead learn some generalities about Mie scattering!

Some new realizations. For given index and a parameters, there will be values of k for which the coefficients blow up -- **poles**! I did not think this could happen. Perhaps the numerators compensate so things don't go infinite, but these poles at least tell you where there will be lots of large field action! In experiments with a laser, you let droplets evaporate so radius a decreases, and then you can pass through **resonances**. No doubt these are the resonances of the dielectric sphere just as we had in the case of the conducting cavity! The Q values quoted are finite, even for lossless dielectrics, so the numerators must

be providing cancellation. (?) Author states that you can generate fields near the object that are a million times stronger than the incident field, due to the resonance effects!

This paper is no doubt correct, but it is not the right place to learn this subject. I did manage to learn a few things from it, however, such as the N and M basis functions. Recall that there are whole BOOKS just on the subject of this kind of scattering!

I notice on the web discussion of s and p polarization states!

Fact: I cannot find a real Mie derivation on the web! I think it will be in my new book arriving soon. Otherwise I will have to go to Marriott do dig something up on this subject! // I did just that! I did find one little paper that had similar M and N notation, and as I was just giving up and storing PDF documents away, I looked in the krugal one and found how that plane wave expansion works!

Plane Wave Expansion Explained. See details in Appendix A, but this is the basic idea. Write down Jackson's exact result 16.139. Then take the appropriate linear combination of his results that corresponds to $\mathbf{E} = (\mathbf{E}_+ + \mathbf{E}_-)/2 = \hat{\mathbf{x}}\, e^{ikz}$. His plane wave formulas then read,

$$\mathbf{E} = \Sigma_\ell \; (1/2)(i)^\ell \sqrt{4\pi(2\ell+1)} \quad [\quad j_\ell(kr)\,[\,\mathbf{X}_{\ell,+1} + \mathbf{X}_{\ell,-1}] \;+\; (1/k)\,\nabla \times \{\, j_\ell(kr)\,[\,\mathbf{X}_{\ell,+1} - \mathbf{X}_{\ell,-1}]\,\}\,]$$
$$\mathbf{B} = \Sigma_\ell \; (1/2)\,(i)^\ell \sqrt{4\pi(2\ell+1)} \quad [\; -\, i\, j_\ell(kr)\,[\,\mathbf{X}_{\ell,+1} - \mathbf{X}_{\ell,-1}] \;-\; (i/k)\,\nabla \times \{\, j_\ell(kr)\,[\,\mathbf{X}_{\ell,+1} + \mathbf{X}_{\ell,-1}]\,\}\,]$$

Then make the following *definitions* for o = odd, and e = even:

$$\mathbf{M}_{o1\ell} \equiv j_\ell(kr)\;\; \mathbf{r} \times \nabla\psi_{o1\ell} \qquad \text{where} \qquad \psi_{o1\ell} \equiv \; P_\ell^1(\cos\theta)\,\sin\phi$$
$$\mathbf{M}_{e1\ell} \equiv j_\ell(kr)\;\; \mathbf{r} \times \nabla\psi_{e1\ell} \qquad \text{where} \qquad \psi_{e1\ell} \equiv \; P_\ell^1(\cos\theta)\,\cos\phi$$

(note that $\sin\phi$ is "odd" in ϕ) with corresponding formulas for the **N** functions

$$\mathbf{N}_{o1\ell} \equiv (1/k)\,\nabla \times \mathbf{M}_{o1\ell} \;= (1/k)\,\nabla \times [\,j_\ell(kr)\;\; \mathbf{r} \times \nabla\psi_{o1\ell}\;]$$
$$\mathbf{N}_{e1\ell} \equiv (1/k)\,\nabla \times \mathbf{M}_{e1\ell} \;= (1/k)\,\nabla \times [\,j_\ell(kr)\;\; \mathbf{r} \times \nabla\psi_{e1\ell}\;]$$

The vector form of M is the same as above, but we have something *other than* $Y_{\ell m}$ as the pilot function. It is in fact the sum of two $Y_{\ell m}$'s of opposite m values, allowing us to make this replacement in Jackson's formula,

$$[\, Y_{\ell,+1}(\theta,\phi) + Y_{\ell,-1}(\theta,\phi)] \;=\; 2\,i \sqrt{\frac{2\ell+1}{4\pi\ell(\ell+1)}}\;\; \psi_{o1\ell}$$

$$[\, Y_{\ell,+1}(\theta,\phi) - Y_{\ell,-1}(\theta,\phi)] \;=\; 2 \sqrt{\frac{2\ell+1}{4\pi\ell(\ell+1)}}\;\; \psi_{e1\ell}$$

Then Jackson's plane wave formulas shown above boil down to this:

$$\mathbf{E}_p \;=\; \hat{\mathbf{x}}\; e^{\,ikz} = \;\; \Sigma_\ell\, i^\ell \frac{2\ell+1}{\ell(\ell+1)} [\; \mathbf{M}^{(1)}_{o1\ell} \;-\; i\, \mathbf{N}^{(1)}_{e1\ell}\;]$$

$$\mathbf{B}_p \;=\; \hat{\mathbf{y}}\; e^{\,ikz} = \;\; \Sigma_\ell\, i^\ell \frac{2\ell+1}{\ell(\ell+1)} [\; -\, \mathbf{M}^{(1)}_{e1\ell} \;-\; i\;\; \mathbf{N}^{(1)}_{o1\ell}\;]$$

In the B equation, the first term is a (- i) from the Jackson formula, but then we divide by i because the difference of Y's does not have the i that the sum did have, and we change o to e as well. For the *second* term, we get -i times the N function that matches the first term in the E equation. // I now have a general rule for doing B from E in any expansion: cross mult coeff's by -i and steal the nature.

This result agrees with (1.6) in all respects except one: in the derivation above, we clearly see that the m-index should have the value 1, as shown in the subscript labels above. But in (1.6), this subscript appears as the general m (which they happen to call ℓ). This is an error in (1.6)! In the Krugal PDF paper, the above formula appears exactly as I have stated it as equation 2.44. The superscript (1) implies that we are using the j_ℓ radial functions.

Recall that Jackson's "first term" in the E equation 16.139 is the "magnetic" or TE term, and this is still true when we combine the two helicity solutions. Thus, we can associate "magnetic" with "odd", and of course "electric" with "even". That is a nice label to have on the M and N functions. So if the first term is odd, we always expect the second term to be odd.

Aside on Krugel: I stumbled onto the contents and first three chapters on-line of the following book:

E. Krugel, *The Physics of Interstellar Dust*, 12/02 $135 Amazon

He is the one who cleared up the above plane wave mystery. Krugel is at the Max Planck Institute (MPI) for Radio Astronomy in Bonn. I notice that the PDF documents cannot be printed, nor can anything be selected for cut and paste!

Attempt to set up the scattering problem in the N and M world.

First, let's look back at our earlier derivation where I wrote down the *four* boundary conditions. My two curl conditions implied matching of tangential field components. The r x B condition is problematical for a conductor because there really is a surface current. However, If you use H instead, I guess H does not see the surface current, the way D does not see surface charge. So I guess I accept Krugel's two r x boundary conditions in the general case of arbitrary isotropic μ and ε, which is what he is talking about. Krugel then notes that each of these r x boundary conditions implie 2 equations if you use the $\hat{\boldsymbol{\phi}}$ and $\hat{\boldsymbol{\theta}}$ directions for tangentialness. He says you then get four equations for the four unknown coefficients and you can then solve.

When I did this problem, I was indeed finding redundancy with my boundary conditions. So I guess the answer is that the two r x conditions are all you need! Here is Krugel's setup of the problem:

$$\mathbf{E_p} = \Sigma_\ell E_\ell \left[\mathbf{M}^{(1)}_{o1\ell} - i\,\mathbf{N}^{(1)}_{e1\ell} \right] \qquad E_\ell = i^\ell \frac{2\ell+1}{\ell(\ell+1)} \qquad \text{// incident plane wave}$$

$$\mathbf{E_s} = \Sigma_\ell E_\ell \left[- b_\ell \mathbf{M}^{(3)}_{o1\ell} + a_\ell \mathbf{N}^{(3)}_{e1\ell} \right] \qquad \text{// incident plane wave, (3) means } h^{(1)}_\ell(kr)$$

$$\mathbf{E_i} = \Sigma_\ell E_\ell \left[c_\ell \mathbf{M}^{(1)}_{o1\ell} - i\,d_\ell \mathbf{N}^{(1)}_{e1\ell} \right] \qquad \text{// incident plane wave, (1) means } j_\ell(kr)$$

I presume the expressions for the **H** fields are the same as the above, except I expect to see the even and odd indices switched. This is because the "first term" in Jackson's B fields are always going to be the "other kind" from what you see in the E field.

What is the general rule for converting from an E equation to a B equation? I did it for the plane wave above:

(1) negate the first term's sign, and replace odd with even

(2) keep the second term's sign and replace even with odd.

So applying these rules to the above,

$$\mathbf{H_p} = \Sigma_\ell H_\ell \left[- \mathbf{M}^{(1)}_{e1\ell} - i \mathbf{N}^{(1)}_{o1\ell} \right] \qquad H_\ell = i^\ell \frac{2\ell+1}{\ell(\ell+1)}$$

$$\mathbf{H_s} = \Sigma_\ell H_\ell \left[b_\ell \mathbf{M}^{(3)}_{e1\ell} + a_\ell \mathbf{N}^{(3)}_{o1\ell} \right]$$

$$\mathbf{H_i} = \Sigma_\ell H_\ell \left[- c_\ell \mathbf{M}^{(1)}_{e1\ell} - i d_\ell \mathbf{N}^{(1)}_{o1\ell} \right]$$

Now the boundary conditions are

$$H_{i\theta} = H_{s\theta} + H_{p\theta} \qquad r \times H = 0$$
$$H_{i\phi} = H_{s\phi} + H_{p\phi}$$

$$E_{i\theta} = E_{s\theta} + E_{p\theta} \qquad r \times E = 0$$
$$E_{i\phi} = E_{s\phi} + E_{p\phi}$$

This wil be continued in another paper!

Appendix A. Trying to get a simple <u>plane wave</u> expansion in terms of the N and M functions

Attempt #1 (not successful)

Let's just try this as an exercise in the VSH X,Y,Z that we think we know about. We start with

$$\mathbf{E} = e^{ikz} \hat{\mathbf{x}}$$

Our full expansions are written,

$$\mathbf{B} = \Sigma_{\ell m} \left[a_E(\ell m) f_\ell(r) \mathbf{X}_{\ell\mathbf{m}}(\theta,\phi) + (-i/k) a_M(\ell m) \nabla \times \{ g_\ell(r) \mathbf{X}_{\ell\mathbf{m}}(\theta,\phi)\} \right]$$

$$\mathbf{E} = \Sigma_{\ell m} \left[(i/k) a_E(\ell m) \nabla \times \{ f_\ell(r) \mathbf{X}_{\ell\mathbf{m}}(\theta,\phi)\} + a_M(\ell m) g_\ell(r) \mathbf{X}_{\ell\mathbf{m}}(\theta,\phi) \right]$$

but try to shorten it by writing,

$$\mathbf{B} = \Sigma_{\ell m} \left[\quad a_E \, f \, \mathbf{X} \quad + (-i/k)\, a_M \{ \nabla \times (g \mathbf{X}) \} \right]$$

$$\mathbf{E} = \Sigma_{\ell m} \left[(i/k)\, a_E \{ \nabla \times (f \mathbf{X}) \} + a_M \, g \mathbf{X} \right]$$

Use the fact that (again in compact notation just so we know where factors are sitting!)

$$\nabla \times (f \mathbf{X}) = - C [(f' + 1/r) \mathbf{Z} + \ell(\ell+1) f/r \mathbf{Y}]$$

and we then have

$$\mathbf{B} = \Sigma_{\ell m} \left[\quad a_E \, f \, \mathbf{X} \quad + (-i/k)\, a_M \{ - C [(g' + 1/r) \mathbf{Z} + \ell(\ell+1) g/r \mathbf{Y}] \} \right]$$

$$\mathbf{E} = \Sigma_{\ell m} \left[(i/k)\, a_E \{ - C [(f' + 1/r) \mathbf{Z} + \ell(\ell+1) f/r \mathbf{Y}] \} + a_M \, g \mathbf{X} \right]$$

which we then group by term as

$$\mathbf{B} = \Sigma_{\ell m} \left[\{ a_E \, f \} \mathbf{X} + \{ (i/k)\, C\, a_M \, \ell(\ell+1)\, g/r \} \mathbf{Y} + \{ (i/k)\, C \, a_M (g' + 1/r) \} \mathbf{Z} \right]$$

$$\mathbf{E} = \Sigma_{\ell m} \left[\{ a_M \, g \} \mathbf{X} - \{ (i/k)\, C\, a_E \, \ell(\ell+1) \, f/r \} \mathbf{Y} - \{ (i/k)\, C \, a_E (f' + 1/r) \} \mathbf{Z} \right]$$

In the lower equation, we swap the E and M labels, the f and g functions, AND the secondary terms have an extra minus sign.

Now let's try to compute the three projections as in (11) - (13). Start with

$$E^2_{\ell m} = [1/\ell(\ell+1)] \int d\Omega \, \mathbf{X}^*_{\ell m}/C \bullet \mathbf{E} = [1/C \, \ell(\ell+1)] \int d\Omega \, \mathbf{X}^*_{\ell m} \bullet \mathbf{E}$$

where in the last form we mean Carleton's X. For the E field we shall use

$$\mathbf{E} = e^{ikz} \, \hat{\mathbf{x}} = \hat{\mathbf{x}} \, \Sigma_{\ell'} (i)^{\ell'} \sqrt{4\pi(2\ell'+1)}\, j_{\ell'}(kr)\, Y_{\ell',0}(\theta,\phi) \qquad \text{// Jackson 16.129}$$

Installing this gives,

$$E^2_{\ell m} = [1/\ell(\ell+1)] \, \Sigma_{\ell'} (i)^{\ell'} \sqrt{4\pi(2\ell'+1)}\, j_{\ell'}(kr) \int d\Omega \, Y_{\ell',0}(\theta,\phi) \, [\mathbf{X}^*_{\ell m}(\theta,\phi) \bullet \hat{\mathbf{x}}]$$

At this point, we need to go off an do an overdue piece of work, which is to relate the Cartesian unit vectors to the X,Y,Z vectors. As shown in Appendix B we have

$$[\mathbf{X}^*_{\ell m}(\theta,\phi) \bullet \hat{\mathbf{x}}] = \left[-S\phi \, \frac{\partial Y^*_{\ell m}}{\partial \theta} - (C_\theta C_\phi / r\, S_\theta) \frac{\partial Y^*_{\ell m}}{\partial \phi} \right]$$

which makes for a fairly unpleasant angular integral,

$$E^2_{\ell m} = \left[\ 1/\ell(\ell+1)\ \right] \Sigma_{\ell'} (i)^{\ell'} \sqrt{4\pi(2\ell'+1)}\, j_{\ell'}(kr) \int d\Omega\, Y_{\ell',0}(\theta,\phi) \left[-S\phi \ \frac{\partial Y^*_{\ell m}}{\partial \theta} \ - (C_\theta C_\phi / r\, S_\theta) \frac{\partial Y^*_{\ell m}}{\partial \phi} \ \right]$$

But now what do we do? Notice that we cannot conclude that m=0 due to the other ϕ factors present. This shows the wisdom of Jackson's approach using the raising and lowering operators on page 568. Although I cannot trivially do the integral shown, I think I can conclude that whatever it is, it is a function of ℓ,ℓ' and m. I suspect the integral will vanish unless m = -1, 0, or +1 based on my intuitive angular momentum addition notions. Also it will vanish unless ℓ' = ℓ-1, ℓ or ℓ+1 But for the moment let's write:

$$E^2_{\ell m} = \left[\ 1/\ell(\ell+1)\ \right] \Sigma_{\ell'} (i)^{\ell'} \sqrt{4\pi(2\ell'+1)}\, j_{\ell'}(kr)\, K^2_{\ell' \ell m} =$$

Presumably the other two components have similar results. Then we have to put these sums over ℓ' into our original sum over ℓm. The result is a mess and I have no direction really as to where I am going with this, so let's can it for now.

Attempt #2 (successful)

Let's here piggyback on Jackson's calculations! We know that:

$$\mathbf{E_\pm} = (\hat{\mathbf{x}} \ \pm\ i\, \hat{\mathbf{y}}\ )\, e^{ikz}$$

gives the result

$$\mathbf{E_\pm} = \Sigma_\ell\ (i)^\ell \sqrt{4\pi(2\ell+1)}\quad \left[\qquad j_\ell(kr)\, \mathbf{X}_{\ell,\pm1}\quad \pm\ (1/k)\, \nabla \times \{\, j_\ell(kr)\, \mathbf{X}_{\ell,\pm1}\,\}\ \right]$$
$$\mathbf{B_\pm} = \Sigma_\ell\ (i)^\ell \sqrt{4\pi(2\ell+1)}\quad \left[\ \mp\ i\, j_\ell(kr)\, \mathbf{X}_{\ell,\pm1}\quad -\quad (i/k)\, \nabla \times \{\, j_\ell(kr)\, \mathbf{X}_{\ell,\pm1}\,\}\ \right]$$

So let's use this

$$\mathbf{E}\ =\ (\mathbf{E}_+ + \mathbf{E}_-)/2\ = \hat{\mathbf{x}}\ e^{ikz}$$
$$\mathbf{B}\ =\ (\mathbf{B}_+ + \mathbf{B}_-)/2\ = \hat{\mathbf{y}}\ e^{ikz} \qquad \text{// see 16.130 and fact that } \mathbf{B}\ = \hat{\mathbf{z}} \times \mathbf{E}$$

Then our results will be

$$\mathbf{E} = \Sigma_\ell\ (1/2)(i)^\ell \sqrt{4\pi(2\ell+1)}\quad \left[\qquad j_\ell(kr)\, [\, \mathbf{X}_{\ell,+1} + \mathbf{X}_{\ell,-1}]\quad +\quad (1/k)\, \nabla \times \{\, j_\ell(kr)\, [\, \mathbf{X}_{\ell,+1} - \mathbf{X}_{\ell,-1}]\,\}\ \right]$$
$$\mathbf{B} = \Sigma_\ell\ (1/2)\ (i)^\ell \sqrt{4\pi(2\ell+1)}\quad \left[\ -\ i\, j_\ell(kr)\, [\, \mathbf{X}_{\ell,+1} - \mathbf{X}_{\ell,-1}]\quad -\quad (i/k)\, \nabla \times \{\, j_\ell(kr)\, [\, \mathbf{X}_{\ell,+1} + \mathbf{X}_{\ell,-1}]\,\}\ \right]$$

I have confirmed this second equation several times, every sign and i factor is important!

Now I am having trouble relating this form to (1.6) in my little Mie web paper.

<u>BUT WAIT!!!!</u> Think about what these things look like:

$$[\, \mathbf{X}_{\ell,+1} + \mathbf{X}_{\ell,-1}] = C(\ell,m) * \mathbf{r} \times \nabla\, [\, Y_{\ell,+1}(\theta,\phi) + Y_{\ell,-1}(\theta,\phi)]$$

Look at Jackson 3.53 and 3.54

$$Y_{\ell,+1}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi\ell(\ell+1)}}\ P_\ell^1(\cos\theta)\ e^{i\phi}$$

$$Y_{\ell,-1}(\theta,\phi) = (-1)^1\ Y_{\ell,+1}^*(\theta,\phi) = -\sqrt{\frac{2\ell+1}{4\pi\ell(\ell+1)}}\ P_\ell^1(\cos\theta)\ e^{-i\phi}$$

Therefore,

$$[\ Y_{\ell,+1}(\theta,\phi) + Y_{\ell,-1}(\theta,\phi)] = \sqrt{\frac{2\ell+1}{4\pi\ell(\ell+1)}}\ P_\ell^1(\cos\theta)\ \{\ e^{i\phi} - e^{-i\phi}\ \} = 2\,i\sqrt{\frac{2\ell+1}{4\pi\ell(\ell+1)}}\ P_\ell^1(\cos\theta)\sin\phi$$

If we leave off the constant, my krugel PDF claims that

$$\psi_{o1\ell} = P_\ell^1(\cos\theta)\sin\phi$$

And then the "odd" M function is the curl of this thing! That is why it is "odd", perhaps. That is to say

$$\mathbf{M_{o1\ell}} = j_\ell(kr)\ \mathbf{r} \times \nabla\psi_{o1\ell}$$

If we keep track of the constants

$$\begin{aligned} j_\ell(kr)\ [\ \mathbf{X}_{\ell,+1} + \mathbf{X}_{\ell,-1}] &= j_\ell(kr)C(\ell,m) * \mathbf{r} \times \nabla\ [\ Y_{\ell,+1}(\theta,\phi) + Y_{\ell,-1}(\theta,\phi)] \\ &= j_\ell(kr)C(\ell,m)\ 2\,i\sqrt{\frac{2\ell+1}{4\pi\ell(\ell+1)}}\ \ \mathbf{r} \times \nabla\ \{\ P_\ell^1(\cos\theta)\sin\phi\} \\ &= \frac{-i}{\sqrt{\ell(\ell+1)}}\ 2\,i\sqrt{\frac{2\ell+1}{4\pi\ell(\ell+1)}}\ \ j_\ell(kr)\ \mathbf{r} \times \nabla\ \psi_{o1\ell} \\ &= \frac{-i}{\sqrt{\ell(\ell+1)}}\ 2\,i\sqrt{\frac{2\ell+1}{4\pi\ell(\ell+1)}}\ \ j_\ell(kr)\ \mathbf{r} \times \nabla\ \psi_{o1\ell} \\ &= 2\ \sqrt{\frac{2\ell+1}{4\pi}}\ \frac{1}{\ell(\ell+1)}\ \mathbf{M_{o1\ell}} \end{aligned}$$

Then the first term in Jackson's sum becomes:

$$\mathbf{E} = \Sigma_\ell\ (1/2)(i)^\ell\sqrt{4\pi(2\ell+1)}\quad [\quad j_\ell(kr)\ [\ \mathbf{X}_{\ell,+1} + \mathbf{X}_{\ell,-1}]\ \ ]$$

$$= \Sigma_\ell\,(i)^\ell\frac{2\ell+1}{\ell(\ell+1)}[\ \mathbf{M_{o1\ell}} + \ldots$$

Now recall that the N function is defined as

$$\mathbf{N} = (1/k)\ \nabla\ \mathbf{x}\ \mathbf{M} = (1/k)\ \nabla \times \{\ j_\ell(kr)\ \ \mathbf{r} \times \nabla\psi\}$$

You can see that the second term has the difference of the two Y functions

$$j_\ell(kr)\,[\,\mathbf{X}_{\ell,+1} - \mathbf{X}_{\ell,-1}] \;= j_\ell(kr)C(\ell,m) * \mathbf{r} \times \nabla\,[\,Y_{\ell,+1}(\theta,\phi) - Y_{\ell,-1}(\theta,\phi)]$$

$$= j_\ell(kr)C(\ell,m) * \mathbf{r} \times \nabla\;\{\,2\sqrt{\frac{2\ell+1}{4\pi\ell(\ell+1)}}\;P_\ell^1(\cos\theta)\cos\phi\,\}$$

$$= (2/i)\sqrt{\frac{2\ell+1}{4\pi}}\;\frac{1}{\ell(\ell+1)}\;\mathbf{M}_{e1\ell} \qquad \mathbf{M}_{e1\ell} = j_\ell(kr)\;\mathbf{r} \times \nabla\psi_{e1\ell} \qquad \psi_{e1\ell} = P_\ell^1(\cos\theta)\cos\phi$$

The second term in the Jackson sum is then

$$+\;\; (1/k)\,\nabla \times \{\, j_\ell(kr)\,[\,\mathbf{X}_{\ell,+1} - \mathbf{X}_{\ell,-1}]\,\} \;=\; (1/k)\,\nabla \times \{\,(2/i)\sqrt{\frac{2\ell+1}{4\pi}}\;\frac{1}{\ell(\ell+1)}\;\mathbf{M}_{e1\ell}\}$$

$$= 2\sqrt{\frac{2\ell+1}{4\pi}}\;\frac{1}{\ell(\ell+1)}\;(\,-i\,\mathbf{N}_{e1\ell})$$

which has the same form as our M result above, so the final result is this:

$$\mathbf{E} \;=\; \hat{\mathbf{x}}\;e^{ikz} \;=\; \Sigma_\ell\,(i)^\ell\,\frac{2\ell+1}{\ell(\ell+1)}\,[\,\mathbf{M}_{o1\ell} \;-\; i\,\mathbf{N}_{e1\ell}\,]$$

And so we have *precisely* produced equation (1.6) now at 5 of midnight 2/14/03. Thus, we may conclude that they are considering fixed polarization in the $\hat{\mathbf{x}}$ direction! The big mystery is finally resolved.

Now what about the corresponding B equation?

Comments. You get the definite feeling that we are combining a spin 1 "thing" (polarization vector, say) with ℓm spatial functions, so we should be doing some kind of $j = \ell \otimes 1$ somewhere. The underlying group theory is being masked in our efforts so far, but I think I have found a source that discusses this subject. I think the idea might be this concerning the plane wave expansion: If you pre-combine into values of j, then you need only have a sum over j. I know this is a vague comment, but it might explain why the polarization is not appearing in the plane wave expansion I keep looking at.

I did a Marriott library trip today on this subject, was reminded that you cannot park before 6 PM. I found a little paper which I copied a bit of, and it repeats the funny plane wave expansion in terms of N and M functions. However, this paper gives a lot more detail on what these functions look like

Appendix A. Relating Cartesian unit vectors to X,Y,Z

$\mathbf{X}_{\ell m} = \mathbf{r} \times \nabla\, Y_{\ell m}$
$\mathbf{Y}_{\ell m} = \hat{\mathbf{r}}\; Y_{\ell m}$
$\mathbf{Z}_{\ell m} = r\,\nabla Y_{\ell m}$

It is possible to do this, but the results are not very nice. For example

$$\mathbf{r} \times \nabla Y_{\ell m} = \hat{\mathbf{x}} \left[-S\phi\, Y_\theta - (C_\theta C_\phi / r\, S_\theta)\, Y_\phi \right] + \hat{\mathbf{y}} \left[C\phi\, Y_\theta - (C_\theta S_\phi / r\, S_\theta)\, Y_\phi \right] + \hat{\mathbf{z}} \left[1/r\, Y_\phi \right]$$

$$\hat{\mathbf{r}}\ Y_{\ell m} = \hat{\mathbf{x}} \left[S_\theta C_\phi\, Y \right] + \hat{\mathbf{y}} \left[S_\theta S_\phi\, Y \right] + \hat{\mathbf{z}} \left[C_\theta Y \right]$$

$$r \nabla Y_{\ell m} = \hat{\mathbf{x}} \left[C_\theta C_\phi\, Y_\theta - S_\phi / S_\theta\ Y_\phi \right] + \hat{\mathbf{y}} \left[C_\theta C_\phi\, Y_\theta + C_\phi / S_\theta\ Y_\phi \right] + \hat{\mathbf{z}} \left[-S_\theta Y_\theta \right]$$

where Y_θ means the θ derivative of Y, etc. As noted elsewhere, this must somehow be the same as a certain rotation acting on our unit vectors.