More on the M and N Functions PhL 2.16.03

1. About the M function

In the previous paper on this topic, I made this identification for the "first term" multipole fields,

$$\mathbf{M}_{\ell m}(r,\theta,\phi) = \nabla \times [f_\ell(r)\, Y_{\ell m}\, \mathbf{r}\,] = -f_\ell(r)\, \mathbf{r} \times \nabla Y_{\ell m} = -f_\ell(r)\, \mathbf{X}_{\ell m}/C(\ell,m)$$

where

$$\mathbf{X}_{\ell m}(\theta,\phi) = C(\ell,m) * \mathbf{r} \times \nabla Y_{\ell m}(\theta,\phi)$$

$C(\ell,m) =$	$+1$	Carleton
	-1	Mie (not sure what this means now)
	$\frac{-i}{\sqrt{\ell(\ell+1)}}$	Jackson

Sometimes, such as in Krugel, we found that $Y_{\ell m}$ was replace with a linear combination of $Y_{\ell m}$ within the same ℓ group.

Let's now prove a few theorems concerning the M functions

Theorem 1: If $\mathbf{M} \equiv \nabla \times (\mathbf{c}\,\psi)$ where **c** is either a constant vector or **r**, then $\mathbf{M} = -\mathbf{c} \times \nabla\psi$.

Proof: $\nabla \times (\mathbf{c}\,\psi) = \nabla\psi \times c + \psi\nabla \times c = \nabla\psi \times c = -\mathbf{c} \times \nabla\psi$

Theorem 2: Let $\mathbf{M} \equiv \nabla \times (\mathbf{c}\,\psi)$. If **c** is either a constant vector or **r**, and if ψ satisfies the wave equation $(\nabla^2 + k^2)\,\psi = 0$, then so does **M**: $(\nabla^2 + k^2)\,\mathbf{M} = 0$

Proof: If we apply ∇^2 onto **M**, it moves through onto the $\mathbf{c}\,\psi$, and then we can apply our identity

$$\nabla^2 (\mathbf{c}\,\psi) = \mathbf{c}\nabla^2\psi + \psi\,\nabla^2\,\mathbf{c} + 2\,\nabla c \bullet \nabla\psi$$

where we think of a component of c at a time here, but write it as a vector. Now if **c** is a constant, the last two terms vanish and we get our result,

$$\nabla^2\,\mathbf{M} \equiv \nabla \times \nabla^2 (\mathbf{c}\,\psi) = \nabla \times (\mathbf{c}\,\nabla^2\psi) = -k^2\,\nabla \times (\mathbf{c}\,\psi) = -k^2\,\mathbf{M}$$

It *turns out* that this still works when $\mathbf{c} = \mathbf{r}$ even though r is not a constant vector. Consider,

$$\nabla^2 (\mathbf{r}\,\psi) = \mathbf{r}\nabla^2\psi + \psi\,\nabla^2\,\mathbf{r} + 2\,\nabla r \bullet \nabla\psi = -k^2 (\mathbf{r}\,\psi) + 0 + 2\,\nabla\psi$$

where we have used the fact that $\nabla^2 \mathbf{r} = 0$ since $\partial_j \partial_j r_i = \partial_j \delta_{j,i} = 0$. Obviously since $\mathbf{r}$ is linear, it cannot have a Laplacian. In the second term, we note that $\nabla \mathbf{r} = \mathbf{I}$, the identity matrix, which we might write in the form $\overleftrightarrow{\mathbf{1}}$ and then we would say that $\overleftrightarrow{\mathbf{1}} \bullet \nabla\psi = \nabla\psi$ (or do as components!) . So at this point we have

$$\nabla^2 \mathbf{M} \equiv \nabla \times \nabla^2 (\mathbf{r}\,\psi) = \nabla \times [-k^2 (\mathbf{r}\,\psi) + 2\,\nabla\psi] = -k^2 \mathbf{M} + 2\,\nabla \times \nabla\psi = -k^2 \mathbf{M} \qquad \text{QED}$$

Theorem 3: Let $\mathbf{M} \equiv \nabla \times (\mathbf{r}\,\psi) = -\mathbf{r} \times \nabla\psi$. If ψ satisfies the wave-*like* equation $\nabla^2\psi = f(r)\psi$, then $\mathbf{M}$ satisfies the same equation, $\nabla^2\mathbf{M} = f(r)\,\mathbf{M}$. { This theorem is not true if we replace $\mathbf{r}$ with constant **c.**}

Proof: $\nabla^2 \mathbf{M} = \nabla \times \nabla^2 (\mathbf{r}\,\psi) = \nabla \times (\mathbf{r}\nabla^2\psi) = \nabla \times [(f(r)\,\psi)\,\mathbf{r}\,] = \nabla(f(r)\,\psi) \times r + (f(r)\,\psi)\,\nabla \times r$
$= \nabla(f(r)\,\psi) \times r = [\, f(r)\,\nabla\psi + \psi\,\nabla f(r)\,] \times \mathbf{r} = -\mathbf{r} \times f(r)\,\nabla\psi + 0 = -f(r)\,\mathbf{r} \times \nabla\psi = f(r)\,\mathbf{M}$ by Theorem 1.

Example of Theorem 3. Consider $\mathbf{X}_{\ell m} = \mathbf{r} \times \nabla Y_{\ell m}$ so that $\psi = -Y_{\ell m}$. We know that $\nabla^2 Y_{\ell m} = -\ell(\ell+1)/r^2\, Y_{\ell m}$ so that $f(r) = -\ell(\ell+1)/r^2$. Thus we know that $\nabla^2 \mathbf{X}_{\ell m} = -\ell(\ell+1)/r^2\, \mathbf{X}_{\ell m}$. We proved this fact in gory detail in our "vector harmonics.doc" paper. We might think of this as "the wave equation within a partial wave", loosely speaking.

Summary of what we have so far:

- Define $\mathbf{M} \equiv \nabla \times (\mathbf{r}\,\psi)$. Then:
- $\mathbf{M} = -\mathbf{r} \times \nabla\psi$, which is one of our usual VSH basis functions $\mathbf{X}$, up to a constant
- $\nabla \bullet \mathbf{M} = 0$
- If ψ satisfies a wave *or* wave-like equation , then so does **M.**

Theorem 4. Suppose we define $\mathbf{M} \equiv f_\ell(r)\,\mathbf{M_o} = f_\ell(r)\,\nabla \times (\mathbf{r}\,\psi) = -f_\ell(r)\,\mathbf{r} \times \nabla\psi$, where $\mathbf{M_o} = -\mathbf{r} \times \nabla\psi$ and where:

(1) $\psi = \psi(\theta,\phi)$;

(2) ψ satisfies the wave-like equation $\nabla^2\psi = g(r)\psi$ where $g(r) = -\ell(\ell+1)/r^2$;

(3) $f_\ell(r)$ is a linear combination of spherical Bessel functions such as $j_\ell(kr)$, where k is the constant appearing in the Bessel equation.

Then $(\nabla^2 + k^2)\,\mathbf{M} = 0$.

Corollary: If ψ is any linear combination of spherical harmonics $Y_{\ell m}$ *within* the ℓ manifold, theorem 4 is true.

Proof of Theorem 4: From Theorem 2 we know that $\nabla^2\,\mathbf{M_o} = g(r)\,\mathbf{M_o}$. So consider,

$$\nabla^2 \mathbf{M} = \nabla^2 [f_\ell(r)\,\mathbf{M_o}\,] = [\nabla^2 f_\ell(r)]\,\mathbf{M_o} + f_\ell(r)\,\nabla^2 \mathbf{M_o} + 2\,\nabla f_\ell(r) \bullet \nabla \mathbf{M_o}$$

$$= \{\, 1/r^2\,\partial_r [\, r^2\,\partial_r f_\ell(r)] + g(r)\, f_\ell(r)\,\}\,\mathbf{M_o} + 2\, f'_\ell(r)\,\partial_r \mathbf{M_o}$$

Now comes a somewhat unobvious fact that I got caught on. Consider:

$$- \mathbf{M_o} = \mathbf{r} \times \nabla\psi = \mathbf{r} \times \left(1/r \ \psi_\theta \ \hat{\boldsymbol{\theta}} + 1/r \ 1/S_\theta \ \psi_\phi \hat{\boldsymbol{\phi}} \right)$$

$$= \psi_\theta \hat{\boldsymbol{\phi}} - 1/S_\theta \ \psi_\phi \hat{\boldsymbol{\theta}} .$$

So $\mathbf{M_o}$ is not even a function of r, and therefore $\partial_r \mathbf{M_o} = 0$. Thus we get:

$$\nabla^2 \mathbf{M} = \{ 1/r^2 \ \partial_r [r^2 \ \partial_r f_\ell(r)] - \ell(\ell+1)/r^2 \ f_\ell(r) \} \ \mathbf{M_o} = \{ - k^2 \ f_\ell(r) \} \ \mathbf{M_o} = - k^2 \ \mathbf{M} \qquad \text{QED}$$

Since $f_\ell(r)$ is a Bessel function as noted, we get the simplification of {...} shown on the right above.

Comment: The above theorem shows an interesting result. We know that $f_\ell(r)\psi_\ell$ satisfies the **scalar** wave equation (where $f_\ell(r)$ is a spherical Bessel of parameter k, and ψ_ℓ is a linear combination of spherical harmonics within ℓ). If we are interested in solving the **vector** wave equation, then $\mathbf{M} = - f_\ell(r) \ \mathbf{r} \times \nabla\psi_\ell$ works. It has the extra nice property that $\nabla \bullet \mathbf{M} = 0$.

Observation: The multipole field "main terms" for **E** and **B** are of exactly this form. Usually we have ψ_ℓ as a straight $Y_{\ell m}$, but it is OK to instead use something like $\psi_\ell = C [Y_{\ell m} \pm Y_{\ell - m}]$ if that is convenient. For the expansion of an $\hat{\mathbf{x}}$ polarized plane wave, this does turn out to be convenient, and m = 1. We are also satisfying $\nabla \bullet \mathbf{E,B} = 0$ with this solution.

2. About the N function

Now let's look at N defined by:

$$\mathbf{N} = \nabla \times \mathbf{M}$$

where **M** satisfies the wave equation. Then **N** must also satisfy the wave equation, since

$$\nabla^2 \mathbf{N} = \nabla^2 \nabla \times \mathbf{M} = \nabla \times (\nabla^2 \mathbf{M}) = \nabla \times (-k^2 M) = -k^2 \nabla \times \mathbf{M} = -k^2 \mathbf{N}$$

Notice also that $\nabla \bullet \mathbf{N} = 0$.

Usually, a multiplicative constant is used out in front,

$$\mathbf{N} = (1/k) \nabla \times \mathbf{M}$$

since this removes a factor of 1/k in the "second terms" in the multipole expansions for E and B. We have just shown, then, that these "second terms" satisfy both the divergence condition and wave equation, so they represent a second solution to the wave equation mentioned earlier. So, we can now extend our earlier comment:

Comment: We know that $f_\ell(r)\psi_\ell$ satisfies the **scalar** wave equation (where $f_\ell(r)$ is a spherical Bessel of parameter k, and ψ_ℓ is a linear combination of spherical harmonics within ℓ). If we are interested in solving the **vector** wave equation, then we know of two solutions:

(1) $\mathbf{M} = -f_\ell(r)\,\mathbf{r} \times \nabla\psi_\ell$, and we also know that $\nabla \bullet \mathbf{M} = 0$.

(2) $\mathbf{N} = (1/k)\,\nabla \times \mathbf{M} = -(1/k)\,\nabla \times [f_\ell(r)\,\mathbf{r} \times \nabla\psi_\ell]$, and $\nabla \bullet \mathbf{N} = 0$.

These are exactly the first and second terms we see in Jackson's multipole expansion formulas, apart from overall constants.

We can think of **M** as **M** = constant x $f_\ell(r)$ * **X** where **X** is one of Carleton's three VSH's. We know that we can explode **N** into a linear combination of the **Y** and **Z** VSH basis vectors. Only **X** of these three basis vectors satisfies the wave equation! However, the **N** linear combination of **Y** and **Z** also solves it!

One other fact. With **N** scaled by the (1/k) factor as shown

$$\nabla \times \mathbf{N} = k\,\mathbf{M}$$

So we really have this symmetrical situation:

$$\mathbf{N} = (1/k)\,\nabla \times \mathbf{M}$$
$$\mathbf{M} = (1/k)\,\nabla \times \mathbf{N}$$

Also we know that (from inspection of M)

$\mathbf{r} \bullet \mathbf{M} = 0$ so M is the "transverse" solution

Of course we know this since **M** ~ **X** and we know that **X** and **Z** are transverse, **Y** radial. Note that , since **N** is a mixture of **Y** and **Z**, it has both transverse and radial pieces.