Summary of Carelton's VSH Handout Notes PhL 2.13.03

The Basis Functions and Expansion Theorem

Define these three vector functions on the surface of a sphere θ,ϕ which has radius r:

$$\mathbf{X}_{\ell m} = \mathbf{r} \times \nabla Y_{\ell m} \qquad 2 \qquad = \mathbf{X}^1_{\ell m}\sqrt{\ell(\ell+1)}$$
$$\mathbf{Y}_{\ell m} = \hat{\mathbf{r}}\, Y_{\ell m} \qquad r \qquad = \mathbf{X}^2_{\ell m}$$
$$\mathbf{Z}_{\ell m} = r\nabla Y_{\ell m} \qquad 1 \qquad = \mathbf{X}^3_{\ell m}\sqrt{\ell(\ell+1)}$$

where $Y_{\ell m}$ is the usual spherical harmonic as in Jackson. These vectors are mutually perpendicular, with the **Y** function being radial and **Z** and **X** being transverse to the sphere. In an E&M problem, one is familiar with the two transverse directions, and 1 and 2 are exactly those directions.

Within the ℓm manifold (partial wave) of the multipole expansion of a vector function **A**(θ,ϕ), the three vectors above form a true set of basis functions, and we end up with the following expansion theorem including the projection formula

$$\mathbf{A}(\theta,\phi) = \Sigma_{\ell mn}\ A_{\ell mn}\,\mathbf{X}^n_{\ell m}(\theta,\phi) \qquad A_{\ell mn} = \int d\Omega\ \mathbf{X}^n_{\ell m}{}^*(\theta,\phi)\bullet\,\mathbf{A}(\theta,\phi)$$

expansion projection

These imply the following orthogonality and completeness relations:

$$\int d\Omega\ \ \mathbf{X}^{n'}_{\ell' m'}(\theta,\phi)^*\bullet\,\mathbf{X}^n_{\ell m}(\theta,\phi) = \delta_{\ell',\ell}\,\delta_{m',m}\,\delta_{n',n} \qquad \text{orthogonality}$$

$$\Sigma_{\ell mn}\ \ [\mathbf{X}^n_{\ell m}{}^*(\theta,\phi)]_i\,[\,\mathbf{X}^n_{\ell m}(\theta',\phi')\,]_{i'} == \delta_{i,i'}\ \ \delta(\Omega-\Omega') \qquad \text{completeness}$$

In the detailed notes, we prove orthogonality, and the fact that the three vectors are perpendicular to each other, but we have to simply accept as being awfully reasonable the fact that the basis functions are complete, so the expansion is valid for any reasonable vector function. I could not find a trivial proof of this fact, but there probably is one.

The above result is then trivially extended to vector functions of r, θ, ϕ by adding r as a ride-along parameter, and we then get,

$$\mathbf{A}(r,\theta,\phi) = \Sigma_{\ell mn}\ A_{\ell mn}(r)\,\mathbf{X}^n_{\ell m}(\theta,\phi) \qquad A_{\ell mn}(r) = \int d\Omega\ \mathbf{X}^n_{\ell m}{}^*(\theta,\phi)\bullet\,\mathbf{A}(r,\theta,\phi)$$

In our applications we will often think of **A** as being **E**, **B** or **J**. The above is the generalization of the partial wave expansion of a *scalar* function, which we know works like this:

$$A(r,\theta,\phi) = \Sigma_{\ell m}\ A_{\ell m}(r)Y_{\ell m}(\theta,\phi) \qquad A_{\ell m}(r) = \int d\Omega\ Y_{\ell m}{}^*(\theta,\phi)\,A(r,\theta,\phi)$$

$\int d\Omega\, Y_{\ell'm'}^*(\theta,\phi)\, Y_{\ell m}(\theta,\phi) = \delta_{\ell',\ell}\, \delta_{m',m}$ orthogonality

$\Sigma_{\ell m}\, Y_{\ell m}^*(\theta,\phi)\, Y_{\ell m}(\theta',\phi') = \delta(\Omega - \Omega')$ completeness

We are familiar with the use of this multipole expansion in electrostatic potential theory for the potential ϕ.

Some Useful Properties of the Basis Functions

$\nabla \bullet \mathbf{X}_{\ell m} = 0$	$\hat{\mathbf{r}} \bullet \mathbf{X}_{\ell m} = 0$
$\nabla \bullet \mathbf{Y}_{\ell m} = (2/r)\, Y_{\ell m}$	$\hat{\mathbf{r}} \bullet \mathbf{Y}_{\ell m} = Y_{\ell m}$
$\nabla \bullet \mathbf{Z}_{\ell m} = -\ell(\ell+1)\,(1/r)\, Y_{\ell m}$	$\hat{\mathbf{r}} \bullet \mathbf{Z}_{\ell m} = 0$
$\nabla \times \mathbf{X}_{\ell m} = -[\ell(\ell+1)/r]\, \mathbf{Y}_{\ell m} - (1/r)\, \mathbf{Z}_{\ell m}$	$\hat{\mathbf{r}} \times \mathbf{X}_{\ell m} = -\mathbf{Z}_{\ell m}$
$\nabla \times \mathbf{Y}_{\ell m} = -(1/r)\, \mathbf{X}_{\ell m}$	$\hat{\mathbf{r}} \times \mathbf{Y}_{\ell m} = 0$
$\nabla \times \mathbf{Z}_{\ell m} = (1/r)\, \mathbf{X}_{\ell m}$	$\hat{\mathbf{r}} \times \mathbf{Z}_{\ell m} = \mathbf{X}_{\ell m}$

It turns out that all operations you can think of to apply to the basis functions, or to scalar functions of r times the basis functions, always keep you within the ℓm manifold. The reason for this fact is summarized by the following exhaustion of all possible dimensionless vectors you can make out of the building blocks **r** and ∇,

$\hat{\mathbf{r}}\, Y_{\ell m}$	**Y**
$r\nabla Y_{\ell m}$	**Z**
$\mathbf{r} \times \nabla Y_{\ell m}$	**X**
$\hat{\mathbf{r}}\, (\mathbf{r} \bullet \nabla Y_{\ell m})$	0
$r^2\, \nabla \times \nabla Y_{\ell m}$	0
$\hat{\mathbf{r}}\, r^2\, \nabla \bullet \nabla Y_{\ell m}$	$\hat{\mathbf{r}}\, r^2\, \nabla^2\, Y_{\ell m} = -\hat{\mathbf{r}}\, L^2\, Y_{\ell m} = -\ell(\ell+1)\, \hat{\mathbf{r}}\, Y_{\ell m} = -\ell(\ell+1)\mathbf{Y}$

The last item is the only but crucial place where the properties of the $Y_{\ell m}$ are required, and no other functions will do! The $Y_{\ell m}$ work because they are coordinate-space representations of angular momentum states $|\ell m\rangle$ which diagonalize L^2 and L_3, which is possible because the Hamiltonian is a rotational scalar.

From now on, we will write the expansion formula using Carleton's X,Y,Z form, to wit,

$$\mathbf{A} = \Sigma_{\ell m}\, [\, A_2\, \mathbf{X} + A_\tau\, \mathbf{Y} + A_1\, \mathbf{Z}\,]$$

in a shorthand notation where the three coefficients really have ℓm labels and are functions or r, and of course the basis vectors also have these labels. Note that 1 and 2 indicate the transverse vector coefficient functions of r, while τ is the radial vector coefficient function.

Diagonalizing Maxell's equations involves applying $\nabla \bullet$ and $\nabla \times$ applied to the above expansion for the E and B fields. Using the properties noted above, one can show these essential facts:

$$\nabla \bullet \mathbf{A} = \Sigma_{\ell m} \left[\nabla \bullet (A_2 \mathbf{X}) + \nabla \bullet (A_\tau \mathbf{Y}) + \nabla \bullet (A_1 \mathbf{Z}) \right]$$

$$= \Sigma_{\ell m} \left\{ 1/r^2 \, \partial_r (r^2 A_\tau) - A_1 \, \ell(\ell+1) \, (1/r) \right\} Y_{\ell m}$$

The result is of course a scalar, and the A_2 coefficient does not appear in the result, indicating that no constraint is put on A_2 by a divergence condition one might find in Maxell's equations. Similarly we have

$$\nabla \times \mathbf{A} = \Sigma_{\ell m} \left[\nabla \times (A_2 \mathbf{X}) + \nabla \times (A_\tau \mathbf{Y}) + \nabla \times (A_1 \mathbf{Z}) \right]$$

$$= \Sigma_{\ell m} \{ -(1/r)\partial_r (rA_2) \mathbf{Z}_{\ell m} - \ell(\ell+1)/r \, A_2 \mathbf{Y}_{\ell m} + [(1/r)(\partial_r (r A_1) - A_\tau/r] \mathbf{X}_{\ell m} \}$$
$$1 \qquad\qquad \tau \qquad\qquad 2$$

This result is more complicated and all three coefficients are involved. We have shown the correct expansion labels for the resulting coefficient combinations.

At this point in the handout, Carleton does two warm-up exercises.

Electrostatics Exercise. Use $\nabla \bullet \mathbf{E} = 4\pi\rho$ and $\nabla \times \mathbf{E} = 0$ and see what happens. The result is that $E_2 = 0$ and we get two coupled first order DEs for E_2 and E_τ which we can combine to get the Poisson radial equation for the quantity (rE^1) which we can interpret as the potential $-\phi$. This equation is driven by source term $4\pi\rho_{\ell m}$. We find in fact that $E^1{}_{\ell m} = - \phi_{\ell m} / r$ where the $\phi_{\ell m}$ are the *usual* scalar multipole coefficients used in electrostatics multipole expansion.

Magnetostatics Exercise. Use $\nabla \bullet \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = 4\pi/c \, \mathbf{J}$ and see what happens. The $\nabla \times \mathbf{B}$ equation results in three equations (39)-(41), and the $\nabla \bullet \mathbf{B}$ equation gives a fourth (42). By combining two of these, we obtain again Poisson's radial equation, but it is now for the quantity (rB^τ) and the equation is driven by a J^2 term. Carleton provides in (44) the Green's Function solution for B^τ. Via a separate path, he writes the conventional scalar multipole expansion for *magnetic* potential ϕ in terms of moments $\mu_{\ell m}$. Then $\mathbf{B} = -\nabla\phi$ is expanded in VSH giving equations (36)-(38). Comparing our Green's solution with (36) gives a traditional formula for the magnetic moment as an integral over the current distribution J.

After these warm-up exercises, we go for the whole nine yards. On page 6 we write out the four Maxwell equations (with sources), and applying the above div and curl expansions, we end up with 8 little equations for the 6 VSH component functions. Now comes *the* key fact. These 8 equations can be grouped into two sets of 4 equations which I have called the α group and the β group. Each group involves only 3 of the 6 unknown functions, so we have really *two completely independent solution sets*.

TE = magnetic: α: $E_\tau = E_1 = B_2 = 0$ β: B_τ , B_1, E_2 all active

TM = electric: β: $B_\tau = B_1 = E_2 = 0$ α: E_τ , E_1, B_2 all active

The name TE arises because $E_\tau = 0$ in this solution set -- there is no radial E field. We know from Jackson that the conventional names magnetic and electric arise later on when we see what the radiation solutions look like. In general, you solve the TE and TM problems independently, and then of course you could

superpose solutions together. Notice this fact: in each active set, you solve for a #2 field, and that is ALL of that kind of field. The other two fields in the set are of the OTHER type. So in TE the entire E field is simply E^2 , for example, and the other two fields are B type.

Think now about what we have done! We take the four very complicated Maxwell vector equations with nasty curls and vector components, and we reduce them in the partial wave expansion to 2 independent sets of 4 scalar equations for three field components. In each set, we can show that the #2 field component function must solve the radial wave equation driven by an appropriate current mixture, as shown in (59) and (60).

Full Maxwell Exercise: the cavity resonator. Imagine no sources and a conducting metal cavity. What E&M modes (resonances) can exist in there? The BC's are the usual $\hat{\mathbf{r}} \bullet \mathbf{B} = 0$ and $\mathbf{r} \times \mathbf{E} = 0$ at r = a, which boil down to the extremely simple (61). Since no sources, solutions of the radial wave equation are spherical Bessel's like $j_\ell(kr)$. The TE and TM solutions are just multiples of $j_\ell(kr)$ as in (62,3). However, the BC's cause the solution values of k to be quantized, and these are then the cavity normal modes. The conditions for TM and TE are different, (65) and (66), so modes have different eigenfrequencies. This is a fairly complex problem to even think about, but the VSH method provides an almost instant solution! Of course the geometry is optimal for VSH work!

By the way, these same BC's apply to Jackson's scattering from a conducting sphere. In that problem, we have extra BC's in the full coordinate space of an incoming plane wave and outgoing spherical wave. I now realize that Jackson 16.139 is nothing more than the expansion of a plane wave traveling in the z direction onto the VSH basis functions. He uses circular polarizations. I recall at the time how completely obscure this seemed! You can see how Jackson prefers to use the combination $\nabla \times f_\ell(r)\, \mathbf{X}_{\ell m}$ to incorporate the **Y** and **Z** components of the expansion, and he has even provided a little orthogonality for this combination in 16.132 which is unbelievably obtuse! Carleton has shown how this can all be understood in a systematic fashion. I could resolve this problem using the VSH from the start, project the plane wave onto the basis functions from scratch, but of course I would end up with the same 16.139 with three terms in place of Jackson's two.

Radiation from a localized source. Here we just take those two #2 component wave equations and do the Green's function solutions, as done in Jackson. The B^2 solution is shown in (69) (this is of course the particular solution, to which we can add solutions of the sourceless problem, like those cavity modes, in order to match some boundary conditions.)

Now on page 9 we come to the normal "statement" of multipole fields. For TM, we know that the B field has only the B_2 component, and that is what (70) says. Here we use the Hankel 1 function because we are implicitly matching the spherical outgoing boundary condition, since our topic is "radiation from a localized source". Jackson in 6.42 is slightly more general, not specifying the type of radial function. Away from the source, you get the E field from the B field in trivial fashion just from the usual Maxwell. By installing our specific Green's solution (69) for B_2, we get a specific result for the radiation field in terms of newly minted coefficient a_E as shown. It is an integral over the curl of J in the source times other stuff as in (72), and this is the result that took me 5 pages of notes to convert into form (73) where you see that a TM mode is driven by charge ρ if it is present, hence the name "electric".

At this point, declaration of the form of the TE mode follows at once with (76), just looking at the different source in the radial equation. In this case, we don't have to do the elaborate conversion, we see that a_M is an integral of J as shown.

So fine, we have recovered Jackson's multipole expansion results in a much more organized approach, with much less mystery. I presume Jackson has held firm in his 2nd and 3rd editions.

Comment on Wave Equations. We know that components of E and B in free space solve the wave equation, let there be light! In the presence of sources ρ and **J**, these wave equations are not so nice. Here is a quick shot,

$$(\nabla^2 + k^2)\,\mathbf{E} = 4\pi\nabla\rho - 4\pi ik/c\ \mathbf{J} \qquad \text{and} \qquad ik\mathbf{B} = \nabla \times \mathbf{E}$$

$$(\nabla^2 + k^2)\,\mathbf{B} = -4\pi/c\ \nabla \times \mathbf{J} \qquad \text{and} \qquad ik\mathbf{E} = -\nabla \times \mathbf{B} + 4\pi/c\ \mathbf{J}$$

This is why we introduce the vector potential A.

Now in the VSH approach, we find that in each mode, the #2 component satisfies a *radial* wave equation. The B_2 solution for the TM is driven by the #2 VSH expansion coefficient of $\nabla \times \mathbf{J}$, in analogy with the second line above, giving us result (72). However, after the transformation to (73) using continuity, we find that in fact ρ is really the main driver of the TM mode **B**, contrary to what appears above, where we don't even see ρ in the second line.

Similarly, the E_2 solution for TE is driven by the #2 coefficient of **J**, as shown in (76). This dimly reflects the first line above, but again we marvel at the absence of ρ in affecting E_2 directly. Of course in light of continuity, the two sources ρ and J are always connected and all these comments are not very productive I guess.

I don't think the vector basis functions satisfy any wave equations, though I saw someone claim this once. See other notes on the N and M functions.