Vector Spherical Harmonics PhL 2.8.03

These are my *extended* notes on the 1995 Physics 712 handout on the subject of VSH which was assembled by Carleton from notes of Richard and Tracy. Lots of calculations are recorded in my appendix. I plan to summarize this set of notes in *another* document, leaving out all the details, and trying to focus on the main points.

1. The Scalar Function ψ defines an axis rotation.

Consider the three vectors as follows:

$$\mathbf{r} \times \nabla\psi \qquad \hat{\mathbf{r}}\,\psi \qquad r\nabla\psi$$

where ψ is any *complex* function ψ(θ,ϕ). Notice that all three vectors are dimensionless. These three vectors are mutually perpendicular:

$r \bullet \nabla\psi = r\partial_r\psi = 0$ since ψ is not a function of r

$r \bullet (\mathbf{r} \times \nabla\psi) = 0$ since $A\bullet(\mathbf{A}\times\mathbf{B}) = 0$ for any vectors

$\nabla\psi \bullet (\mathbf{r} \times \nabla\psi) = 0$ same reason

What are these vectors? The radial one is just of magnitude ψ pointing radially outward, that is pretty clear. The rΔψ vector points in a direction *tangent* to the sphere's surface in the direction in which ψ has its maximum increase. And **r** x Δψ is also tangent also and is perp to the other two vectors.

Now let's try to expand an arbitrary complex function **A**(θ,ϕ) as follows:

$$\mathbf{A} = \alpha\, r\nabla\psi + \beta\, \hat{\mathbf{r}}\,\psi + \gamma\, \mathbf{r} \times \nabla\psi$$

where α, β and γ are complex functions of θ and ϕ. Is this a completely general expansion of **A**? I will argue that it is, and here are the components:

$$\alpha = A \bullet (r\nabla\psi) \qquad \beta = (A \bullet \hat{\mathbf{r}}\,)\,\psi \qquad \gamma = A \bullet (\mathbf{r} \times \nabla\psi)$$

All we are doing is changing from x,y,z to x',y',z' where the new coordinate system is a function of our pilot function ψ.

2. Specific choice for ψ as $Y_{\ell m}(\theta,\phi)$ and some reasons for this choice.

We could, for example, set $\psi = Y_{\ell m}(\theta,\phi)$, then we get:

$$\mathbf{r} \bullet \nabla\, Y_{\ell m} = 0$$

$\mathbf{r} \bullet (\mathbf{r} \times \nabla Y_{\ell m}) = 0$

$\nabla Y_{\ell m} \bullet (\mathbf{r} \times \nabla Y_{\ell m}) = 0$

Let's now give these three vector functions Carleton's names,

$$\mathbf{X}_{\ell m} = \mathbf{r} \times \nabla Y_{\ell m} \qquad \mathbf{Y}_{\ell m} = \hat{\mathbf{r}}\, Y_{\ell m} \qquad \mathbf{Z}_{\ell m} = r\nabla Y_{\ell m}$$

In passing, we note that Jackson also uses an X function, but his is scaled differently, see (16.45).

$$\mathbf{X}_{\ell m}^{\text{JACKSON}} = \frac{1}{i\sqrt{\ell(\ell+1)}}\, \mathbf{X}_{\ell m} = \frac{1}{\sqrt{\ell(\ell+1)}}\, \mathbf{L}\, Y_{\ell m} \qquad iL = r \times \nabla$$

The i here comes from Jackson's inclusion of an i in his definition of L, making it consistent with the L that we use in quantum mechanics. For our purposes here, however, this factor causes only trouble.

I might have called these more suggestive names, like $\mathbf{M}_{\ell m}$, $\mathbf{R}_{\ell m}$ and $\mathbf{G}_{\ell m}$ (angular Momentum, Radial, Gradient) . One problem is that you can never remember which letter goes with which function. In retrospect, I think I like the simple X,Y,Z choice.

Now, for each choice of values ℓm, we have a little perpendicular vector triad where one vector is radial and the other two are tangential to the surface of the sphere. Notice, however, that the orientation of the two tangential vectors relative to the sphere is going to be dependent on the choice of ℓ and m, whereas the radial vector is always radial.

Why are we interested in these vectors with $\psi = Y_{\ell m}$? Well, first off, in our appendix at the end of this paper we painstakingly calculate many things, and in doing so we learn many things. One fact we find is that application of any of the standard operators like curl, div, grad, lapl, r $\bullet$ and r x to the three basis functions produces either a linear combination of the same basis functions (which reinforces our idea of them spanning the 3-space), or produces a multiple of $Y_{\ell m}$ in the case of an operation that produces a scalar result like div. In these proofs, we sometimes encounter the ∇^2 operator and we replace it with the operator $-1/r^2\, L^2$ when it acts on a function of θ and ϕ, where iL is r x ∇. It is *at this point* that we make use of the specific properties of the $Y_{\ell m}$ functions, that $L^2\, Y_{\ell m} = \ell(\ell+1)\, Y_{\ell m}$. It is this fact that keeps everything closed within the little ℓm manifold. For example, you never encounter an L_+ type operator that tries to raise you to $Y_{\ell m+1}$, for example.

Let's look more closely at this claim that nothing can push us out of the ℓm space. No matter how much vector stuff we apply on the left, when we work through all the vector identities, these are the only dimensionless vector things we can "end up with":

$\hat{\mathbf{r}}\, Y_{\ell m}$	$\mathbf{Y}$
$r\nabla Y_{\ell m}$	$\mathbf{Z}$
$\mathbf{r} \times \nabla Y_{\ell m}$	$\mathbf{X}$
$\hat{\mathbf{r}}\, (\mathbf{r} \bullet \nabla Y_{\ell m})$	0
$r^2\, \nabla \times \nabla Y_{\ell m}$	0
$\hat{\mathbf{r}}\; r^2\, \nabla \bullet \nabla Y_{\ell m}$	$\hat{\mathbf{r}}\, r^2\, \nabla^2\, Y_{\ell m} = -\, \hat{\mathbf{r}}\, L^2\, Y_{\ell m} = -\, \ell(\ell+1)\, \hat{\mathbf{r}}\, Y_{\ell m} = -\, \ell(\ell+1)\mathbf{Y}$

The only things we have to "play with" are the vector $\mathbf{r}$ or $\hat{\mathbf{r}}$, the scalar r, and the $\boldsymbol{\nabla}$ operator. Now for *any* pilot function $\psi(\theta,\phi)$ in place of Y, the first 6 items would still be true, and our space would be closed under these operations. It is the last item that is critical. With the generic ψ function we would end up with $r^2\,\nabla^2\psi = r^2\,g$, some new function, and we could then end up with $\hat{\mathbf{r}}\,g$ as a new vector.

As an example, we know (by explicit calc in our appendix) that ∇^2 applied to any of the vector basis functions just yields a linear combination of those basis functions. We also know that any purely-r operator can only shuffle the powers of r around in some way, without affecting the $Y_{\ell m}$ -- certainly we will not shift any ℓm by applying combinations of ∂_r and r. Therefore, it is true that the L^2 operator when acting on any of the vector basis functions will give at worst a linear combination of same. Since $[\,L^2\,,\,\mathbf{L}\,]=0$, we can see that $L^2\,\mathbf{X} = \ell(\ell+1)\,\mathbf{X}$. For **Y** and **Z** I don't think this is true, but L^2 will certainly yield a linear combination of the vector basis functions as per our discussion above.

In particular, the vector operators in Maxwell's Equations won't take you out of ℓm, so each ℓm manifold represents a "separate problem", as is the goal in any diagonalization. Of course the underlying fact is that Maxwell's equations are invariant under SO(3) and this is why $L^2\;L_3$ diagonalization is possible. The Hamiltonian or Lagrangian would be something like $F^{\mu\upsilon}F_{\mu\upsilon} + A^{\mu}J_{\mu}$ and is clearly SO(3) invariant, not to mention SO(3,1) invariant. So these comments I think provide a little extra motivation for the underpinning of the vector multipole expansion.

Before going on, we should note that Jackson's "grand finale multipole expansion of **E** or **B**" 16.47 has two vector terms, but according to 16.143, the second term itself breaks into two other terms, and you then end up with three vector terms in each equation, and these are precisely the vector basis functions we are here talking about. Jackson however does not really discuss the fact that these three functions span the space; he is just silent about that and grinds out the results.

3. Orthogonality Proofs

Let's think of orthogonality not just in terms of dot products, but in terms of complex dot products with the left item complex conjugated, and integrated over all solid angle $d\Omega$. A special "inner product".

Y with Y. Orthogonality of the **Y**'s is obvious from the ψ's, since $\hat{\mathbf{r}}\bullet\hat{\mathbf{r}} = 1$. That is, we know that our choice of $\psi = Y_{\ell m}(\theta,\phi)$ precisely have this orthonormality, namely (stated as (5) in the paper)

$$\int d\Omega\;\hat{\mathbf{r}}\;Y_{\ell'm'}^{*}(\theta,\phi)\bullet\hat{\mathbf{r}}\;Y_{\ell m}(\theta,\phi) = \int d\Omega\;Y_{\ell'm'}^{*}(\theta,\phi)\;Y_{\ell m}(\theta,\phi) = \delta_{\ell',\ell}\;\delta_{m',m}$$

X with X. Orthogonality of the **X**'s requires the following work: (there is probably an easier way, see below)

$$\mathbf{L} = \hat{\mathbf{x}}\,L_x + \hat{\mathbf{y}}\,L_y + \hat{\mathbf{z}}\,L_z$$

Define the following

$$\boldsymbol{\varepsilon}_{\pm} = \hat{\mathbf{x}} \pm i\,\hat{\mathbf{y}} \qquad \Rightarrow \qquad \boldsymbol{\varepsilon}_{\pm}^{*}\bullet\boldsymbol{\varepsilon}_{\pm} = 2 \qquad \text{and} \qquad \boldsymbol{\varepsilon}_{\pm}^{*}\bullet\boldsymbol{\varepsilon}_{\mp} = 0$$

$$L_{\pm} = L_x + L_y$$

You then find that

$$\mathbf{L} = (1/2i)[\ \boldsymbol{\varepsilon}_+ L_+ + \boldsymbol{\varepsilon}_- L_-] \ + \hat{\mathbf{z}}\, L_z$$

and then

$$(\mathbf{L}\ \ )^* \bullet (\mathbf{L}\ \ ) = (1/2)\ \{\ L_+^*\ \ L_+\ + \ L_-^*\ \ L_-\ ) + L_3^*\ \ L_3$$

where I have left space for you to write in $Y_{\ell'm'}$ and $Y_{\ell m}$ in the spaces. The two raising or two lowering operators do the same change to the two Y functions, so the final delta form is not changed, but you pick up the square of the raising algebra coefficients Jackson shows on page 542. Then added to the last squared coefficient m^2 you get $\ell(\ell+1)$, as Carleton shows in his (4). So,

$$\int d\Omega\ \mathbf{L}Y_{\ell'm'}^*(\theta,\phi) \bullet \mathbf{L}\ Y_{\ell m}(\theta,\phi) \ = \ell(\ell+1)\ \delta_{\ell',\ell}\ \delta_{m',m}$$

Notice that this proof does not work with just any old ψ functions! Here is another proof that is more to the point:

$$\int d\Omega\ \mathbf{L}Y_{\ell'm'}^*(\theta,\phi) \bullet \mathbf{L}\ Y_{\ell m}^*(\theta,\phi) = \int d\Omega\ \langle\ell'm'\,|\boldsymbol{\mathcal{L}}|\,\theta\phi\rangle \bullet \langle\theta\phi|\boldsymbol{\mathcal{L}}|\,\ell m\rangle = \langle\ell'm'\,|\boldsymbol{\mathcal{L}} \bullet \boldsymbol{\mathcal{L}}\,|\,\ell\boldsymbol{m}\rangle$$

$$= \langle\ell'm'\,|\boldsymbol{\mathcal{L}}^2|\,\ell\boldsymbol{m}\rangle = \ell(\ell+1)\,\langle\ell'm'\,|\,\ell\boldsymbol{m}\rangle = \ \ell(\ell+1)\ \delta_{\ell',\ell}\ \delta_{m',m}$$

where $\boldsymbol{\mathcal{L}}$ is the abstract operator in the usual Hilbert space we deal with in these matters. Again, it is the Casimir operator of the group and all that stuff.

Z with Z. Finally, what about orthogonality of the Z's? I got stuck on that for a while, but then I realized this theorem

$$\int d\Omega\ (\boldsymbol{\nabla} \bullet \mathbf{f}) = 0$$

as long as $f(\theta,\phi)$ is finite at $\theta = \pm\pi$, and is periodic in ϕ. See proof on scratch. You basically treat the divergence as two terms and in each term you have a perfect differential with respect to one of the two angle variables and then the assumptions given make the result zero. Using a simple vector identity, a corollary to this theorem is

$$\int d\Omega\ (\nabla f \bullet \nabla g) \ = - \int d\Omega\ (f\, \boldsymbol{\nabla}^2 g)$$

just as if you were doing a parts integration and throwing away the parts. We know from Jackson p 543 that $-\boldsymbol{\nabla}^2 g \ = (1/r^2)\ L^2\ g$ and if g is a $Y_{\ell m}$ then L^2 becomes $\ell(\ell+1)$. This then proves orthogonality of the Z's, equation (6),

$$r^2 \int d\Omega \, \nabla Y_{\ell'm'}^*(\theta,\phi) \bullet \nabla Y_{\ell m}^*(\theta,\phi) = \ell(\ell+1)\, \delta_{\ell',\ell}\, \delta_{m',m}$$

Y with Z and Y with X. These function sets are orthogonal simply because the vectors are orthogonal from our earlier observations, namely, that

$$\mathbf{r} \bullet \nabla\, Y_{\ell m} = 0 \qquad \Rightarrow Y \text{ with } Z = 0$$
$$\mathbf{r} \bullet (\mathbf{r} \times \nabla\, Y_{\ell m}) = 0 \qquad \Rightarrow Y \text{ with } X = 0$$

Z with X. This is the one remaining combination, and it is not *trivially* true because, although

$$\nabla\, Y_{\ell m} \bullet (\mathbf{r} \times \nabla\, Y_{\ell m}) = 0$$

we have

$$\nabla\, Y_{\ell m}^* \bullet (\mathbf{r} \times \nabla\, Y_{\ell m}) \neq 0$$

Here is a proof that with the angular integral included, Z is orthogonal to X. Start with

$$\nabla \bullet [\, f \;\; \mathbf{L}g \;] = \nabla f \bullet \mathbf{L}g + f\, \nabla \bullet \mathbf{L}g \qquad \text{// just a vector identity, thinking } f = Y_{\ell m}^*$$

Integration over solid angle gives 0 for the left factor from our theorem above, assuming f and g are finite etc. The first term on the right is the one we *want* to show is 0 (it is ~ $\mathbf{Z}^* \bullet \mathbf{X}$) , so we will have proven (7) if we can show that the *second* term on the right is zero. But

$$\nabla \bullet \mathbf{L}g = \nabla \bullet (\mathbf{r} \times \nabla g) = \nabla g \bullet (\nabla \times r) - r \bullet (\nabla \times \nabla g) = 0 + 0 = 0$$

since $(\nabla \times r) = 0$ and $\nabla \times (\nabla g) = 0$, QED.

4. The Expansion Theorem

We can combine all the orthogonality conditions proven above involving the X,Y,Z into a single statement. To do this, let's redefine the vector functions as follows

$$\mathbf{X}^1_{\ell m} = k_\ell\, \mathbf{X}_{\ell m} = k_\ell\, \mathbf{r} \times \nabla\, Y_{\ell m} = i\, k_\ell\, \mathbf{L}\, Y_{\ell m}$$
$$\mathbf{X}^2_{\ell m} = \mathbf{Y}_{\ell m} = \hat{\mathbf{r}}\, Y_{\ell m}$$
$$\mathbf{X}^3_{\ell m} = k_\ell\, \mathbf{Z}_{\ell m} = k_\ell\, r\, \nabla Y_{\ell m}$$

where we have defined

$$k_\ell \equiv 1/\sqrt{\ell(\ell+1)}$$

Then in the previous section we have *proven* the following fact:

$$\int d\Omega \;\; \mathbf{X}^{n'}_{\ell'm'}(\theta,\phi)^* \bullet \mathbf{X}^{n}_{\ell m}(\theta,\phi) \;= \delta_{\ell',\ell}\, \delta_{m',m}\, \delta_{n',n} \qquad \text{orthogonality}$$

Furthermore, we have shown that the functions $\mathbf{X}^{n}_{\ell m}(\theta,\phi)$ as vectors in 3-space form a triad of perpendicular vectors which, if normalized to unity, would just be some rotation of the canonical unit vectors, with the specific rotation being a function of ℓ, m, θ and ϕ. Therefore, in terms of 3-space, we know that the vectors $\mathbf{X}^{n}_{\ell m}(\theta,\phi)$ clearly "span" the 3-space. Any vector could be written as a linear combination of these three vectors.

Now, we want to make this conjecture: any vector function of θ and ϕ can be expanded as follows:

$$\mathbf{A}(\theta,\phi) \;=\; \Sigma_{\ell mn} \;\; A_{\ell mn}\, \mathbf{X}^{n}_{\ell m}(\theta,\phi) \qquad (1)$$

The radial projection of this equation is,

$$\hat{\mathbf{r}} \bullet \mathbf{A}(\theta,\phi) \;= A_r(\theta,\phi) = \;\; \Sigma_{\ell mn} \;\; A_{\ell mn}\, (\hat{\mathbf{r}} \bullet \mathbf{X}^{n}_{\ell m}(\theta,\phi)) \;= \Sigma_{\ell m} \;\; A_{\ell m2}\, Y_{\ell m}(\theta,\phi)$$

and we know from scalar theory that this is in fact a "complete" expansion for the scalar function $A_r(\theta,\phi)$. So at least the conjecture seems possibly true. The orthogonality makes us think it is true! At least within each partial wave, we can see that we are spanning the full 3-vector space with our three basis functions.

If we assume the above expansion is true, we can use the orthonormality condition as follows. Start with the expansion,

$$\mathbf{A}(\theta,\phi) \;=\; \Sigma_{\ell mn} \;\; A_{\ell mn}\, \mathbf{X}^{n}_{\ell m}(\theta,\phi)$$

Peruse the orthonormality,

$$\int d\Omega \;\; \mathbf{X}^{n'}_{\ell'm'}(\theta,\phi)^* \bullet \mathbf{X}^{n}_{\ell m}(\theta,\phi) \;= \delta_{\ell',\ell}\, \delta_{m',m}\, \delta_{n',n}$$

Apply integral operator to both sides

$$\int d\Omega\, \mathbf{X}^{n'}_{\ell'm'}(\theta,\phi)^* \bullet \mathbf{A}(\theta,\phi) \;=\; \Sigma_{\ell mn} \;\; A_{\ell mn} \int d\Omega \;\; \mathbf{X}^{n'}_{\ell'm'}(\theta,\phi)^* \bullet \mathbf{X}^{n}_{\ell m}(\theta,\phi)$$

$$= \Sigma_{\ell mn} \;\; A_{\ell mn}\, \delta_{\ell',\ell}\, \delta_{m',m}\, \delta_{n',n} = A_{\ell'm'n'}$$

And there is our projection,

$$A_{\ell'm'n'} = \int d\Omega\, \mathbf{X}^{n'}_{\ell'm'}{}^{*}(\theta,\phi) \bullet \mathbf{A}(\theta,\phi)$$

I have proven nothing of course, but it does seem reasonable. Here is the implied completeness condition:

$$\Sigma_{\ell mn} \;\; [\mathbf{X}^{n}_{\ell m}{}^{*}(\theta,\phi)]_i \; [\; \mathbf{X}^{n}_{\ell m}(\theta',\phi') \;]_{i'} = \delta_{i,i'} \;\; \langle \theta\phi | \theta'\phi' \rangle = \delta_{i,i'} \;\; \delta(\Omega - \Omega')$$

which would be called an addition theorem for the X's. If we could prove this directly, I think our expansion conjecture would then be justified. However, I think the proof would be very difficult. The usual proof is to show completeness some other way, then the above has to be true!

In a different notation we might say,

$\langle n\ell m|i\theta\phi\rangle \langle i\theta\phi|n'\ell'm'\rangle = \langle n\ell m|n'\ell'm'\rangle$ orthogonality

$\langle i\theta\phi|n\ell m\rangle\langle n\ell m|'i'\theta'\phi'\rangle = \langle i\theta\phi|'i'\theta'\phi'\rangle$ completeness

Finally, if $A = A(r,\theta,\phi)$, treat r as a parameter being carried along, and we then get

$$\mathbf{A}(r,\theta,\phi) = \Sigma_{\ell mn}\ A_{\ell mn}(r)\ \mathbf{X}^{n}_{\ell m}(\theta,\phi)$$

$$A_{\ell mn}(r) = \int d\Omega\ \mathbf{X}^{n}_{\ell m}(\theta,\phi)^{*} \bullet \mathbf{A}(r,\theta,\phi)$$

This is really a multipole or partial wave expansion of a vector field, and we are going to apply it to the electric and magnetic fields soon. What we are doing is similar to Jackson Chapter 16 but maybe a little more organized. We shall find that we can deal with each partial wave separately and thereby simplify our attack on certain problems.

5. Apply divergence to the expansion

In these notes, I mistakenly used the symbol τ instead of r as the label for the coefficient of the *radial* function Y. I then stayed consistent with that use. The two transverse vectors Z and X get labels 1 and 2.

Since I want to check things against Carleton's writeup, let's use his form of the expansion

$$\mathbf{A} = \Sigma_{\ell m}\ [\ A_2\,\mathbf{X}\ + A_\tau\,\mathbf{Y}\ + A_1\,\mathbf{Z}\ ]$$

$$\nabla \bullet \mathbf{A} = \Sigma_{\ell m}\ [\nabla \bullet (A_2\,\mathbf{X})\ + \nabla \bullet (A_\tau\,\mathbf{Y})\ + \nabla \bullet (A_1\mathbf{Z})\ ] = \Sigma_{\ell m}\ [\ 1 + 2 + 3\ ]$$

From our appendix we have

$\nabla \bullet \mathbf{X}_{\ell m} = 0$	$\hat{\mathbf{r}} \bullet \mathbf{X}_{\ell m} = 0$
$\nabla \bullet \mathbf{Y}_{\ell m} = (2/r)\ Y_{\ell m}$	$\hat{\mathbf{r}} \bullet \mathbf{Y}_{\ell m} = Y_{\ell m}$
$\nabla \bullet \mathbf{Z}_{\ell m} = -\ \ell(\ell+1)\ (1/r)\ Y_{\ell m}$	$\hat{\mathbf{r}} \bullet \mathbf{Z}_{\ell m} = 0$

So we get

$$1 = \nabla \bullet (A_2\,\mathbf{X}) = \nabla A_2 \bullet \mathbf{X} + A_2\,\nabla \bullet \mathbf{X} = \partial_r\,A_2\,(\,\hat{\mathbf{r}} \bullet \mathbf{X}\,) + A_2\,(\nabla \bullet \mathbf{X}\,)$$
$$= \partial_r\,A_2\,(\,\hat{\mathbf{r}} \bullet \mathbf{X}\,) + A_2\,(\nabla \bullet \mathbf{X}\,) = \partial_r\,A_2\,(\,0\,) + A_2\,(0\,) = 0$$

$$2 = \nabla \bullet (A_\tau \mathbf{Y}) = \partial_r A_\tau (\hat{\mathbf{r}} \bullet \mathbf{Y}) + A_\psi (\nabla \bullet \mathbf{Y}) = (\partial_r A_\tau) Y_{\ell m} + A_\tau (2/r) Y_{\ell m}$$
$$= 1/r^2 \partial_r (r^2 A_\tau) Y_{\ell m}$$

$$3 = \nabla \bullet (A_1 \mathbf{Z}) = \partial_r A_1 (\hat{\mathbf{r}} \bullet \mathbf{Z}) + A_1 (\nabla \bullet \mathbf{Z}) = \partial_r A_1 (\mathbf{0}) + A_1 (- \ell(\ell+1) (1/r) Y_{\ell m})$$
$$= - A_1 \ell(\ell+1) (1/r) Y_{\ell m}$$

So here is the result

$$\mathbf{A} = \Sigma_{\ell m} [A_2 \mathbf{X} + A_\tau \mathbf{Y} + A_1 \mathbf{Z}]$$

$$\nabla \bullet \mathbf{A} = \Sigma_{\ell m} [\nabla \bullet (A_2 \mathbf{X}) + \nabla \bullet (A_\tau \mathbf{Y}) + \nabla \bullet (A_1 \mathbf{Z})]$$

$$= \Sigma_{\ell m} \{ 1/r^2 \partial_r (r^2 A_\tau) - A_1 \ell(\ell+1) (1/r) \} Y_{\ell m}$$ which agrees with (21)

Now, suppose someone imposes this requirement,

$$\nabla \bullet \mathbf{A} = 4\pi\rho$$

Then this really must be satisfied in each partial wave, so we find that

$$\{ 1/r^2 \partial_r (r^2 A_\tau) - A_1 \ell(\ell+1) (1/r) \} = 4\pi\rho_{\ell m}$$

Thus, this divergence condition (perhaps $\nabla \bullet \mathbf{B} = 0$ for a magnetic field, or $\nabla \bullet \mathbf{E} = \rho$ for E field) puts no restriction at all on the function A_2 , but requires the other two functions to be related by this little DE. This begins to show how we might apply Maxwell's equations and end up with constraints on the multipole coefficients of the fields within each partial wave.

5. Apply curl to the expansion

Since I want to check things against Carleton's writeup, let's use his form of the expansion

$$\mathbf{A} = \Sigma_{\ell m} [A_2 \mathbf{X} + A_\tau \mathbf{Y} + A_1 \mathbf{Z}]$$

$$\nabla \times \mathbf{A} = \Sigma_{\ell m} [\nabla \times (A_2 \mathbf{X}) + \nabla \times (A_\tau \mathbf{Y}) + \nabla \times (A_1 \mathbf{Z})] = \Sigma_{\ell mn} [1 + 2 + 3]$$

From our appendix we have

$$\nabla \times \mathbf{X}_{\ell m} = - [\ell(\ell+1)/r] \mathbf{Y}_{\ell m} - (1/r) \mathbf{Z}_{\ell m} \qquad \hat{\mathbf{r}} \times \mathbf{X}_{\ell m} = - \mathbf{Z}_{\ell m}$$
$$\nabla \times \mathbf{Y}_{\ell m} = - (1/r) \mathbf{X}_{\ell m} \qquad \hat{\mathbf{r}} \times \mathbf{Y}_{\ell m} = 0$$
$$\nabla \times \mathbf{Z}_{\ell m} = (1/r) \mathbf{X}_{\ell m} \qquad \hat{\mathbf{r}} \times \mathbf{Z}_{\ell m} = \mathbf{X}_{\ell m}$$

So we get

$$1 = \nabla \times (A_2\,\mathbf{X}) = \nabla A_2 \times \mathbf{X} + A_2\,\nabla \times \mathbf{X} = \partial_r A_2\,(\hat{\mathbf{r}} \times \mathbf{X}) + A_2\,(\nabla \times \mathbf{X})$$
$$= \partial_r A_2\,(-\mathbf{Z}_{\ell m}) + A_2\,(-[\ell(\ell+1)/r]\,\mathbf{Y}_{\ell m} - (1/r)\,\mathbf{Z}_{\ell m})$$
$$= -[\partial_r A_2 + (1/r)]\,\mathbf{Z}_{\ell m} - A_2\,\ell(\ell+1)/r\,\mathbf{Y}_{\ell m}$$
$$= -(1/r)\partial_r (rA_2)\mathbf{Z}_{\ell m} - A_2\,\ell(\ell+1)/r\,\mathbf{Y}_{\ell m}$$

$$2 = \nabla \times (A_\tau\,\mathbf{Y}) = \partial_r A_\tau\,(\hat{\mathbf{r}} \times \mathbf{Y}) + A_\tau\,(\nabla \times \mathbf{Y}) = (\partial_r A_\tau)\;0 + A_\tau(-1/r)\,\mathbf{X}_{\ell m}$$
$$= -A_\tau(1/r)\,\mathbf{X}_{\ell m}$$

$$3 = \nabla \times (A_1\mathbf{Z}) = \partial_r A_1\,(\hat{\mathbf{r}} \times \mathbf{Z}) + A_1\,(\nabla \times \mathbf{Z}) = \partial_r A_1\,(\mathbf{X}_{\ell m}) + A_1\,(1/r)\,\mathbf{X}_{\ell m}$$
$$= [\partial_r A_1 + A_1/r]\,\mathbf{X}_{\ell m} = (1/r)(\partial_r (r\,A_1)\,\mathbf{X}_{\ell m}$$

So here is the result

$$\mathbf{A} = \Sigma_{\ell m}\,[\,A_2\,\mathbf{X} + A_\tau\,\mathbf{Y} + A_1\,\mathbf{Z}\,]$$

$$\nabla \times \mathbf{A} = \Sigma_{\ell m}\,[\nabla \times (A_2\,\mathbf{X}) + \nabla \times (A_\tau\,\mathbf{Y}) + \nabla \times (A_1\mathbf{Z})\,]$$

$$= \Sigma_{\ell m}\,\{\; -(1/r)\partial_r (rA_2)\,\mathbf{Z}_{\ell m} - \ell(\ell+1)/r\,A_2\mathbf{Y}_{\ell m} + [(1/r)(\partial_r (r\,A_1) - A_\tau/r]\,\mathbf{X}_{\ell m}\;\}$$
$$\qquad\qquad 1 \qquad\qquad\qquad \tau \qquad\qquad\qquad\qquad 2$$

and amazingly enough, this exactly agrees with (22)-(25).

Now, suppose someone imposes this requirement,

$$\nabla \times \mathbf{A} = \mathbf{0}$$

Then this really must be satisfied in each partial wave, and for each component! So we get

$-(1/r)\partial_r (rA_2) = 0$	=> nothing since $A_2 = 0$ from next equation
$-\ell(\ell+1)/r\,A_2 = 0$	=> $A_2 = 0$
$(1/r)(\partial_r (r\,A_1) - A_\tau/r = 0$	=> $\partial_r (r\,A_1) = A_\tau$

So with this condition, in each partial wave we have knocked out the A_2 component! And the other two components are related by this simple differential equation!

6. Doing Electrostatics with VSH

We have our two equations already noted, repeat them here:

$$\nabla \bullet \mathbf{E} = 4\pi\rho$$

which implies

$$\{\ 1/r^2\,\partial_r(\ r^2\ E_\tau\ )\ -\ E_1\ \ell(\ell+1)\ (1/r)\ \}\ =4\pi\rho_{\ell m} \qquad \text{(a)} \qquad \text{verifies (31)}$$

and

$$\nabla \times \mathbf{E}\ =\mathbf{0}$$

which implies

$$-\ \ell(\ell+1)/r\ E_2 = 0 \qquad \Rightarrow\ E_2 = 0$$
$$(1/r)(\partial_r(r\ E_1)-\ E_\tau/r = 0 \qquad \Rightarrow \partial_r(r\ E_1) = E_\tau \qquad \text{(b)}$$

The two DE's both involved E_1 and E_τ. If we combine these we get (32) as a single DE for E_1, and if we then define $\phi = -\ r\ E_1$ then equation (31) reads as follows:

$$\partial_r^2\ \phi\ +(2/r)\ \partial_r\ \phi\ -\ \ell(\ell+1)/r^2\phi = -\ 4\pi\rho_{\ell m}$$

Compare this to Jackson 16.5 with $k^2 = 0$ and a source added, and you see that we have discovered our friend the Poisson equation where the "potential" seems to be $\phi = -\ r\ E_1$. So think of this as saying that we solve the Poisson equation for ϕ, and then we have $E_1 = -\phi/r$, $E_\tau\ = -\partial_r\ \phi$, $E_2 = 0$. Then our E field must be,

$$\mathbf{E}\ =\ \Sigma_{\ell m}\ [\ E_\tau\ \mathbf{Y}\ \ +E_1\ \mathbf{Z}\ ]\ \ =\Sigma_{\ell m}\ [-\partial_r\ \phi\ \mathbf{Y}\ -\ \phi/r\ \ \mathbf{Z}\ ]\ =$$
$$-\ \Sigma_{\ell m}\ [\ \partial_r\ \phi_{\ell m}\ \hat{\mathbf{r}}\ \ Y_{\ell m}\ +\ \phi_{\ell m}\ \nabla Y_{\ell m}\ ]$$

where now we have shown the subscripts on ϕ which we were too lazy to add before. Do we recognize this solution ? Go to Jackson (4.1) on the multipole expansion for ϕ, and assume we have found some solution

$$\phi(r,\theta,\phi) = \Sigma_{\ell m}\ \ (4\pi/(2\ell+1))\ q_{\ell m}\ 1/r^{\ell+1}\ Y_{\ell m}\ = \Sigma_{\ell m}\ \phi_{\ell m}(r)\ Y_{\ell m}(\theta.\phi)$$

where

$$\phi_{\ell m}(r)\ =\ (4\pi/(2\ell+1))\ q_{\ell m}\ 1/r^{\ell+1}$$

Then the electric field should be

$$E = -\nabla\phi\ = -\ \Sigma_{\ell m}\ \ \nabla[\ \phi_{\ell m}(r)\ Y_{\ell m}(\theta.\phi)\ ] = -\ \Sigma_{\ell m}\ \{\ \nabla[\ \phi_{\ell m}(r)]\ Y_{\ell m}(\theta.\phi)\ \ +\phi_{\ell m}(r)\ \nabla\ Y_{\ell m}(\theta.\phi)\ \}$$

$$= -\ \Sigma_{\ell m}\ \{\ \partial_r\phi_{\ell m}(r)\ \hat{\mathbf{r}}\ \ Y_{\ell m}(\theta.\phi)\ \ +\phi_{\ell m}(r)/r\ \ \ r\nabla\ Y_{\ell m}(\theta.\phi)\ \}$$

$$= -\ \Sigma_{\ell m}\ \{\ \partial_r\phi_{\ell m}(r)\ \mathbf{Y}\ +\ \phi_{\ell m}(r)/r\ \ \mathbf{Z}\ \}$$

which agrees with our VSH result.

7. Doing Magnetostatics with VSH

Before looking at Maxwell's here, (35) is just the electrostatic expansion restated for magnetic quantities, including the magnetic potential. If we set $\mathbf{B} = -\nabla\phi$ we instantly end up with (36) through (38), no problem here. Now, let's look at Maxwell's equations.

We have our two equations already noted, repeat them here:

$$\nabla \bullet \mathbf{B} = 0$$

which implies

$$\{ 1/r^2\, \partial_r (r^2 B_\tau) - B_1\, \ell(\ell+1)\, (1/r) \} = 0 \qquad \text{see (42)}$$

and

$$\nabla \times \mathbf{B} = 4\pi/c\, \mathbf{J}$$

which implies

$$-(1/r)\partial_r (rB_2) = 4\pi/c\, J_1 \qquad / Z \qquad (40)$$
$$-\ell(\ell+1)/r\, B_2 = 4\pi/c\, J_\tau \qquad / Y \qquad (39)$$
$$(1/r)(\partial_r (r\, B_1) - B_\tau/r = 4\pi/c\, J_2 \qquad / X \qquad (41)$$

The two DE's both involved B_1 and B_τ. I have combined these to get (43) for B^τ driven by J^2.

Now derivation of (44) requires a bit of a digression! Jackson (3.120) shows the definition of the Green's function for this equation and the form of the solution. I abbreviate as follows:

$$\text{op}[g] = -4\pi/r^2\, \delta(r-r') \qquad (3.120)$$

Then if you have

$$\text{op}[F] = s$$

the solution will be

$$F(r) = -\int_0^a g(r,r')\, s(r')\, (r'^2/4\pi)\, dr'$$

as you know by applying op to both sides. We show the source function s as localized. For r outside this region, we have $r > r'$ and we know that $g(r,r') = B'(r')/r^{\ell+1}$. Similarly, if we look at very small r near r=0, we conclude that for $r < r'$ we have $g(r,r') = A(r')r^\ell$. From the required symmetry of g, this tells us that

$$g(r,r') = C\,[\ r_<^{\ell}\ ]\ [\ 1 / r_>^{\ell+1}\ ] \qquad = 4\pi/(2\ell+1)\ \ r_<^{\ell} / r_>^{\ell+1}$$

We determine the constant according to page 80 and we find that $C = 4\pi/(2\ell+1)$ as shown above. The solution of our posed problem is then

$$F(r) = -\ 1/(2\ell+1)\ \ *\ 1/r^{\ell+1} * \int_0^a s(r')\ r'^{\ \ell+2}\ dr'$$

Now in our particular problem we have $s(r) = 4\pi/c\ \ \ell(\ell+1)\ J_2(r)$ and $F = rB_\tau$ and this gives (44) !!

So the point is that we did our VHS thing for this magnetic problem and we ended up with an equation (43) for B_τ driven by J_2 and we have solved it in (44) for B_τ. Comparing (44) to (36) gives

$$\mu_{\ell m} = -\ell/c \int_0^a J_{2\ell m}(r')\ r'^{\ \ell+2}\ dr'$$

but of course we are still in ℓm partial waves here. We can use (11) to replace $J_{2\ell m}(r')$ with the projection integral, and this then gives (45) which can then be rewritten easily as (46).

This result (46) claims to tell us the magnetic moment of a current distribution. In Jackson this appears in (16.95), but the forms don't exactly match. Here is how we fix things up. Consider:

$$\nabla \bullet [\ r^{\ell}Y_{\ell m}\ \ \mathbf{r} \times \mathbf{J}\ ]\ \ = \mathbf{r} \times \mathbf{J} \bullet \nabla\ [r^{\ell}Y_{\ell m}]\ \ + r^{\ell}\ Y_{\ell m} \nabla \bullet (\mathbf{r} \times \mathbf{J})\ \ = 1 + 2$$

$$1 = \mathbf{r} \times \mathbf{J} \bullet \nabla\ (r^{\ell})\ Y_{\ell m}\ \ + \mathbf{r} \times \mathbf{J} \bullet r^{\ell}\ \nabla Y_{\ell m} = 0 + r^{\ell}\ \nabla Y_{\ell m} \bullet \mathbf{r} \times \mathbf{J}$$

So we get

$$\nabla \bullet [\ r^{\ell}Y_{\ell m}\ \ \mathbf{r} \times \mathbf{J}\ ]\ \ = \ r^{\ell}\ Y_{\ell m} \nabla \bullet (\mathbf{r} \times \mathbf{J})\ \ + r^{\ell}\ \nabla Y_{\ell m} \bullet \mathbf{r} \times \mathbf{J}$$

In (16.96 a) the divergence integral will vanish, so we get 0 = Jackson + Carleton. The only disagreement is a - sign (both have minus signs). It is a little hazy how Jackson defined his moments, perhaps this minus arises because a_M has a minus sign in 16.47.

Ignoring this minor detail, Carleton's exercise here was to consider magnetostatics, use a potential to define the moments, and then solve for the B field using the VSH and do a comparison to end up with a classic result for the moments.

8. Maxwell's Equations with VSH.

We list off the four fully coupled standard Maxwell equations with sources, and then we write down all 8 VSH equations. These follow trivially from my div and curl results above, all are correct. I then did exactly what Carleton said, and got the two driven wave equations shown in (59) and (60). But we hold now to do another exercise section.

9. Fields in conducting cavity.

(1) Duplicate stuff from above, but set to **B** and replace ∇ with $\hat{\mathbf{r}}$ since we want $\hat{\mathbf{r}} \bullet \mathbf{B} = 0$

$$\mathbf{B} = \Sigma_{\ell m} \left[B_2 \mathbf{X} + B_\tau \mathbf{Y} + B_1 \mathbf{Z} \right]$$

$$\hat{\mathbf{r}} \bullet \mathbf{B} = \Sigma_{\ell m} \left[\hat{\mathbf{r}} \bullet (B_2 \mathbf{X}) + \hat{\mathbf{r}} \bullet (B_\tau \mathbf{Y}) + \hat{\mathbf{r}} \bullet (B_1 \mathbf{Z}) \right] = \Sigma_{\ell m} B_\tau Y_{\ell m}$$
$$\Rightarrow B_\tau = 0$$

using

$$\nabla \bullet \mathbf{X}_{\ell m} = 0 \qquad \hat{\mathbf{r}} \bullet \mathbf{X}_{\ell m} = 0$$
$$\nabla \bullet \mathbf{Y}_{\ell m} = (2/r)\, Y_{\ell m} \qquad \hat{\mathbf{r}} \bullet \mathbf{Y}_{\ell m} = Y_{\ell m}$$
$$\nabla \bullet \mathbf{Z}_{\ell m} = -\ell(\ell+1)\,(1/r)\, Y_{\ell m} \qquad \hat{\mathbf{r}} \bullet \mathbf{Z}_{\ell m} = 0$$

(2) Duplicate stuff from above, but set to **E** and replace ∇ with $\hat{\mathbf{r}}$ since we want $\hat{\mathbf{r}} \times \mathbf{E} = 0$

$$\mathbf{E} = \Sigma_{\ell m} \left[E_2 \mathbf{X} + E_\tau \mathbf{Y} + E_1 \mathbf{Z} \right]$$

$$\hat{\mathbf{r}} \times \mathbf{E} = \Sigma_{\ell m} \left[\hat{\mathbf{r}} \times (E_2 \mathbf{X}) + \hat{\mathbf{r}} \times (E_\tau \mathbf{Y}) + \hat{\mathbf{r}} \times (E_1 \mathbf{Z}) \right] = \Sigma_{\ell mn} \left[-E_2 \mathbf{Z}_{\ell m} + E_1 \mathbf{X}_{\ell m} \right]$$
$$\Rightarrow E_2 = E_1 = 0$$

using

$$\nabla \times \mathbf{X}_{\ell m} = -\left[\ell(\ell+1)/r\right] \mathbf{Y}_{\ell m} - (1/r)\, \mathbf{Z}_{\ell m} \qquad \hat{\mathbf{r}} \times \mathbf{X}_{\ell m} = -\mathbf{Z}_{\ell m}$$
$$\nabla \times \mathbf{Y}_{\ell m} = -(1/r)\, \mathbf{X}_{\ell m} \qquad \hat{\mathbf{r}} \times \mathbf{Y}_{\ell m} = 0$$
$$\nabla \times \mathbf{Z}_{\ell m} = (1/r)\, \mathbf{X}_{\ell m} \qquad \hat{\mathbf{r}} \times \mathbf{Z}_{\ell m} = \mathbf{X}_{\ell m}$$

So, the boundary conditions for either a spherical conducting cavity or a spherical conducting sphere are very simple:

$$\hat{\mathbf{r}} \bullet \mathbf{B} = 0 \quad \Rightarrow B_\tau(a) = 0$$

$$\hat{\mathbf{r}} \times \mathbf{E} = 0 \quad \Rightarrow E_2(a) = E_1(a) = 0$$

where these fields are evaluated at the surface of the sphere, r = a. Certainly a simple result!

Now if there are no sources, we are just looking for stand-alone solutions that can exist, they refer to these as resonances. They will of course have surface currents and charges, but these are not the "free" sources implied by the equations.

Now we are going to "look for solutions" of the 8 equations (51)-(58) subject to the boundary conditions (61), everything is sourceless. Recall that E_r and B_r refer to RADIAL components of these fields. For each field, we know that the other two components are TRANSVERSE. So NOW finally we

are seeing the meaning of the indices 1,2 and r -- better late than never. 1 and 2 are the two transverse directions that we associate with the functions Z and X.

Our equations divide into the α group and the β group as shown.

Suppose in the α group we look for solutions with $E_\tau = 0$. These would be "transverse electric" or TE, since there is no radial E in a solution of this sort. In this case, looking at the α group, we get $E_1 = 0$ from (51) and then $B_2 = 0$ from (55) . Then equation (56) and (57) are both 0 = 0 and add nothing new. The surviving fields are E_2 and B_1 and B_τ which are all in the β group. So the TE solution means that all three fields in the α group are null everywhere.

TE = magnetic:	α: $E_\tau = E_1 = B_2 = 0$	β: B_τ , B_1, E_2 all active
TM = electric:	β: $B_\tau = B_1 = E_2 = 0$	α: E_τ , E_1, B_2 all active

For TE, to find the non-zero β fields, we see that E_2 must solve (60), and for the cavity case, it has to be a $j_\ell(kr)$. Notice that of the three conditions (61), the middle one is in the α group, the other two are in the β. So for our TM solution in the cavity which involves β fields only, the condition $E_2(a) = 0$ implies $j_\ell(ka) = 0$ which gives the k mode eigenvalues from the zeros of this function. But we also have that $B_\tau(a) = 0$ to think about, but equation (53) shows that this is satisfied at once if $E_2(a) = 0$.

Suppose we look for TM solutions with the whole α group being zero. Again, just assuming $B_\tau = 0$ forces $E_2 = 0$ from (53) and then $B_1 = 0$ from 54. In this case, we take in the α group that B_2 is a $j_\ell(kr)$ for the same reason, finite at the origin. But now the BC is that $E_1(a) = 0$ and from (57) this puts the condition that $\partial_r (r\, B_2(a)) = 0 = \partial_r (r\, j_\ell(kr))|^{r=a}$ as claimed in (65).

So from this little exercise, we see the power of the entire formalism here. We have the two decoupled equation sets which we can think of completely independently, this is why we have the famous TE and TM "modes" in all these problems, I never quite understood that before. In Jackson page 543, he produced two 3-equation sets but then he had to do some arm-waving on the next page.

Solving the conducting cavity problem was easy! The boundary conditions were trivial, we thought about each of the two modes, and for each mode were able to write down "by inspection" the solution for the primary active field, either E_2 or B_2. These are the **X** or $\mathbf{L}\psi$ components, really $E_{2\ell m}(r)$ keep in mind. These are the primary E and M fields in Jackson's 16.47 grand finale. Now in either case, once you have the primary field, the only other existing fields are of the other type. For example, in TE you find that E_2 is your Bessel function, and then the only other existing fields are B_1 and B_τ . These are trivial to write down using (53) and (54). And similarly for the other case. So again, once you go into partial waves, this spherically symmetric problem is trivial to solve, and you do get eigenvalues for k. This is a very excellent problem to consider.

10. Radiation from a Localized Source

So now we put the sources back in and we try to solve (59), (60) using the Green's Function method. From Jackson we know that the Green's is (68), and this gives solution (69). You can always verify a result like this by applying op[] to both sides which I just did. I modified (69) with an upper endpoint on the integral, but now see not necessary.

Now you have to recognize the J-function there as being the 2-coefficient (the X coefficient) in the expansion you could get for ∇ x J. You could then use the inversion formula for this coefficient as I have shown after equation (69). Inserting this inversion into (69) then gives the result (70) with (72). And (71) is trivial from (50).

Stop! We have now derived the standard Jackson formula for an electric or TM partial wave multipole field! We are reminded that it is called electric since ρ appears in the coefficient.

I just spent about 4 hours deriving Jackson's identity (16.90). It took 5 pages, and I made about 5 mistakes along the way, each of which had to be identified and corrected. Very painful, retain in notes. So I am now happy with Carleton's (72) and (73). You have to realize that both Carleton's X and his a_E are scaled relative to Jackson.

Now look back at (59) and (60). We just used the Green's method to solve (59) for the TM modes and our result was (70)-(73). Solving (60) instead for the TE mode is simpler since the J structure is more straightforward. In analogy with (69) we get the square current bracket replaced with [-ikr J^2] (which cancels the first 1/r' factor) , and of course B^2 is replaced with E^2. Therefore, our result will be (72) with curl J replaced by -ik**J** , and this gives precisely (76).

I just read the trailing part of the handout on dipoles and quadrupole and long λ limits and Cartesian moments and all that, it is fine, but this is not what interests me in this handout, so no notes.

Done! Now I would like to write a "summary" of what all happened here! Forest for trees. Amazingly it took me 18 pages of notes to digest this little handout which is only 12 pages!

Appendix 1: Horrendous operations on the Basis Functions

$$\mathbf{r} \bullet \mathbf{X}^1_{\ell m} \sim \hat{\mathbf{r}} \bullet \mathbf{X}_{\ell m} \sim \hat{\mathbf{r}} \bullet \mathbf{L}\, Y_{\ell m} \sim \mathbf{r} \bullet (\mathbf{r} \times \nabla)\, Y_{\ell m} = 0$$

$$\mathbf{r} \bullet \mathbf{X}^2_{\ell m} = \mathbf{r} \bullet \mathbf{Y}_{\ell m} = \mathbf{r} \bullet \hat{\mathbf{r}}\, Y_{\ell m} = r\, Y_{\ell m}$$

$$\mathbf{r} \bullet \mathbf{X}^3_{\ell m} = \sim \mathbf{r} \bullet \mathbf{Z}_{\ell m} = \sim \mathbf{r} \bullet \nabla Y_{\ell m} \sim r\, \nabla_r\, Y_{\ell m} = 0 \text{ because } Y_{\ell m} \text{ is not a function of r}$$

$$\mathbf{r} \times \mathbf{X}^1_{\ell m} = k_\ell\, \mathbf{r} \times \mathbf{X}_{\ell m} = k_\ell\, \mathbf{r} \times \mathbf{r} \times \nabla Y_{\ell m} = k_\ell\, [\, (\mathbf{r} \bullet \nabla Y_{\ell m})\, \mathbf{r} - r^2 \nabla Y_{\ell m}\,] = -\, k_\ell\, r^2\, \nabla Y_{\ell m}$$
$$= -\, k_\ell\, r\, \mathbf{Z}_{\ell m} = -\, r\, \mathbf{X}^3_{\ell m}$$

$$\mathbf{r} \times \mathbf{X}^2_{\ell m} = \mathbf{r} \times \mathbf{Y}_{\ell m} = \mathbf{r} \times \hat{\mathbf{r}}\, Y_{\ell m} = 0$$

$$\mathbf{r} \times \mathbf{X}^3_{\ell m} = k_\ell\, \mathbf{r} \times \mathbf{Z}_{\ell m} = k_\ell\, \mathbf{r} \times (r\, \nabla Y_{\ell m}) = k_\ell r\, \mathbf{r} \times \nabla Y_{\ell m} = k_\ell r\, \mathbf{X}_{\ell m} = r\, \mathbf{X}^1_{\ell m}$$

$$\nabla \bullet \mathbf{X}^1_{\ell m} = k_\ell\, \nabla \bullet \mathbf{X}_{\ell m} = k_\ell\, \nabla \bullet (\mathbf{r} \times \nabla\, Y_{\ell m}) = k_\ell\, [\nabla\, Y_{\ell m} \bullet (\nabla \times \mathbf{r}) - r \bullet (\nabla \times \nabla Y_{\ell m})] = 0$$

$$\nabla \bullet \mathbf{X}^2_{\ell m} = \nabla \bullet \mathbf{Y}_{\ell m} = \nabla \bullet (\hat{\mathbf{r}}\, Y_{\ell m}) = [\nabla\, Y_{\ell m} \bullet \hat{\mathbf{r}} + Y_{\ell m} (\nabla \bullet \hat{\mathbf{r}})] = Y_{\ell m}(2/r) = (2/r)\, Y_{\ell m}$$

$$\nabla \bullet \mathbf{X}^3_{\ell m} = k_\ell\, \nabla \bullet \mathbf{Z}_{\ell m} = k_\ell\, \nabla \bullet (r\, \nabla Y_{\ell m}) = k_\ell\, [\, \nabla r \bullet \nabla Y_{\ell m} + r\, (\nabla \bullet \nabla Y_{\ell m})] = r\, k_\ell\, \nabla^2\, Y_{\ell m}$$
$$= -\, k_\ell\, \ell(\ell+1)\, (1/r)\, Y_{\ell m} = -\, (1/\, r\, k_\ell\,)\, Y_{\ell m}$$

$\nabla \times \mathbf{X^1_{\ell m}} = k_\ell \nabla \times \mathbf{X}_{\ell m} = k_\ell \nabla \times (\mathbf{r} \times \nabla Y_{\ell m}) = ?$ // This is a harder one to do, to wit: $f = f(\theta,\phi)$

$\nabla \times (\mathbf{r} \times \nabla f) = \mathbf{r}\ (\nabla \bullet \nabla f) - \nabla f\ (\nabla \bullet \mathbf{r}) + (\nabla f \bullet \nabla)\ \mathbf{r} - (\mathbf{r} \bullet \nabla)\nabla f \quad = 1 + 2 + 3 + 4$

$1 = \mathbf{r}\nabla^2 f$ any f
$2 = -3\nabla f$ any f
$3 = \partial_i f\ \partial_i\ r_j = \partial_i f\ \delta_{ij} = \partial_j f = \nabla f$ any f
$4 = -r_i \partial_i\ \partial_j f = -r_i\ \partial_j \partial_i f = -\partial_j (r_i\ \partial_i f) + (\partial_j r_i)(\partial_i f)$
$= -\partial_j (\mathbf{r} \bullet \nabla f) + \partial_j f = 0 + \nabla f = \nabla f$ true only for $f = f(\theta,\phi)$

So $\nabla \times (\mathbf{r} \times \nabla f) = \mathbf{r}\nabla^2 f - \nabla f$ and for $f = Y_{\ell m}$ get $= -[\ \ell(\ell+1)/r\ ]\ Y_{\ell m}\ \hat{\mathbf{r}} - \nabla\ Y_{\ell m}$ so repeat:

$\nabla \times \mathbf{X^1_{\ell m}} = k_\ell \nabla \times \mathbf{X}_{\ell m} = k_\ell \nabla \times (\mathbf{r} \times \nabla\ Y_{\ell m}) = k_\ell \{ -[\ell(\ell+1)/r\ ]\ Y_{\ell m}\ \hat{\mathbf{r}} - \nabla\ Y_{\ell m} \}$
$= k_\ell \{ -[\ell(\ell+1)/r\ ]\ \mathbf{Y}_{\ell m} - (1/r)\ \mathbf{Z}_{\ell m} \}$
$= -[k_\ell\ \ell(\ell+1)/r\ ]\ \mathbf{X^2_{\ell m}} - (1/r)\ \mathbf{X^3_{\ell m}}$

$\nabla \times \mathbf{X^2_{\ell m}} = \nabla \times \mathbf{Y}_{\ell m} = \nabla \times (\hat{\mathbf{r}}\ Y_{\ell m}) = \nabla\ Y_{\ell m} \times \hat{\mathbf{r}} + Y_{\ell m}\ (\nabla \times \hat{\mathbf{r}}) = -(1/r)\ \mathbf{r} \times \nabla\ Y_{\ell m}$
$= -(1/r)\ \mathbf{X}_{\ell m} = (-1/k_\ell r)\ \mathbf{X^1_{\ell m}}$

$\nabla \times \mathbf{X^3_{\ell m}} = \nabla \times k_\ell\ \mathbf{Z}_{\ell m} = k_\ell \nabla \times (r\ \nabla Y_{\ell m}) = k_\ell\ [\ \nabla r \times \nabla Y_{\ell m} + r\ (\nabla \times \nabla Y_{\ell m})] =$
$k_\ell\ [\hat{\mathbf{r}} \times \nabla Y_{\ell m} + 0] = k_\ell (1/r)\ \mathbf{r} \times \nabla Y_{\ell m} = k_\ell (1/r)\ \mathbf{X}_{\ell m} = (1/r)\ \mathbf{X^1_{\ell m}}$

Now lets engage the curl of curl battle!

$\nabla \times \nabla \times \mathbf{X^1_{\ell m}} = \nabla \times (-[k_\ell\ \ell(\ell+1)/r\ ]\ \mathbf{X^2_{\ell m}} - (1/r)\ \mathbf{X^3_{\ell m}})$
$= -k_\ell\ \ell(\ell+1)\ \nabla \times [\ (1/r)\ \mathbf{X^2_{\ell m}}] - \nabla \times [(1/r)\ \mathbf{X^3_{\ell m}}] \quad = -k_\ell\ \ell(\ell+1)\ [1] - [2]$

$[1] = \nabla \times [\ (1/r)\ \mathbf{X^2_{\ell m}}] = \nabla(1/r) \times \mathbf{X^2_{\ell m}} + (1/r)\ \nabla \times \mathbf{X^2_{\ell m}}$
$= (-1/r^2)\ \{\ \hat{\mathbf{r}} \times \mathbf{X^2_{\ell m}}\ \} + (1/r)\ \{\ \nabla \times \mathbf{X^2_{\ell m}}\ \}$
$= (-1/r^2)\ \{\ \mathbf{0}\ \} + (1/r)\ \{ -(1/k_\ell r)\ \mathbf{X^1_{\ell m}}\ \}$
$= (-1/k_\ell r^2)\ \mathbf{X^1_{\ell m}}$

$[2] = \nabla \times [(1/r)\ \mathbf{X^3_{\ell m}}] = \nabla(1/r) \times \mathbf{X^3_{\ell m}} + (1/r)\ \nabla \times \mathbf{X^3_{\ell m}}$
$= (-1/r^2)\{\ \hat{\mathbf{r}} \times \mathbf{X^3_{\ell m}}\ \} + (1/r)\ \{\ \nabla \times \mathbf{X^3_{\ell m}}\}$
$= (-1/r^2)\{\ \mathbf{X^1_{\ell m}}\ \} + (1/r)\ \{\ (1/r)\ \mathbf{X^1_{\ell m}}\ \}$
$= 0$

$\Rightarrow \quad \nabla \times \nabla \times \mathbf{X^1_{\ell m}} = -k_\ell\ \ell(\ell+1)\ [1] - [2] = -k_\ell\ \ell(\ell+1)\ (-1/k_\ell r^2)\ \mathbf{X^1_{\ell m}} = \ell(\ell+1)/r^2\ \mathbf{X^1_{\ell m}}$
$\Rightarrow \quad \nabla \times \nabla \times \mathbf{X^1_{\ell m}} = \ell(\ell+1)/r^2\ \mathbf{X^1_{\ell m}}$

$\nabla \times \{\ \nabla \times \mathbf{X^2_{\ell m}}\ \} = \nabla \times \{\ -(1/k_\ell r)\ \mathbf{X^1_{\ell m}}\ \} \quad = (-1/k_\ell) \nabla \times \{(1/r)\ \mathbf{X^1_{\ell m}}\}$
$= (-1/k_\ell)\ [(-1/r^2)\ \{\ \hat{\mathbf{r}} \times \mathbf{X^1_{\ell m}}\ \} + (1/r)\ \{\ \nabla \times \mathbf{X^1_{\ell m}}\ \}\ ]$
$= (-1/k_\ell)\ [(-1/r^2)\ \{ -\mathbf{X^3_{\ell m}}\ \} + (1/r)\ \{ -[k_\ell\ \ell(\ell+1)/r\ ]\ \mathbf{X^2_{\ell m}} - (1/r)\ \mathbf{X^3_{\ell m}}\ \}\ ]$
$= (-1/k_\ell)\ [(1/r)\ \{ -[k_\ell\ \ell(\ell+1)/r\ ]\ \mathbf{X^2_{\ell m}}\ \}\ ]$

$= \quad \ell(\ell+1)/r^2\ \mathbf{X^2_{\ell m}}$ // interesting!

$\nabla \times \{ \nabla \times \mathbf{X^3_{\ell m}} \} = \nabla \times \{ (1/r)\ \mathbf{X^1_{\ell m}} \} =$

$= [(-1/r^2) \{ \hat{\mathbf{r}} \times \mathbf{X^1_{\ell m}} \} + (1/r) \{ \nabla \times \mathbf{X^1_{\ell m}} \}]$

$= [(1/r) \{ - [k_\ell\, \ell(\ell+1)/r]\ \mathbf{X^2_{\ell m}} \}]$ from previous calculation

$= - k_\ell\, \ell(\ell+1)/r^2\ \mathbf{X^2_{\ell m}}$ // breaking the rule set by the previous two!

Summarize these double-curl results:

$\nabla \times \{ \nabla \times \mathbf{X^1_{\ell m}} \} = \ell(\ell+1)/r^2\ \mathbf{X^1_{\ell m}}$

$\nabla \times \{ \nabla \times \mathbf{X^2_{\ell m}} \} = \ell(\ell+1)/r^2\ \mathbf{X^2_{\ell m}}$

$\nabla \times \{ \nabla \times \mathbf{X^3_{\ell m}} \} = - k_\ell\, \ell(\ell+1)/r^2\ \mathbf{X^2_{\ell m}}$

Now let's try some Laplacians:

$\nabla^2\ \mathbf{X^1_{\ell m}} = \nabla \{ \nabla \bullet \mathbf{X^1_{\ell m}} \} - \{ \nabla \times \nabla \times \mathbf{X^1_{\ell m}} \}$

$= \nabla \{0 \} - \{ \ell(\ell+1)/r^2\ \mathbf{X^1_{\ell m}} \}$

$= - \ell(\ell+1)/r^2\ \mathbf{X^1_{\ell m}}$

$\nabla^2\ \mathbf{X^2_{\ell m}} = \nabla \{ \nabla \bullet \mathbf{X^2_{\ell m}} \} - \{ \nabla \times \nabla \times \mathbf{X^2_{\ell m}} \}$

$= \nabla \{(2/r)\ Y_{\ell m} \} - \{\ell(\ell+1)/r^2\ \mathbf{X^2_{\ell m}} \}$

$= 2\ \{\nabla [(1/r)\ Y_{\ell m}]\} - \ell(\ell+1)/r^2\ \mathbf{X^2_{\ell m}}$

$= 2 \{ (-1/r^2)\ \hat{\mathbf{r}}\ Y_{\ell m} + (1/r)\ \nabla\ Y_{\ell m} \} - \ell(\ell+1)/r^2\ \mathbf{X^2_{\ell m}}$

$= 2 \{ (-1/r^2)\ \mathbf{X^2_{\ell m}} + (1/r^2)\ \mathbf{X^3_{\ell m}} / k_\ell \} - \ell(\ell+1)/r^2\ \mathbf{X^2_{\ell m}}$

$= (2/ k_\ell r^2)\ \mathbf{X^3_{\ell m}} - [\ell(\ell+1) +2] /r^2\ \mathbf{X^2_{\ell m}}$ /not very nice

$\nabla^2\ \mathbf{X^3_{\ell m}} = \nabla \{ \nabla \bullet \mathbf{X^3_{\ell m}} \} - \{ \nabla \times \nabla \times \mathbf{X^3_{\ell m}} \}$

$= \nabla \{ - k_\ell\ \ell(\ell+1)\ (1/r)\ Y_{\ell m} \} - \{ - k_\ell\, \ell(\ell+1)/r^2\ \mathbf{X^2_{\ell m}} \}$

$= - k_\ell\ \ell(\ell+1)\ \{ \nabla [(1/r)\ Y_{\ell m}] \} + k_\ell\, \ell(\ell+1)/r^2\ \mathbf{X^2_{\ell m}}$

$= - k_\ell\ \ell(\ell+1)\ \{ (-1/r^2)\ \hat{\mathbf{r}}\ Y_{\ell m} + (1/r)\ \nabla\ Y_{\ell m} \} + k_\ell\, \ell(\ell+1)/r^2\ \mathbf{X^2_{\ell m}}$

$= - k_\ell\ \ell(\ell+1)\ \{ (-1/r^2)\ \mathbf{X^2_{\ell m}} + (1/r^2)\ \mathbf{X^3_{\ell m}} / k_\ell \} + k_\ell\, \ell(\ell+1)/r^2\ \mathbf{X^2_{\ell m}}$

$= 2\ k_\ell\, \ell(\ell+1)/r^2\ \mathbf{X^2_{\ell m}} - \ell(\ell+1)/\ r^2\ \mathbf{X^3_{\ell m}}$

I am getting tired of all this stuff. Wondering if Maple can do it? No, but I found a package that I can add to my Maple, looks interesting... Well, the problem is related to "inert" versus "do it" . I don't really want the thing to be calculated, I want it to remain in symbolic form, like ∇Y. Back to hand work!

The Laplacians look suspicious, I bet they should all come out the same, but there is so much algebra here I have no real way to check things, and early errors propagate! At least I have a written derivation record so I should be able to repair things at some point.

Let's summarize some of these result in a table:

$\mathbf{X^1_{\ell m}} = k_\ell\ \mathbf{X}_{\ell m} = k_\ell\ \mathbf{r} \times \nabla\ Y_{\ell m} = i\ k_\ell\ \mathbf{L}\ Y_{\ell m}$ $\qquad\qquad$ $\mathbf{X}_{\ell m} = \mathbf{r} \times \nabla\ Y_{\ell m}$

$\mathbf{X^2_{\ell m}} = \mathbf{Y}_{\ell m} = \hat{\mathbf{r}}\ Y_{\ell m}$
$\mathbf{X^3_{\ell m}} = k_\ell \mathbf{Z}_{\ell m} = k_\ell\ r\ \nabla Y_{\ell m}$

$\mathbf{Y}_{\ell m} = \hat{\mathbf{r}}\ Y_{\ell m}$
$\mathbf{Z}_{\ell m} = r\ \nabla Y_{\ell m}$

$\hat{\mathbf{r}} \bullet \mathbf{X^1_{\ell m}} = 0$
$\hat{\mathbf{r}} \bullet \mathbf{X^2_{\ell m}} = Y_{\ell m}$
$\hat{\mathbf{r}} \bullet \mathbf{X^3_{\ell m}} = 0$

$\hat{\mathbf{r}} \bullet \mathbf{X}_{\ell m} = 0$
$\hat{\mathbf{r}} \bullet \mathbf{Y_{\ell m}} = Y_{\ell m}$
$\hat{\mathbf{r}} \bullet \mathbf{Z}_{\ell m} = 0$

$\hat{\mathbf{r}}\ \mathbf{x}\ \mathbf{X^1_{\ell m}} = -\ \mathbf{X^3_{\ell m}}$
$\hat{\mathbf{r}}\ \mathbf{x}\ \mathbf{X^2_{\ell m}} = 0$
$\hat{\mathbf{r}}\ \mathbf{x}\ \mathbf{X^3_{\ell m}} = \mathbf{X^1_{\ell m}}$

$\hat{\mathbf{r}}\ \mathbf{x}\ \mathbf{X}_{\ell m} = -\ \mathbf{Z}_{\ell m}$
$\hat{\mathbf{r}}\ \mathbf{x}\ \mathbf{Y_{\ell m}} = 0$
$\hat{\mathbf{r}}\ \mathbf{x}\ \mathbf{Z}_{\ell m} = \mathbf{X}_{\ell m}$

$\hat{\mathbf{r}}\ \mathbf{x}\ \hat{\mathbf{r}}\ \mathbf{x}\ \mathbf{X^1_{\ell m}} = -\ \hat{\mathbf{r}}\ \mathbf{x}\ \mathbf{X^3_{\ell m}} = -\ \mathbf{X^1_{\ell m}}$
$\hat{\mathbf{r}}\ \mathbf{x}\ \hat{\mathbf{r}}\ \mathbf{x}\ \mathbf{X^2_{\ell m}} = 0$
$\hat{\mathbf{r}}\ \mathbf{x}\ \hat{\mathbf{r}}\ \mathbf{x}\ \mathbf{X^3_{\ell m}} = \hat{\mathbf{r}}\ \mathbf{x}\ \mathbf{X^1_{\ell m}} = -\ \mathbf{X^3_{\ell m}}$

$\hat{\mathbf{r}}\ \mathbf{x}\ \hat{\mathbf{r}}\ \mathbf{x}\ \mathbf{X_{\ell m}} = -\ \mathbf{X_{\ell m}}$
$\hat{\mathbf{r}}\ \mathbf{x}\ \hat{\mathbf{r}}\ \mathbf{x}\ \mathbf{Y_{\ell m}} = 0$
$\hat{\mathbf{r}}\ \mathbf{x}\ \hat{\mathbf{r}}\ \mathbf{x}\ \mathbf{Z_{\ell m}} = -\ \mathbf{Z_{\ell m}}$

$\nabla \bullet \mathbf{X^1_{\ell m}} = 0$
$\nabla \bullet \mathbf{X^2_{\ell m}} = (2/r)\ Y_{\ell m}$
$\nabla \bullet \mathbf{X^3_{\ell m}} = -\ k_\ell\ \ell(\ell+1)\ (1/r)\ Y_{\ell m}$

$\nabla \bullet \mathbf{X}_{\ell m} = 0$
$\nabla \bullet \mathbf{Y_{\ell m}} = (2/r)\ Y_{\ell m}$
$\nabla \bullet \mathbf{Z}_{\ell m} = -\ \ell(\ell+1)\ (1/r)\ Y_{\ell m}$

$\nabla\ \mathbf{x}\ \mathbf{X^1_{\ell m}} = -\ [k_\ell\ \ell(\ell+1)/r\ ]\ \mathbf{X^2_{\ell m}} - (1/r)\ \mathbf{X^3_{\ell m}}$
$\nabla\ \mathbf{x}\ \mathbf{X^2_{\ell m}} = -\ (1/k_\ell r)\ \mathbf{X^1_{\ell m}}$
$\nabla\ \mathbf{x}\ \mathbf{X^3_{\ell m}} = (1/r)\ \mathbf{X^1_{\ell m}}$

$\nabla\ \mathbf{x}\ \mathbf{X}_{\ell m} = -\ [\ell(\ell+1)/r\ ]\ \mathbf{Y}_{\ell m} - (1/r)\ \mathbf{Z}_{\ell m}$
$\nabla\ \mathbf{x}\ \mathbf{Y_{\ell m}} = -\ (1/r)\ \mathbf{X}_{\ell m}$
$\nabla\ \mathbf{x}\ \mathbf{Z}_{\ell m} = (1/r)\ \mathbf{X}_{\ell m}$

$\nabla \bullet (\ r\ \mathbf{x}\ \nabla f) = 0$ for any f