The Rainbow PhL 1.22.03

Look at the pencil drawing. It shows a single ray coming in at impact parameter h (angle x) and emerging at an angle ψ relative to the initial direction. Now imagine an observing eye viewing the emerging ray at a great distance. Model the eye as a circular pupil that catches rays. The number of rays captured by the eye is proportional to $d\psi$ because the pupil has a finite size. Since we are so far away, the fact that the location on the sphere of the emerging ray varies slightly as ψ varies does not matter.

In the other direction, imagine that a ray comes in at some $d\phi$ azimuth (north pole is left end of sphere) below the plane of paper. Then just tilt the entire disk drawn in the figure by rotating it about the horizontal axis by angle ϕ. The angle between the emerging ray and the untilted emerging ray is $\delta\theta = \sin\psi\ \delta\phi$ as proven in our little appendix below. This is obvious when $\psi=\pi/2$ so $\sin\psi=1$.

Imagine a square (edge d) into which is inscribed the pupil (diameter d) of a viewing eye (at distance D from the sphere). The square forces $\delta\theta = \delta\psi = d/D$. All rays within these small angles hit the square. Of these, the fraction $\pi/4$ hit the disk of the retina, but we don't bother with this factor in the following since we are not looking for an absolute answer, just the shape of the curve.

Now, the number of rays which pass through an area $R^2\ d\Omega$ on the sphere (x = polar angle, ϕ = azimuthal angle) is σ(rays/area)* $R^2\ d\Omega$ * cos(x), this last factor since $R^2\ d\Omega$ is tilted relative to the incoming plane wave of rays. We have assumed a uniform incoming plane wave ray density σ; these rays are coming from the left in the figure. We are also implicitly assuming polarization perpendicular to the plane of paper. The other polarization direction will have some extra angle complications and will probably have a weaker output, but the nature of the result won't be different. Thus we have

$$\delta N(x,\phi) = \sigma R^2 \cos x \sin x\ \delta x\ \delta\phi = \text{number of rays hitting our differential area.}$$

Then set $\delta x = \delta\psi/\psi'(x)$ and $\delta\phi = \delta\theta/\sin\psi$. Finally, use $\delta\theta = \delta\psi = d/D$ and the result is

$$\delta N(\text{number of rays reaching retina}) = \frac{\sigma R^2 \sin x \cos x\ d}{D \sin\psi} \frac{d}{D\psi'(x)} \qquad (1)$$

where the second fraction is δx which we have kept separate for the moment. We have an equation for $\psi'(x)$ on our pencil sheet, and we know $\psi'(x) = 0$ at $\psi = 137.48^\circ$ or so. At this point, everything else in the equation is finite, so dN has an infinite peak here! This makes the point quite well that the critical angle completely dominates the reflected light, and this was Descartes idea. But we know the result cannot really be infinite. To correct the situation , let's replace $\delta x = \delta\psi/\psi'(x)$ with a more accurate expression. We can expand $\psi(x)$ around the point x and get this result, where now we keep an extra term:

$$\delta\psi = \delta x \{ \psi' + (\delta x/2) * \psi'' \}$$

Rewrite this as

$$(\psi''/2)(\delta x)^2 + (\psi')(\delta x) - \delta\psi = 0 \qquad A(\delta x)^2 + B(\delta x) + C = 0$$

solve this for δx ,

$$\delta x = [- \psi' \pm \text{sqrt}((\psi')^2 + 2\psi''\delta\psi)] / \psi''$$

Which root do we want here? We want the root that causes cancellation between the two large terms because the result we are looking for is differentially small. We know that $\psi''>0$, so assume that $\delta\psi > 0$ so the inside of the square root is larger than $|\psi'|$. Then we want this solution:

$$\delta x = | - \psi' + \text{sign}(\psi') * \text{sqrt}((\psi')^2 + 2\,\psi''\,\delta\psi) | / \psi''$$

We have added | | because we want $\delta x > 0$ since we are just doing a ray density scaling.

We set $\delta\psi = d/D$ inside and then our formula becomes

$$dN(\text{number of rays reaching retina}) = \frac{\sigma R^2 \sin x \cos x\, d}{D \sin\psi} \frac{|-\psi' + \text{sign}(\psi')*\text{sqrt}(\psi'^2+2\psi''(d/D))|}{\psi''} \quad (2)$$

Right AT the critical point where $\psi'=0$ we can write this as

$$dN(\text{number of rays reaching retina}) = \frac{\sigma R^2 \sin x \cos x\, d}{D \sin\psi} \frac{d}{D\,\text{sqrt}(\psi''d/2D)} \quad (3)$$

Comparing with (1) above, we see that ψ' in the denominator has been replaced by $\text{sqrt}(\psi''d/2D)$. This is small, but not 0, so the result for dN is large but finite.

So let's plot the above general result for dN using Maple! The result is very impressive and highly peaked as expected. We could compute the width of the peak, but the Maple plot shows it to be about .04 radians with our assumed $f = d/D = .0001$. This is about 2.4 degrees, very narrow, a thumb width which is 2.4 degrees as well for me (1" at 24"). This might agree with the observed width of a rainbow. Probably the color bands overlap a lot, etc.

Colors are there because the index of water varies with frequency. Since visible is below the electronics resonance area, index (and dielectric constant) increases with increasing frequency, so red light is bent least and has the lowest index, and will have a smaller ψ angle, so red appears at the bottom of the rainbow band (on the inside of the big circle). To see why a rainbow is a circle, hold the piece of paper with the figure on it and rotate it such that rays coming from behind you all parallel cause reflection into your eye. The paper must be attached as a paddle on a wheel whose axis goes through your eye.

In the single bounce situation which we have looked at, the output angle ψ ranges from 180 degrees down to 137 degrees and our drawing is typical. Suppose the ray does two integral bounces. I have shown a second bounce as a dotted line in the figure. You can see that this ray emerges on the top, not on the bottom of the figure. In this case, red light with less bend comes out at *larger* ψ because now ψ is measured around on the top side of the figure. So the second rainbow has the opposite coloring order. For n bounces, you have to study the geometry and see whether ray comes out on the top or the bottom. I have seen three rainbows in Torrey.

Appendix. What is the angle between the tilted emergent ray and the original untilted emergent ray? Go out to the eye a large distance D away. Make a little coordinate system centered on the emergence point, with x to the right and y down and z out of paper. The coordinates of the point at the eye that is hit by this ray are: $z = D\sin\psi d\phi$, $x = D\cos\psi$, $y = D\sin\psi$, or $\mathbf{r} = \mathbf{a}$. The coordinates of the point with no tilt are $z = 0$,

x = Dcosψ, y = Dsinψ or **r** = **b.** The corresponding rays are given by **r1** = α**a** and **r2** = α**b** where α is a scaling factor. The angle between these rays is given by cosθ = r1-hat • r2-hat = a-hat • b-hat. Now |a| = D sqrt(1 + (sinψdϕ)^2) while |b| = D. And a•b = D. Thus, cosθ = 1/ sqrt(1 + (sinψdϕ)^2). But dϕ is small, so we can expand RHS to get 1 - 1/2((sinψdϕ)^2 . But we know that for small angle dθ we have cosdθ = 1 - 1/2dθ^2 . This we get our answer which is dθ = sinψdϕ.