Lorentz Group Representations PhL 9.25.08

Today I hunted around on the web for the explicit Dirac representation generator and finite matrices, which appear half-baked in BD1. I found **Guy Moore** at McGill who teaches QFT and had some nice stuff on his web site. One of his courses is QFT Physics 610,

http://www.physics.mcgill.ca/~guymoore/ph610/index.html

A little info on him, PhD Princeton 1997. Wonder if Crodt knows him?

University of Washington, 1999-2002 [Postdoc]
McGill University, 1997-1999 [Postdoc]
Princeton University, 1992-1997 [Grad. Student]
Harvey Mudd College, 1988-1992 [Undergrad]
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1. His comments on books in the field.

The course will follow the textbook

M. Peskin and D. Schroeder, An Introduction to Quantum Field Theory

and you can get it, for instance, from Amazon.ca or Chapters/Indigo (same price)

He goes not say why he picked this book, but he does comment on other books:

For those of you who would like additional or alternative textbooks, I will make a few recommendations. Shop around for prices, I just gave links to show you exactly what book I meant.

Mark Srednicki's book covers the same material (almost) in a different and in my opinion **better order**, is very readable, and in many ways is **a better textbook.**

Lowel Brown's book introduces QFT in a **completely different way** than Peskin which you may prefer but which means it does not get as far in the same number of pages. Beware, he uses a **different metric** convention.

Ryder's book is a slightly less advanced text which many people like but which I don't know well myself.

Zinn-Justin is the **bible of Quantum Field Theory** but is much more **formal** than I intend to be in our class. Buy it if you intend to take Quantum Field Theory really seriously but not if you are on a tight budget.

Steven Weinberg's book is a place to go if you want **everything spelled out in complete detail,** but not if you want clear expository explanations; and it uses a different metric convention than we will.

He says nothing about path quantization which seems odd. Maybe all books do it this way now.

2. His offered notes

Lectures are not available that I can see, but he offers three items as follows:

Various Notes

A set of notes on symmetries, representations, and group theory can be found here in postscript.

A similar set of notes for Lorentz symmetry, spinors, and the Dirac algebra is here in postscript. You may find the following useful:

Quick lookup sheet of commonly needed equations, in postscript or pdf

I have downloaded these as Moore1,2,3. Of particular interest to me today is item #2 which contains the information I am seeking!

3. His own book.

Moore has written a book called **"The Standard Model: a Primer"** which is $60, 560 pages, 2006. I suspect this is a Levitt-like book. Amazon has no reviewers yet. You can "look inside" at Amazon. Here is the table of contents:

List of illustrations		*page* viii
List of tables		ix
Preface		x
Acknowledgments		xv
	Part I Theoretical framework	1
1	**Field theory review**	3
1.1	Hilbert space, creation and annihilation operators	3
1.2	General properties of interactions	7
1.3	Free field theory	15
1.4	Implications of symmetries	29
1.5	Renormalizable interactions	34
1.6	Some illustrative examples	38
1.7	Problems	46
2	**The standard model: general features**	53
2.1	Particle content	53
2.2	The Lagrangian	58
2.3	The perturbative spectrum	61
2.4	Interactions	71
2.5	Symmetry properties	84
2.6	Problems	104
3	**Cross sections and lifetimes**	111
3.1	Scattering states and the S-matrix	111
3.2	Time-dependent perturbation theory	115
3.3	Decay rates and cross sections	118

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Part II Applications: leptons 125

4 Elementary boson decays 127
4.1 Z^0 decay 127
4.2 $W^\pm$ decays 142
4.3 Higgs decays 146
4.4 Problems 151

5 Leptonic weak interactions: decays 153
5.1 Qualitative features 153
5.2 The calculation 160
5.3 The large-mass expansion 163
5.4 Feynman rules 169
5.5 Problems 183

6 Leptonic weak interactions: collisions 188
6.1 The Mandelstam variables 189
6.2 e^+e^- annihilation: calculation 192
6.3 e^+e^- annihilation: applications 197
6.4 The Z boson resonance 202
6.5 t-channel processes: crossing symmetry 209
6.6 Interference: Møller scattering 212
6.7 Processes involving photons 214
6.8 Problems 223

7 Effective Lagrangians 231
7.1 Physics below M_w: the spectrum 232
7.2 The Fermi theory 233
7.3 Physics below M_w: qualitative features 239
7.4 Running couplings 242
7.5 Higgsless effective theory 257
7.6 Problems 269

Part III Applications: hadrons 273

8 Hadrons and QCD 275
8.1 Qualitative features of the strong interactions 276
8.2 Heavy quarks 284
8.3 Light quarks 290
8.4 Problems 318

9 Hadronic interactions 320
9.1 Quasi-elastic scattering 321
9.2 Hard inelastic scattering: partons 329

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9.3 Soft inelastic scattering: low-energy mesons 350
9.4 Neutral-meson mixing 367
9.5 Problems 386

Part IV Beyond the standard model 393

10 Neutrino masses 395
10.1 The kinematics of massive neutrinos 396
10.2 Neutrino oscillations 398
10.3 Neutrinoless double-beta decay 418
10.4 Gauge-invariant formulations 421
10.5 Problems 433

11 Open questions, proposed solutions 434
11.1 Effective theories (again) 435
11.2 Dimension zero: cosmological-constant problem 437
11.3 Dimension two: hierarchy problem 439
11.4 Dimension four: triviality, $\theta_{\rm QCD}$, flavor problems 455
11.5 Dimension six: baryon-number violation 471
11.6 Problems 481

***Appendix A* Experimental values for the parameters** 484
***Appendix B* Symmetries and group theory review** 487
***Appendix C* Lorentz group and the Dirac algebra** 494
***Appendix D* ξ-gauge Feynman rules** 507
***Appendix E* Metric convention conversion table** 518
Select bibliography 525
Index 535

So we learn that two of his free downloads are appendices B and C of this book, which are called Appendices 2 and 3 in the actual download.

One item I notice in his first chapter is this, something I had been trying to remember a while ago.

Second, the momentum delta functions we have written are accompanied by factors of $2E_{\mathbf{p}}$, and the same $2E_{\mathbf{p}}$ appears in the denominator in momentum integrations. This normalization, called relativistic normalization, is convenient in a Lorentz invariant theory, because it makes it easier to make Lorentz invariance manifest. Note in particular, that

$$\int \frac{\mathrm{d}^3\mathbf{p}}{2E_{\mathbf{p}}(2\pi)^3} = \int \frac{\mathrm{d}^4p}{(2\pi)^4} 2\pi\delta(p^2+m^2)\theta(p^0) \qquad (1.15)$$

which is manifestly Lorentz invariant. [Note that our metric convention is that $\eta_{\mu\nu} = \mathrm{Diag}[-1,1,1,1]$, so $p^2 \equiv Z_{\mu\nu}p^\mu p^\nu = -(p^0)^2 + \mathbf{p}^2$.] This expression can be verified by performing the p^0 integration, using the δ function. Its Lorentz invariance is not quite manifest, since the step function $\theta(p^0)$ does not look invariant, as it refers to the time component; but the $2\pi\delta(p^2+m^2)$ forces p^μ to be timelike for $m^2 > 0$ and lightlike for $m^2 = 0$, which ensures that the sign of p^0 does not change under (orthochronous) Lorentz transformations. Throughout this book, whenever there is an integral $\int \mathrm{d}^3p/(2\pi)^3 2E_{\mathbf{p}}$, we will always implicitly define $p^0 = E_{\mathbf{p}}$ inside the integral.

4. Moore's Lorentz Notes

He uses $\eta^{\mu\nu} = g^{\mu\nu}$ matching BD. In his opening section, he talks about the coordinate transformation and for a scalar to stay scalar, we must have this be true:

$$\eta_{\mu\nu} = \Lambda^{\alpha}{}_{\mu}\eta_{\alpha\beta}\Lambda^{\beta}{}_{\nu}\,, \qquad (3.1.5) \qquad x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$$

If you do infinitesimal you find

$$\begin{aligned} \eta_{\mu\nu} &= \left(\delta^{\alpha}_{\mu} + \omega^{\alpha}{}_{\mu}\right)\eta_{\alpha\beta}\left(\delta^{\beta}_{\nu} + \omega^{\beta}{}_{\nu}\right) \\ &= \eta_{\mu\nu} + (\omega_{\nu\mu} + \omega_{\mu\nu}) + O(\omega^2)\,, \\ 0 &= \omega_{\nu\mu} + \omega_{\mu\nu}\,. \qquad (3.1.8) \end{aligned}$$

So in this section, the object $\omega^{\alpha}{}_{\mu}$ is a <u>combination of generator and parameter</u>, which I always find confusing, same as done in BD. I agree that this thing with both indices lowered is antisymmetric. But then later on he writes 3.2.2 below, using the same symbol ω, but now ω is just the parameter part, not the generator part. I suspect that 3.2.2 is the general case, and for the vector representation we get the above form. As shown below, the vector representation has these generators

$(J^{\mu\nu})^{\alpha}{}_{\beta} = i\,(\,g^{\mu\alpha}g^{\nu}{}_{\beta} - g^{\nu\alpha}g^{\mu}{}_{\beta})$ = explicitly antisymmetric in μ and ν

in which case we get:

$[(i/2)\,\omega_{\mu\nu}\,J^{\mu\nu}]^{\alpha}{}_{\beta} = -\,\tfrac{1}{2}\,\omega_{\mu\nu}\,(\,g^{\mu\alpha}g^{\nu}{}_{\beta} - g^{\nu\alpha}g^{\mu}{}_{\beta}) = -\,\tfrac{1}{2}\,(\,\omega^{\alpha}{}_{\beta} - \omega_{\beta}{}^{\alpha})$

Well, lower the α on both sides to get:

$[(i/2)\,\omega_{\mu\nu}\,J^{\mu\nu}]_{\alpha\beta} = -\,\tfrac{1}{2}\,\omega_{\mu\nu}\,(\,g^{\mu}{}_{\alpha}g^{\nu}{}_{\beta} - g^{\nu}{}_{\alpha}g^{\mu}{}_{\beta}) = -\,\tfrac{1}{2}\,(\,\omega_{\alpha\beta} - \omega_{\beta\alpha}) = -\,\omega_{\alpha\beta}$

where I use the antisymmetry of $\omega_{\mu\nu}$ as discussed below. So for the vector representation we can write

$U = 1 - i/2\;\omega_{\mu\nu}J^{\mu\nu} \quad \Rightarrow \quad U^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta} - i/2\;\omega_{\mu\nu}(J^{\mu\nu})^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta} + \omega^{\alpha}{}_{\beta}$

which is the usual way you see things for the vector representation where "parameter and generator are combined". The generator is really there, it is just hard to see it! The point is that this ω thing really is the parameter matrix in both the vector and general case, I guess I had not realized that. Also, the logic above shows the reason for the ½ factor. Then, given the antisymmetry of $\omega_{\alpha\beta}$, we have this most general form for the parameter matrix $\omega^{\alpha}{}_{\beta}$:

$$\omega^{\mu}{}_{\nu} = \begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & 0 & -r_3 & r_2 \\ b_2 & r_3 & 0 & -r_1 \\ b_3 & -r_2 & r_1 & 0 \end{pmatrix} , \qquad (3.1.9)$$

where Guy uses the suggestive boost and rotation letters b and r. *This* matrix of course is not antisymmetric due to the metric tensor. It is symmetric in the space-time parts and antisymmetric in the space-space parts, as you can see.

Now on to the general situation. He considers small transformations and defines abstract (hatted) generators as follows:

$$U(\omega) = \mathbf{1} - \frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu} + O(\omega^2) , \qquad (3.2.2)$$

$$U(\xi) = \mathbf{1} - i\xi_{\mu}\hat{P}^{\mu} . \qquad (3.2.3)$$

Without showing details, he claims that by considering the known finite transformations, you can deduce the Lie algebra which is this: (in fact, this is the complete Poincare Lie algebra, of which the Lorentz is a subset, or subalgebra).

$$\left[\hat{J}^{\mu\nu} , \hat{P}^{\alpha}\right] = -i\left(\eta^{\mu\alpha}\hat{P}^{\nu} - \eta^{\nu\alpha}\hat{P}^{\mu}\right) . \qquad (3.2.10)$$

By a completely analogous but more involved procedure one can also show,

$$\left[\hat{J}^{\mu\nu} , \hat{J}^{\alpha\beta}\right] = -i\left(\eta^{\nu\beta}\hat{J}^{\mu\alpha} + \eta^{\mu\alpha}\hat{J}^{\nu\beta} - \eta^{\mu\beta}\hat{J}^{\nu\alpha} - \eta^{\nu\alpha}\hat{J}^{\mu\beta}\right) , \qquad (3.2.11)$$

and (this is simpler)

$$\left[\hat{P}^{\mu} , \hat{P}^{\nu}\right] = 0 . \qquad (3.2.12)$$

Now I think I can show that the **generators must be antisymmetric** in their labels. Here is a compact notation for the Lorentz algebra:

$[\ \mu\nu, \alpha\beta] = -i\ (\ \mu\alpha + \nu\beta - \nu\alpha - \mu\beta)$

On the RHS, I show only the J labels, because we can always figure out the g indices from the J labels shown. We can see "by inspection" that the RHS here is *antisymmetric* in $\mu\leftrightarrow\nu$ and in $\alpha\leftrightarrow\beta$. So we know that

$[J^{\mu\nu}, J^{\alpha\beta}] = [J^{\mu\nu}, -J^{\beta\alpha}]$

For a fixed $\alpha\beta$, this is true for ALL $\mu\nu$. I think this is enough to claim then that $J^{\alpha\beta} = -J^{\beta\alpha}$. So, I am happy to accept that the generators are antisymmetric in their labels, just from the look of the Lie algebra.

Then given this fact, when we think about the combination of parameter and generator $\omega_{\mu\nu}J^{\mu\nu}$, if we tried some "general" $\omega_{\mu\nu} = \omega_{\mu\nu}^{A} + \omega_{\mu\nu}^{S}$, we could find that $\omega_{\mu\nu}^{S}J^{\mu\nu} = 0$ and so $\omega_{\mu\nu}^{S}$ has no significance -- it does nothing at all. So we might as well take $\omega_{\mu\nu}$ = antisymmetric and just set $\omega_{\mu\nu}^{S} = 0$.

Comment: the $J^{\mu\nu}$ shown above agree with the $M^{\mu\nu}$ of BD according to my notes, but I can't really find a full statement of the Lie algebra in BD either volume. Other people like Weinberg might use $M^{\mu\nu} = \pm iJ^{\mu\nu}$ which removes the "i" from the Lie algebra. (Then maybe the generators are symmetric and real and hence Hermitian?)

Theorem: the parameter matrix $\omega_{\mu\nu}$ and the generators $J^{\mu\nu}$ are each antisymmetric. For $\omega_{\mu\nu}$, the two letters are in fact matrix indices, whereas for $J^{\mu\nu}$ the two letters are labels, not matrix indices.

Then, just as I would have done, he restates things in terms of some J and K generators:

To make contact with the more familiar generators of rotations and boosts, it is convenient to define

$$\hat{J}_i \equiv \frac{\epsilon_{ijk}}{2}\hat{J}_{jk}\,, \tag{3.2.13}$$

$$\hat{K}_i \equiv \hat{J}_{i0}\,, \tag{3.2.14}$$

which are respectively the generator of rotations about the i axis and of boosts along the i axis, so a rotation by $\vec{\theta}$ is $\exp(-iJ_i\theta_i)$ and a boost by $\vec{v}$ is $\exp(-iK_iv_i)$. They satisfy the commutation relations, following from Eq. (3.2.11),

$$\left[\hat{J}_i\,,\,\hat{J}_j\right] = i\epsilon_{ijk}\hat{J}_k\,, \tag{3.2.15}$$

$$\left[\hat{J}_i\,,\,\hat{K}_j\right] = i\epsilon_{ijk}\hat{K}_k\,, \tag{3.2.16}$$

$$\left[\hat{K}_i\,,\,\hat{K}_j\right] = -i\epsilon_{ijk}\hat{J}_k\,. \tag{3.2.17}$$

and these agree with my old notes. Notice that the first two are really just saying that J and K transform as vectors under rotation, the interesting one is the third that boosts do not commute. Notice, by the way, that everything here is "representation independent", could talk vector or scalar or spinor, etc.

He then starts talking about "representations" in this sense

$$U(\omega)\varphi_a(x)U^*(\omega) = D^{-1}_{ab}(\omega)\varphi_b(\Lambda x)\,, \tag{3.3.1}$$

where he is transforming a "field" and where Λ is how the coordinate transforms. Here is how a vector field must transform

$$U(\omega)A^\mu(x)U^*(\omega) = (\Lambda^{-1})^\mu{}_\nu A^\nu(\Lambda x)\,. \tag{3.3.3}$$

Note: He uses * for † when we are in the abstract Hilbert space, so * is the mathematical adjoint symbol in this case.

Note added 10-3-08: I had to undertake a 7-day side trip before I was able to really understand the above transformation equations! I wrote two documents on the subject, which of course bring in many related ideas as well.

Answer: As Moore points out in his Moore1 Appendix a notes, you can define a "symmetry" as follows in the Quantum Mechanics Hilbert Space:

$|\varphi'\rangle = U|\varphi\rangle \qquad O' = UOU^\dagger$

$\Rightarrow \langle\varphi'| O' |\psi'\rangle = \langle\varphi| U^\dagger (UOU^\dagger) U |\psi\rangle = \langle\varphi| (U^\dagger U)O(U^\dagger U) |\psi\rangle = \langle\varphi| O |\psi\rangle$ if $U^\dagger U = 1$

Sometimes this is phrased in terms of saying " If you do $|\varphi\rangle \rightarrow U|\varphi\rangle$ and $O \rightarrow UOU^\dagger$, and if nothing changes in the theory, meaning $\langle\varphi| O |\psi\rangle \rightarrow \langle\varphi| O |\psi\rangle$, then U defines a *symmetry* and it is a unitary transformation in the Hilbert Space with $U^\dagger U = 1$". The could be an internal or a spacetime related symmetry. In all of the above, operators like O and U are "abstract operators" in the QM Hilbert space.

Now an example would be to take a field operator like $A^\mu(x)$. Then the 3.3.3 above says this:

$$A'^\mu(x') = UA^\mu(x)U^\dagger = (\Lambda^{-1})^\mu{}_\nu A^\mu(\Lambda x)$$

Now why do we want an *inverse* on this thing? More generally he is saying

$$\varphi'_a(x') = U(\omega)\varphi_a(x)U^+(\omega) = D^{-1}{}_{ab}(\omega)\, \varphi_b(\Lambda x)$$

where $D_{ab}(\omega)$ is a representation of the group element ω.

Regarding this inverse, my little rotation group notes have always said $R\, V\, R^\dagger = R^{-1}V$ and have this same notion of inversion. Why do I have that inverse there? [This was the question that started my 7-day tangential voyage.]

In general, the matrix D shown above is represented in terms of the generators as follows:

$$D_{ab}(\omega) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}_{ab}\right), \qquad (3.3.4)$$

where there is a minus sign, and where ab are the matrix indices, and where $\omega_{\mu\nu}$ are the parameters of the transformation which live in a symmetric 4x4 matrix with only 6 distinct terms, as we know. At first I regard this as a little strange, but I guess it is good. We know why the ½ is in there. The exponent is a world-scalar which is the main point I think. By the way, notice that the matrix $J^{\mu\nu}$ does not have a hat, but the general operator does. He repeats the generator algebra without hats and says the matrices must respect it. Fine.

Comment: Moore does not write it anywhere, but I presume this is true:

$$U(\omega) = \exp(-i/2\ \omega_{\mu\nu}\hat{J}^{\mu\nu})$$

Both this and the matrix form above are functions of a matrix argument ω. We know that U will be Unitary for rotations but not for boosts, so the letter U is perhaps a bit misleading.

Note added 10.3.08. But, U will be unitary for boosts in the QM Hilbert Space with covariant ket normalization, something I learned on my 7-day digression. In this case, since $U^{-1} = U^{\dagger}$, we may conclude that that generators $J^{\mu\nu}$ are each Hermitian, that is, $\hat{J}^{\mu\nu} = \{\hat{J}^{\mu\nu}\}^{\dagger}$:

$U(\omega) = \exp(-i/2\ \omega_{\mu\nu}\hat{J}^{\mu\nu})$
$U^{-1}(\omega) = U(-\omega) = \exp(+i/2\ \omega_{\mu\nu}\hat{J}^{\mu\nu})$
$U^{\dagger}(\omega) = \exp(+i/2\ \omega_{\mu\nu}\{\hat{J}^{\mu\nu}\}^{\dagger})$ // since $\omega_{\mu\nu}$ are real

Remember that this fact is not true for the boost generators in the 4-vector represenation!

Now comes the part that is completely new to me, but which Geoff Chew was talking about at Torrey. It is the left and right business. If we *redefine* generators as shown here, the Lie algebra's completely decouple from each other:

$$\hat{\mathcal{L}}_i \equiv \frac{\hat{J}_i + i\hat{K}_i}{2}, \qquad \hat{\mathcal{R}}_i \equiv \frac{\hat{J}_i - i\hat{K}_i}{2}, \tag{3.3.6}$$

satisfy the commutation relations,

$$\left[\hat{\mathcal{L}}_i\,,\,\hat{\mathcal{L}}_j\right] = i\epsilon_{ijk}\hat{\mathcal{L}}_k\,, \tag{3.3.7}$$
$$\left[\hat{\mathcal{R}}_i\,,\,\hat{\mathcal{R}}_j\right] = i\epsilon_{ijk}\hat{\mathcal{R}}_k\,, \tag{3.3.8}$$
$$\left[\hat{\mathcal{L}}_i\,,\,\hat{\mathcal{R}}_j\right] = 0\,. \tag{3.3.9}$$

Each decoupled side, the Left and Right, is just a rotation group algebra! So somehow SO(3,1) is equivalent to SO(3) ⊕ SO(3), and this is the basis of the representation notation as (j_L, j_R). He comments that we shall be interested in 4 representations, to wit: scalar = (0,0). Vector = (1/2,1/2):

The second common representation is the vector representation, which transforms as $(\frac{1}{2},\frac{1}{2})$. The Lie algebra is represented as $\mathcal{J}^{\mu\nu}_{\alpha\ \beta} = i(\eta^{\mu\alpha}\eta^{\nu}_{\beta} - \eta^{\mu}_{\beta}\eta^{\nu\alpha})$, and $D = \Lambda$, as already discussed. Note that for both of these representations, the matrix D is always real; therefore it is consistent to consider real valued scalar or vector fields.

Notice that he is giving the explicit generators here and I used them above. I do have this in my notes where generators are $M^{\mu\nu}$.

Now back to the left and right stuff and the "spinor representations" First, we get the following 2D representation of the above L,R algebra

The simplest nontrivial matrices which satisfy Eq. (3.3.7) are the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (3.4.1)$$

which satisfy the commutation relation

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\epsilon_{ijk}\frac{\sigma_k}{2}. \quad (3.4.2)$$

Therefore, if the matrices representing $\hat{\mathcal{L}}_i$ and $\hat{\mathcal{R}}_i$ are $\sigma_i/2$ and 0 respectively, we get a representation of the Lorentz algebra. Inverting Eq. (3.3.6), rotations and boosts are implemented by the matrices,

$$\mathcal{J}_i = \frac{\sigma_i}{2}, \qquad \mathcal{K}_i = -i\frac{\sigma_i}{2}, \qquad \text{(Left handed spinor)} \quad (3.4.3)$$

which it is easy to show satisfy Eq. (3.2.15) through Eq. (3.2.17).

Therefore, a pair of fields $\psi_a, a = 1, 2$ can transform under Lorentz transformations according to

$$U(-\omega)\psi_a U^*(-\omega) = D_{ab}(\omega)\psi_b, \quad D(\omega) = [\exp(-i(r_i - ib_i)\sigma_i/2)], \quad (3.4.4)$$

where r_i, b_i are the amount of rotation and boost performed, as introduced in Eq. (3.1.9). The two fields ψ_a are generally referred to as the components of a single *spinor field* with a the *spinor index*, which is almost always suppressed by writing ψ and D in matrix notation (ψ as a column vector, D as a matrix). Such a spinor field is called a *left handed Weyl spinor* ψ_L.

So in the above, we have explicit 2x2 matrices for the J and K in what I guess must be called the ½ ⊕ 0 representation because he has L = σ/2 and R = 0, as stated in the text above. Notice that we have the parameters b and r as commented on above. A field that transforms as above is a "left-handed Weyl spinor" , sounds good to me.

Obviously, there will be a right-handed one going with 0 ⊕ ½ , and we have

Alternately, $\mathcal{R}_i$ could be represented by the Pauli matrices and $\mathcal{L}_i$ by 0's,

$$\mathcal{J}_i = \frac{\sigma_i}{2}, \qquad \mathcal{K}_i = i\frac{\sigma_i}{2}, \qquad \text{(Right handed spinor)} \quad (3.4.5)$$

in which case a Lorentz transform acts on ψ via

$$U(-\omega)\psi_a U^*(-\omega) = D_{ab}(\omega)\psi_b, \quad D(\omega) = [\exp(-i(r_i + ib_i)\sigma_i/2)]. \quad (3.4.6)$$

A field transforming this way is called a *right handed Weyl spinor* ψ_R.

The difference is the sign on the K matrices. Nothing is arbitrary here, it all follows from the equations shown earlier that relate K,J and R,L.

So at this point we have two distinct 2D spinor representations of the Lorentz group. He then shows that you can convert a ψ_L spinor into a ψ_R and vice versa according to these rules:

$\psi_L' = -\varepsilon\, \psi_R^*$ $\qquad$ $\varepsilon = i\,\sigma_2$ $\qquad$ $\psi_R' = +\varepsilon\, \psi_L^*$

I put a prime on things because all we are saying is that ψ_L' transforms as a left handed spinor, we don't know otherwise what it might be.

He then comments on the "Weyl notation" which does seem a bit strange:

> One, called *Weyl notation*, expresses the fields as two component objects, and then specifies whether one is referring to ψ_L or to its right handed conjugate $\varepsilon\psi_L^*$ by using either an undotted or a dotted index: $\psi_\alpha = \psi_L$ and $\psi^{\dot{\alpha}} = \varepsilon\psi_L^*$. (Indices are raised and lowered using ε and dotted and undotted according to whether they are conjugated.) This notation is common in the supersymmetry and string theory literature.

As he says, ε on a ψ-lower raises the index, and complex conjugation is a dot on the index. This would be a little hard for me to use in my Word system, and I have never seen this notation before.

Next is the "Majorana notation"

> An alternative which we will use, *Majorana notation*, writes a single *four* component field ψ_M, defined as
>
> $$\psi_M = \begin{pmatrix} \psi_L \\ \varepsilon\psi_L^* \end{pmatrix}, \qquad (3.4.9)$$
>
> that is, ψ_M redundantly records both the left handed and the right handed ways of writing the field. The individual pieces can be accessed separately by using the projection operators
>
> $$P_L \equiv \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad P_R \equiv \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}. \qquad (3.4.10)$$
>
> The action of rotations and boosts on ψ_M are respectively,
>
> $$\mathcal{J}_i = \begin{pmatrix} \frac{\sigma_i}{2} & 0 \\ 0 & \frac{\sigma_i}{2} \end{pmatrix}, \qquad \mathcal{K}_i = \begin{pmatrix} \frac{-i\sigma_i}{2} & 0 \\ 0 & \frac{i\sigma_i}{2} \end{pmatrix}. \qquad (3.4.11)$$

He has not said why we would want to make a 4-column vector or why we would want to do any of the above. But if you do it, then he is quoting the 4x4 generator matrices, which is what I have been looking for today. The big question is how this might relate to the Dirac equation.

Notice that the J_i are Hermitian whereas the K_i matrices are anti-hermitian meaning $K_i^\dagger = -K_i$. Also, right here, let's take note of these matrix facts for our 4-spinor Majorana thing

$$J_i\beta = \begin{pmatrix} \sigma_i/2 & 0 \\ 0 & \sigma_i/2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_i/2 \\ \sigma_i/2 & 0 \end{pmatrix}$$

$$\beta J_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sigma_i/2 & 0 \\ 0 & \sigma_i/2 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_i/2 \\ \sigma_i/2 & 0 \end{pmatrix} = J_i\beta$$

$$K_i\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i\sigma_i/2 \\ i\sigma_i/2 & 0 \end{pmatrix}$$

$$\beta K_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -i\sigma_i/2 & 0 \\ 0 & i\sigma_i/2 \end{pmatrix} = \begin{pmatrix} 0 & i\sigma_i/2 \\ -i\sigma_i/2 & 0 \end{pmatrix} = -K_i\beta$$

so we have shown that in this Dirac-type representation,

$$\beta J_i = +J_i\beta \qquad \text{P1}$$
$$K_i\beta = -K_i\beta$$

Then for the finite rotations we have this

$$D(\omega) = \exp[-ir_iJ_i - i\,b_iK_i] = e^{+C}$$
$$D^{-1}(\omega) = \exp[+ir_iJ_i + i\,b_iK_i] = e^{-C} = D(-\omega)$$
$$D^{\dagger}(\omega) = \exp[+ir_iJ_i^{\dagger} + i\,b_iK_i^{\dagger}] = \exp[+ir_iJ_i - i\,b_iK_i]$$

$$D^{\dagger}(\omega)\beta = \exp[+ir_iJ_i - i\,b_iK_i]\,\beta = \beta\exp[+ir_iJ_i + i\,b_iK_i] = \beta\,D^{-1}(\omega) \qquad \text{P2}$$

a fact we shall use below.

He next writes the Dirac 4-spinor as $(\mathcal{E}_L, E_R)$ where he says

> In QED and QCD, if we write the spinor fields as left-handed objects, the fields form pairs with conjugate symmetry transformation properties. For instance, in QED, there is a field $\mathcal{E}_L$ which is charge -1 under $U_{em}(1)$, called the left-handed electron, and a field $-\varepsilon E_R^*$ which is charge 1 under $U_{em}(1)$, called the left-handed positron. In this case it is most convenient to think of the latter as the conjugate of a right-handed field with charge -1, E_R, called the right-handed electron, and to combine them together in a single 4-component object called a *Dirac spinor*, $e = [\mathcal{E}_L\ E_R]^{\top}$.

So things are not very clear at this point! The $\mathcal{E}_L$ is describing a "left handed electron", while he says that $-\varepsilon E_R^*$ is a left-handed positron. Recall from BD that $i\gamma^2$ was the "charge conjugation operator" and you did things like this: $\psi_c = i\gamma^2\psi^*$. There surely is some connection to Moore's discussion here. He goes on to say

> The Lorentz transformation properties of Majorana and Dirac spinors are the same. The two distinctions are that the upper and lower components of a Dirac spinor generally have the same transformation properties under internal symmetries, while for Majorana spinors they have conjugate transformation properties; and the upper and lower components of a Dirac spinor are independent, while for a Majorana spinor they are redundant notations for the same field.

So the 4-spinor really is a (left,right) combination, and that must be where ½ ⊕ 0 + 0 ⊕ ½ is coming from. He is saying that in the Dirac case, the upper and lower subspinors are "independent", whereas in Majorana they are, as he clearly shows above, redundant. Either object (the Majorana spinor, or the Dirac spinor), transforms according to the representation ½ ⊕ 0 + 0 ⊕ ½.

Next here is why the "bar" appears:

The Hermitian conjugate of a spinor field ψ transforms as

$$U(-\omega)\psi^\dagger U^*(-\omega) = (D(\omega)\psi)^\dagger = \psi^\dagger D^\dagger(\omega)\,. \qquad (3.4.12)$$

The $\mathcal{J}_i$ are Hermitian, but the $\mathcal{K}_i$ are anti-Hermitian, so $D(\omega)$ is not in general unitary. Therefore $\psi^\dagger$ does not have the inverse transformation property of ψ. However, there is a Hermitian, unit determinant matrix β,

$$\beta \equiv \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}\,, \qquad \beta\mathcal{J}_i = \mathcal{J}_i\beta\,, \quad \beta\mathcal{K}_i = -\mathcal{K}_i\beta\,, \qquad (3.4.13)$$

which flips the sign of $\mathcal{K}$ but not $\mathcal{J}$ when commuted across $D^\dagger$, so $D^\dagger\beta = \beta D^{-1}$. Therefore, defining $\overline{\psi} = \psi^\dagger\beta$, called the *Dirac conjugate* of ψ,

$$U(-\omega)\overline{\psi}\beta U^*(-\omega) = \psi^\dagger D^\dagger(\omega)\beta = \psi^\dagger\beta D^{-1}(\omega) = \overline{\psi}D^{-1}(\omega)\,, \qquad (3.4.14)$$

so $\overline{\psi}$ has the inverse transformation property of ψ.

Note: last equation has a typo, should say U $\bar{\psi}$ U* with no β.

Note added 10.3.08. Derivation of the above equations. Start with this from our digression document, and compare to Moore's statement of the same thing.

$$\Lambda\, A_{\mathbf{r}}(x)\, \Lambda^{-1} \;=\; S(\mathbf{\Lambda^{-1}})_{\mathbf{rs}}\, A_{\mathbf{s}}(x') \qquad\qquad x' = \Lambda x$$

$$U(\omega)\varphi_a(x)U^*(\omega) = D^{-1}_{ab}(\omega)\varphi_b(\Lambda x)\,, \qquad\qquad (3.3.1)$$

so in Moore notation we have

$$U(\omega)\, \psi_{\mathbf{a}}(x)\, U^{\dagger}(\omega) \;=\; D^{-1}(\omega)_{\mathbf{ab}}\, \psi_{\mathbf{b}}(x')$$

which we can rewrite as follows to expose $D(\omega)_{\mathbf{ab}}$ on the RHS

$$U(-\omega)\, \psi_{\mathbf{a}}(x)\, U^{\dagger}(-\omega) \;=\; D(\omega)_{\mathbf{ab}}\, \psi_{\mathbf{b}}(x')$$

Everything is an operator equation in our QM HS, but of course $D(\omega)_{\mathbf{ab}}$ are just numbers. Think of the LHS as the product of three matrices, and apply † to both sides using the usual matrix rules to get:

$$U(-\omega)\,\psi^{\dagger}_{\mathbf{a}}(x)\,U^{\dagger}(-\omega) \;=\; [D(\omega)_{\mathbf{ab}}\,\psi_{\mathbf{b}}(x')]^{\dagger} \;=\; D(\omega)^{*}_{\mathbf{ab}}\,\psi_{\mathbf{b}}{}^{\dagger}(x') \;=\psi_{\mathbf{b}}{}^{\dagger}(x')\,D(\omega)^{\dagger}_{\mathbf{ba}}$$

or suppressing the 4x4 spinor indices,

$$U(-\omega)\,\psi^{\dagger}(x)\,U^{\dagger}(-\omega) \;=\; [D(\omega)\psi(x')]^{\dagger} \;=\; D(\omega)^{*}\,\psi^{\dagger}(x') \;=\psi^{\dagger}(x')\,D(\omega)^{\dagger}$$

Notice we handle the dagger on the matrix D sort of manually. The † on ψ is with respect to the infinite dim HS, while the † on D is a 4x4 matrix thing. So this is our derivation of (3.4.12) above. I continue however to show the spatial argument.

Now apply the 4x4 matrix $\beta = \gamma^{0}$ from the right. It of course passes through the U operator because it is not in the same space (obvious if you maintain the spinor indices), and we get

$$U(-\omega)\,\psi^{\dagger}(x)\,\beta\,U^{\dagger}(-\omega) \;=\; [D(\omega)\psi(x')]^{\dagger} \;=\; D(\omega)^{*}\,\psi^{\dagger}(x') \;=\psi^{\dagger}(x')\,D^{\dagger}(\omega)\,\beta = \psi^{\dagger}(x')\,\beta\,D^{-1}(\omega)$$

where we used fact P2 to do the last step, and this then agrees with 3.4.14 above. So we can compare:

$$U(-\omega)\,\psi(x)\,U^{\dagger}(-\omega) \;=\; D(\omega)\,\psi(x')$$

$$U(-\omega)\,\bar{\psi}(x)\,U^{\dagger}(-\omega) \;=\; \bar{\psi}(x')\,D^{-1}(\omega)$$

Then of course we get

$$U(-\omega)\,\bar{\psi}(x)\,\psi(x)\,U^{\dagger}(-\omega) \;=\; \bar{\psi}(x')\,D^{-1}(\omega)\,D(\omega)\,\psi(x') \;=\bar{\psi}(x')\,\psi(x')$$

which says that when we rotate our operator expression, we get

$$\bar{\psi}(x)\,\psi(x) \;\rightarrow \bar{\psi}(x')\,\psi(x')$$

Now recall our general field operator transformation rule from the digression document:

$$(A'')_{\mathbf{r}}(x') \;\equiv \Lambda^{-1}\,A_{\mathbf{r}}(x)\,\Lambda \;=\; S\,(\mathbf{\Lambda})_{\mathbf{rs}}\,A_{\mathbf{s}}(x) \qquad\qquad x' = \mathbf{\Lambda}x \qquad\qquad \text{// operator backward}$$

If $A_{\mathbf{r}}(x)$ transformed as a scalar (and had only one component), we would get

$$(A')(x') \equiv\; \Lambda\,A(x)\,\Lambda^{-1} \;=\; A(x') \qquad\qquad x' = \mathbf{\Lambda}x \qquad\qquad \text{// operator forward}$$

which says

$$\Lambda^{-1}\,A(x)\,\Lambda \;=\; A(x')$$

which we compare to the above which says

$$U(-\omega)\,\bar{\psi}(x)\,\psi(x)\,U^{\dagger}(-\omega) = \bar{\psi}(x')\,\psi(x')$$

So we can compare A(x) to $\bar{\psi}(x)\,\psi(x)$ and say that either one transforms as a world-scalar.

I recall Moore commented on **errata**, and I now see that he has a "complete replacement" for this entire appendix! Ouch! In the new version, he has changed to the Weinberg metric for $\eta_{\mu\nu}$ and he has corrected the above typo. Somewhere I have a program that can compare PDF's and indicate the differences. But neither CSDIFF nor my context menu program can do it. Back burner for now, let's just stay with the original appendix.

Side note on the finite rotation in general terms (no specific representation)

$U(\omega) = \exp(-i/2\ \omega_{\mu\nu}\hat{J}^{\mu\nu})$ $\quad J_i = i/2\ \varepsilon_{ijk}J_{jk}$ $\quad K_i = J_{i0}$ $\quad$ // $J_1 = i\ J_{23}$, etc

Now let's write out

$$\omega_{\mu\nu}\hat{J}^{\mu\nu} = \omega^{\mu\nu}J_{\mu\nu} = 2(\Sigma_i\ \omega^{i0}J_{i0} + \omega^{12}J_{12} + \omega^{23}J_{23} + \omega^{31}J_{31})$$

where the 2 comes from the non-listed terms which have all indices reversed. Now make the replacements:

$$\omega^{\mu\nu}J_{\mu\nu} = 2(\Sigma_i\ \omega^{i0}\ K_i + \omega^{12}J_3 + \omega^{23}J_1 + \omega^{31}J_2)$$

Looking back now at 3.1.9, we can write that $\omega^{\alpha\beta} = g^{\beta\nu}\omega^{\alpha}{}_{\nu} = \omega^{\alpha}{}_{\nu}g^{\nu\beta}$ in correct matrix order. Using 3.1.9 above, I find by hand that

$$\omega^{\alpha\beta} = \begin{pmatrix} 0 & -b1 & -b2 & -b3 \\ b1 & 0 & r3 & -r2 \\ b2 & -r3 & 0 & r1 \\ b3 & r2 & -r1 & 0 \end{pmatrix}$$

so that ω^{i0} = row i, col 0 = b_i and $\omega^{12} = +r_3$ and cyclic. This we have

$$\omega^{\mu\nu}J_{\mu\nu} = 2(\Sigma_i\ b_i\ K_i + r_3J_3 + \text{cyclic}) = 2(\mathbf{b}\bullet\mathbf{K} + \mathbf{r}\bullet\mathbf{J})$$

and then finally,

$U(\omega) = \exp(-i/2\ \omega_{\mu\nu}\hat{J}^{\mu\nu}) = \exp(-i\ (\mathbf{b}\bullet\mathbf{K} + \mathbf{r}\bullet\mathbf{J}))$ $\quad$ // which is the way I like to see things.

Of course b and r are real. I don't know anything more about the properties of K and J, such as whether they might be hermitian or symmetric, etc. That probably is representation dependent.

Resuming now,

Since ψ has 4 components, there are sixteen independent 4×4 matrices Γ which can be used to combine spinors, $\overline{\psi}_2 \Gamma \psi_1$. These can all be gotten from four such matrices, called the *gamma matrices* γ^μ, given in Eq. (1.3.38). These satisfy anticommutation relations called the *Clifford algebra*,

$$\left\{\gamma^\mu\,,\,\gamma^\nu\right\} = 2\eta^{\mu\nu}\mathbf{1}\,. \tag{3.4.15}$$

The matrices $\mathcal{J}^{\mu\nu}$ can be expressed in terms of the gamma matrices:

$$\mathcal{J}^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu\,,\,\gamma^\nu\right], \tag{3.4.16}$$

which together with Eq. (3.4.15) is enough to prove that $\mathcal{J}^{\mu\nu}$ satisfies the Lorentz algebra, Eq. (3.3.5). Further, these relations ensure that

$$\left[\mathcal{J}^{\mu\nu}\,,\,\gamma^\alpha\right] = -i\left(\eta^{\mu\alpha}\gamma^\nu - \eta^{\nu\alpha}\gamma^\mu\right)\,, \tag{3.4.17}$$

from which it follows that

$$D^{-1}(\omega)\gamma^\mu D(\omega) = \Lambda^\mu{}_\nu\gamma^\nu\,. \tag{3.4.18}$$

So here we have another form for the generator matrices in terms of γ matrices. And we get a clear statement of what the matrix representation does to the γ^μ.

TBC