The Lorentz Group PhL 10.4.08

This informal exploration is based on the notesy of Guy Moore of McGill, mainly his Appendix 2 or B from his book on the Standard Model. I downloaded this appendix from the web when I was wondering about the Dirac representation. We can look later at formal structures like Dynkin Diagrams, but for now I am interested more in the practical nuts and bolts stuff that is analogous to the rotation group data I have collected.

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1. Transformations U, parameters ω, and generators $J^{\mu\nu}$ for the Lorentz Group

The finite LT can be written in this manner:

$$U(\omega) = \exp(-i/2\ \omega_{\mu\nu}J^{\mu\nu})$$

where there are in theory a total of 16 items in the sum in the exponent. The "parameter" which I often call θ for a specific E^3 rotation is represented here by the real *matrix* ω which contains all 6 LT parameters, as we shall soon see. Each of the $J^{\mu\nu}$ objects is a generator. We usually think of each generator $J^{\mu\nu}$ as a matrix of some dimension N (possibly N=∞) perhaps with some matrix elements $[J^{\mu\nu}]_{ab}$. On the other hand, the object ω is always a 4x4 matrix with matrix elements $\omega_{\mu\nu}$. The sum $\omega_{\mu\nu}J^{\mu\nu}$ is thus a matrix of dimension N, and so is U.

The Lie Algebra for the generators is given by

$$[\ \mu\nu,\ \alpha\beta] = -i\ (\ \mu\alpha + \nu\beta - \nu\alpha - \mu\beta)$$

which is a compact notation standing for the following fully expanded algebra

$$[\ J^{\mu\nu},\ J^{\alpha\beta}] = -i\ (g^{\nu\beta}\ J^{\mu\alpha} + g^{\mu\alpha}\ J^{\nu\beta} - g^{\mu\beta}\ J^{\nu\alpha} - g^{\nu\alpha}\ J^{\mu\beta})$$

where we are using the BD metric g = (1,-1,-1,-1). To get from the compact notation to the full form, add a J in front if each index pair, raise the index pair, then "fill in" with g's in the missing indices. Different authors define the generators differently, sometimes the "i" is missing, we shall see this later. Notice that our ordering of the J terms is similar to that used in (μ-ν)(α-β) = μα + νβ -να - μβ, but of course this ordering is arbitrary.

Examination of the Lie Algebra shows the following to be true

$$[J^{\mu\nu},\ J^{\alpha\beta}] = [J^{\mu\nu},\ -\ J^{\beta\alpha}]$$

Since for any particular $J^{\alpha\beta}$ this is true for ALL $J^{\mu\nu}$, we think there is a proof somewhere that lets you conclude that the generators must be (or can be taken as, without loss of generality) antisymmetric so that

$$-\ J^{\beta\alpha} \quad = J^{\alpha\beta} \qquad \text{// true in any representation}$$

Given this fact, we know at once that, regardless of what the parameter matrix ω might be, we can write it as $\omega^S + \omega^A$ and only the antisymmetric portion can survive in the contraction $\omega_{\mu\nu}J^{\mu\nu}$ since J is antisymmetric. In other words:

$$\omega_{\mu\nu}J^{\mu\nu} = (\omega_{\mu\nu}{}^S + \omega_{\mu\nu}{}^A)\ J^{\mu\nu A} = \omega_{\mu\nu}{}^A J^{\mu\nu A} = \omega_{\mu\nu}{}^A J^{\mu\nu}$$

Thus, we can regard $\omega_{\mu\nu}$ as being antisymmetric and ignore any symmetric part it might have. Thus, the effective matrix ω will have only 6 distinct non-zero elements. For the LT, these 6 real parameters can be called r_i and b_i and the matrix ω can be written as follows (notice that it *is* antisymmetric).

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & b1 & b2 & b3 \\ -b1 & 0 & r3 & -r2 \\ -b2 & -r3 & 0 & r1 \\ -b3 & r2 & -r1 & 0 \end{pmatrix}$$

When we put this into the normal down-tilted index form, it has a different appearance and is no longer antisymmetric [g=diag(1,-1,-1,-1) changes the sign of everything in the last three rows]

$$\omega^{\mu}{}_{\nu} = g^{\mu\alpha}\omega_{\alpha\nu} = \begin{pmatrix} 0 & b1 & b2 & b3 \\ b1 & 0 & -r3 & +r2 \\ b2 & +r3 & 0 & -r1 \\ b3 & -r2 & +r1 & 0 \end{pmatrix}$$

While we're here, we might as well write the form with both indices up [here, since g is on the right in matrix order, we change the sign of everything in the last three *columns* of the above]

$$\omega^{\mu\nu} = \omega^{\mu}{}_{\alpha}\, g^{\alpha\nu} = \begin{pmatrix} 0 & -b1 & -b2 & -b3 \\ b1 & 0 & +r3 & -r2 \\ b2 & -r3 & 0 & +r1 \\ b3 & +r2 & -r1 & 0 \end{pmatrix}$$

and of course $\omega^{\mu\nu}$ is antisymmetric, which follows from $\omega^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}\omega_{\alpha\beta}$. Notice the following facts about this matrix, if we restrict to the 3x3 subspace,

$\omega^{ij} = \varepsilon_{ijk}r_k$ for example $\omega^{12} = \varepsilon_{123}r_3 = +r_3$ // row 1, column 2

We can thus also write

$$\omega^{ij}\,\varepsilon_{ijs} = \varepsilon_{ijk}r_k\,\varepsilon_{ijs} = 2\delta_{ks}r_k = 2r_s$$

Also, we have

$$\omega^{j0} = b_j$$

So we summarize these little facts as follows (they will be used below),

$r_k = \frac{1}{2}\,\omega^{ij}\,\varepsilon_{ijk}$
$b_k = \omega^{k0}$

When U(ω) is considered as a transformation in a covariantly normalized QM Hilbert Space, it is a unitary operator, so we can say

$$U(\omega) = \exp(-i/2\;\omega_{\mu\nu}J^{\mu\nu})$$

$U^{-1}(\omega) = \exp(+i/2\ \omega_{\mu\nu}J^{\mu\nu}) \ = U(-\omega)$
$U^{\dagger}(\omega) \ = \exp(+i/2\ \omega_{\mu\nu}[J^{\mu\nu}]^{\dagger})$

from which we conclude, in the usual way, that the generators in this space must be Hermitian

$[J^{\mu\nu}]^{\dagger} = J^{\mu\nu}$ // for generators in the QM HS

If we combine our last two results, we conclude that

$J^{\mu\nu} = J^{\nu\mu *} = - J^{\mu\nu *}$ // for generators in the QM HS

which says that in some sense, $J^{\mu\nu}$ = imaginary, more on this later.

2. The Vector Representation of the Lorentz Group: The Metric Tensor as a Tensor

In the preceding section, we dealt with the LG generators in the abstract. Here as an example we look at a specific 'representation' known as the vector representation which is at the heart of special relativity.

The LG transformation on 4-vectors plays a special role. It controls the way coordinates and momenta transform, these two quantities being the underpinning basic elements of all physics structures. The symbol Λ is often used for this "vector" representation, so we have

$U(\omega) = \exp(-i/2\ \omega_{\mu\nu}J^{\mu\nu}) = \Lambda$ // vector representation

By examining the known properties of the finite Λ transformation, we can work backwards to find the generators in this 4x4 vector representation, and they are found to be: (I have not proven this here)

$(J^{\mu\nu})^{\alpha}{}_{\beta} = i\ (\ g^{\mu\alpha}g^{\nu}{}_{\beta} - g^{\nu\alpha}g^{\mu}{}_{\beta})$ generators of $\Lambda^{\alpha}{}_{\beta}$

$(J^{\mu\nu})^{\alpha\beta} = i\ (\ g^{\mu\alpha}g^{\nu\beta} - g^{\nu\alpha}g^{\mu\beta})$

From the second form, it is obvious upon inspection that $J^{\mu\nu} = - J^{\nu\mu}$, validating our general rule shown earlier. Because the matrix elements of $J^{\mu\nu}$ are all imaginary, we find that

$[J^{\mu\nu}]^{\dagger} = [J^{\mu\nu}]^{T*} = [J^{\nu\mu}]^{*} = - J^{\nu\mu} = J^{\mu\nu}$

so the $\mathbf{J^{\mu\nu}}$ **are Hermitian** in this representation as well as the QM HS representation, which is in agreement with our previous comments on this issue.

Let's just compare two facts we now know:

(1) $[\ J^{\mu\nu}, J^{\alpha\beta}] = -i\ (g^{\nu\beta}\ J^{\mu\alpha} + g^{\mu\alpha}\ J^{\nu\beta} - g^{\mu\beta}\ J^{\nu\alpha} - g^{\nu\alpha}\ J^{\mu\beta})$ // for all representations

(2) $(J^{\mu\nu})^{\alpha\beta} = i\ (\ g^{\mu\alpha}g^{\nu\beta} - g^{\nu\alpha}g^{\mu\beta})$ // for the vector representation

In (1), g's appear in the Lie Algebra as coefficients. In (2), g's appear in the *matrix elements* of the J for this particular representation. These ideas should not be confused.

From our study of curvilinear coordinates, we learned that the metric *tensor* must transform in this way:

$$g_{kp} = T^{i}{}_{k}\, T^{j}{}_{p}\, g'_{ij}$$

where $T^{i}{}_{k}$ is a general transformation with x' = Tx, and g' is the metric tensor in the primed coordinates. In the case of special relativity, we know that g' = g and we therefore have the following condition on the Λ matrix elements:

$$g_{\alpha\beta} = \Lambda^{\mu}{}_{\alpha}\, \Lambda^{\nu}{}_{\beta}\, g_{\mu\nu}$$

The "condition" says that $g_{\mu\nu}$ must transform as a rank-2 tensor and be the same in both the primed and unprimed coordinate systems. The RHS can be put in normal matrix order as follows:

$$g_{\alpha\beta} = (\Lambda^{T})^{\alpha}{}_{\mu}\, g_{\mu\nu}\, \Lambda^{\nu}{}_{\beta}$$

and sometimes this is useful for calculations in Maple, for example (we do an example below).

In the vector representation we may write the transformation exponential quantity as follows

$$(-i/2)\omega_{\mu\nu}[\, J^{\mu\nu}\,]^{\alpha\beta} = (-i/2)\omega_{\mu\nu}\, i\,(\, g^{\mu\alpha}g^{\nu\beta} - g^{\nu\alpha}g^{\mu\beta}) = \tfrac{1}{2}\,\omega_{\mu\nu}(\, g^{\mu\alpha}g^{\nu\beta} - g^{\nu\alpha}g^{\mu\beta})$$

$$= \tfrac{1}{2}\,\{\, \omega^{\alpha\beta} - \omega^{\beta\alpha}\,\} = \omega^{\alpha\beta}$$

which provides a *considerable simplification* of the finite transformation for the vector representation,

$$\Lambda = \exp(-i/2\ \omega_{\mu\nu}J^{\mu\nu}) = \exp(\omega)$$

The "generators" seem to have disappeared and we are left only with the ω parameter matrix sitting in the exponent. In BD1 page 20 ω is written as $\Delta\omega$ when it is "small". As an exercise, we can examine the metric tensor transformation rule in matrix order given above, in the infinitesimal case,

$$g_{\alpha\beta} = [\exp(\omega)^{T}]^{\alpha}{}_{\mu}\, g_{\mu\nu}\ [\exp(\omega)]^{\nu}{}_{\beta} = [\, 1 + \omega^{T}\,]^{\alpha}{}_{\mu}\, g_{\mu\nu}\, [\, 1 + \omega]^{\nu}{}_{\beta}$$

$$= [\, \delta^{\alpha}{}_{\mu} + \omega^{\mu}{}_{\alpha}\,]\, g_{\mu\nu}\, [\, \delta^{\nu}{}_{\beta} + \omega^{\nu}{}_{\beta}\,]$$

$$= g_{\alpha\beta} + \omega^{\mu}{}_{\alpha}\, g_{\mu\beta} + g_{\alpha\nu}\, \omega^{\nu}{}_{\beta} + \text{order } \omega^{2}$$

$$\approx g_{\alpha\beta} + \omega_{\beta\alpha} + \omega_{\alpha\beta}$$

This calculation, often seen in text books like BD1, just confirms what we already know, which is that ω must be antisymmetric. If we did not know this fact, we could regard this vector representation as proof of

the fact, since the same matrix ω appears as the parameter matrix in ALL representations (that is, in all representations of U and the $J^{\mu\nu}$).

How might we compute the full vector transformation $\Lambda = \exp(\omega)$ with the matrix ω as stated above in Section 1? Let's hold off on this question for now, but we shall return to it.

We now return in the next several sections to the LG in the abstract or general case. Later, however, we shall return to examine more details of the vector representation.

3. The Poincare Group and its Lie Algebra

Before getting on to the J's and K's and other practical things, we pause to state the complete Lie algebra for the Poincare Group which includes the Lorentz Group as a subgroup

$[J^{\mu\nu}, J^{\alpha\beta}] = -i (g^{\nu\beta} J^{\mu\alpha} + g^{\mu\alpha} J^{\nu\beta} - g^{\mu\beta} J^{\nu\alpha} - g^{\nu\alpha} J^{\mu\beta})$ // Lorentz subalgebra

$[J^{\mu\nu}, P^{\alpha}] = -i(g^{\mu\alpha}P^{\nu} - g^{\nu\alpha}P^{\mu})$ // antisymmetric in μν

$[P^{\mu}, P^{\nu}] = 0$

There are 10 generators for the Poincare group, the non-zero $J^{\mu\nu}$ and the P^{μ}.

Schweber refers to the Poincare group as the "inhomogeneous Lorentz group", and shows a simple way to write a Poincare transformation using a 5 vector and a 5x5 matrix, see his page 45 in my binder notes from UCB.

4. The J_i and K_i generators of the Lorentz Group.

These are defined in terms of the generators $J^{\mu\nu}$ as follows

$J_i = \frac{1}{2} \varepsilon_{ijk} J_{jk} \quad => \quad J_1 = \frac{1}{2} (\varepsilon_{123}J_{23} + \varepsilon_{132}J_{32}) = J_{23}$

so we might just say

$J_i = J_{jk}$ where jk completes the cyclic order begun by i

Reversing things, we have

$J_{ij} = \varepsilon_{ijk}J_k$ valid for all i,j

For the K's we have

$K_i = J_{i0}$

Let's see if we can restate the Lie algebra of the $J^{\mu\nu}$ in terms of the J's and K's, the general algebra being

$$[\, J^{\mu\nu}, J^{\alpha\beta}] = -i\,(g^{\nu\beta} J^{\mu\alpha} + g^{\mu\alpha} J^{\nu\beta} - g^{\mu\beta} J^{\nu\alpha} - g^{\nu\alpha} J^{\mu\beta})$$

We can of course lower all indices and the algebra stays the same since it is a tensor equation

$$[\, J_{\mu\nu}, J_{\alpha\beta}] = -i\,(g_{\nu\beta} J_{\mu\alpha} + g_{\mu\alpha} J_{\nu\beta} - g_{\mu\beta} J_{\nu\alpha} - g_{\nu\alpha} J_{\mu\beta})$$

A. Compute $[\, J_i, J_s]$.

For J_j with J_i we can first restate the above with i=1,2,3 type indices

$$[\, J_{jk}, J_{mn}] = -i\,(g_{kn} J_{jm} + g_{jm} J_{kn} - g_{jn} J_{km} - g_{km} J_{jn})$$

Then apply ½ ε_{ijk} and ½ ε_{smn} to the LHS to get

$$[\, J_i, J_s] = -i\,¼\,\varepsilon_{ijk}\,\{\,\varepsilon_{smn}(g_{kn} J_{jm} + g_{jm} J_{kn} - g_{jn} J_{km} - g_{km} J_{jn})\,\}$$

Let's first consider just the 1st and 4th terms in {...}. We have

1st term = $\varepsilon_{smn}\, g_{kn}\, J_{jm} = \varepsilon_{snm}\, g_{km}\, J_{jn}$ (swap mn names) = $-\,\varepsilon_{smn}\, g_{km}\, J_{jn}$ (ε antisym) = 4th term

By the same consideration, we can see that 3rd term = 2nd term. So let's throw out the 2nd and 4th terms and install a factor of 2 and we get

$$[\, J_i, J_s] = -i\,½\;\varepsilon_{ijk}\,\{\,\varepsilon_{smn}(g_{kn} J_{jm} - g_{jn} J_{km})\,\} = -i\,½\,\varepsilon_{smn}\,[\,\varepsilon_{ijk}\,(g_{kn} J_{jm} - g_{jn} J_{km})\;]$$

Now we repeat the same argument, this time with ε_{ijk} and we find that the second term here is equal to the first term, so through out the second term and install another factor of 2, and we get

$$[\, J_i, J_s] = -i\,\varepsilon_{smn}\,\varepsilon_{ijk}\, g_{kn}\, J_{jm} = -i\,\varepsilon_{smn}\,\varepsilon_{ijk}\, g_{kn}\, J_{jm} \qquad J_{jm} = \varepsilon_{jmt} J_t$$

But in our 3x3 world here we know that $g_{kn} = -\,\delta_{kn}$ so the above becomes

$$[\, J_i, J_s] = +i\,\varepsilon_{smk}\,\varepsilon_{ijk}\, J_{jm} = +i\,\Sigma_{mkjt}\;\varepsilon_{smk}\,\varepsilon_{ijk}\,\varepsilon_{jmt} J_t \qquad \text{since } J_{jm} = \varepsilon_{jmt} J_t$$

Let's examine the triple ε product

$\varepsilon_{smk}\,\varepsilon_{ijk}\,\varepsilon_{jmt} = \varepsilon_{smk}\,\varepsilon_{jki}\,\varepsilon_{jmt}$ (cycle forward indices on central ε)

$= \varepsilon_{smk}\,(\,\delta_{km}\delta_{it} - \delta_{kt}\delta_{im}) = -\,\varepsilon_{sit}$ (since first term vanishes) $= +\,\varepsilon_{ist}$

Thus we get

$$[\, J_i, J_s] = +i\,\varepsilon_{ist} J_t$$

and we thus find that the J_i satisfy the usual rotation group Lie Algebra. This in effect justifies our identification of these J_i as the "usual" rotation generators. [The above derivation is probably not the simples one you could do, but it serves as a good exercise on using the ε tensor.]

B. Compute [J_i, K_s].

We start again with the general result

$$[J_{\mu\nu}, J_{\alpha\beta}] = -i (g_{\nu\beta} J_{\mu\alpha} + g_{\mu\alpha} J_{\nu\beta} - g_{\mu\beta} J_{\nu\alpha} - g_{\nu\alpha} J_{\mu\beta})$$

and we write $K_i = J_{i0}$ so set $\alpha = i$ and $\beta = 0$:

$$[J_{\mu\nu}, J_{i0}] = -i (g_{\nu 0} J_{\mu i} + g_{\mu i} J_{\nu 0} - g_{\mu 0} J_{\nu i} - g_{\nu i} J_{\mu 0})$$

Next, set μν = jk from the set 1,2,3 to write

$$[J_{jk}, J_{i0}] = -i (g_{k0} J_{ji} + g_{ji} J_{k0} - g_{j0} J_{ki} - g_{ki} J_{j0})$$

Right off the bat we can toss the terms with g_{k0} and g_{j0} since both these matrix elements are 0, so

$$[J_{jk}, J_{i0}] = -i (g_{ji} J_{k0} - g_{ki} J_{j0})$$

Next, we can set $g_{ji} = - \delta_{ji}$ etc to get

$$[J_{jk}, J_{i0}] = +i (\delta_{ji} J_{k0} - \delta_{ki} J_{j0})$$

Now apply $\frac{1}{2} \varepsilon_{sjk}$ to get

$$[J_s, J_{i0}] = i/2 \, \varepsilon_{sjk}(\delta_{ji} J_{k0} - \delta_{ki} J_{j0}) = i/2 (\varepsilon_{sik} J_{k0} - \varepsilon_{sji} J_{j0})$$

$$= i/2 (\varepsilon_{sik} J_{k0} + \varepsilon_{sij} J_{j0}) = i \varepsilon_{sik} J_{k0}$$

so we have now shown that

$$[J_s, J_{i0}] = i \varepsilon_{sik} J_{k0} \quad \Rightarrow \quad [J_s, K_i] = i \varepsilon_{sik} K_k$$

and this is our second important result. It just says that the K_i transform as a vector operator under rotations, so we could have guessed this result.

C. Compute [K_i, K_j].

We start again with the general result

$$[J_{\mu\nu}, J_{\alpha\beta}] = -i (g_{\nu\beta} J_{\mu\alpha} + g_{\mu\alpha} J_{\nu\beta} - g_{\mu\beta} J_{\nu\alpha} - g_{\nu\alpha} J_{\mu\beta})$$

and we write $K_i = J_{i0}$ and $K_j = J_{j0}$ so replace indices accordingly:

$$[J_{i0}, J_{j0}] = -i (g_{00} J_{ij} + g_{ij} J_{00} - g_{i0} J_{0j} - g_{0j} J_{i0}) = -i (g_{00} J_{ij}) = -i J_{ij} = -i \varepsilon_{ijk} J_k$$

where we toss terms that are obviously 0. Thus we have arrived at the result:

$$[J_{i0}, J_{j0}] = -i \varepsilon_{ijk} J_k \qquad \Rightarrow \qquad [K_i, K_j] = -i \varepsilon_{ijk} J_k$$

which is something new! The previous two just said J and K transform as vector operators, but this says something different.

D. Summary of the J,K Lie Algebra

$$[J_i, J_j] = +i \varepsilon_{ijk} J_k$$
$$[J_i, K_j] = +i \varepsilon_{ijk} K_k$$
$$[K_i, K_j] = -i \varepsilon_{ijk} J_k$$

$$J_i = \tfrac{1}{2} \varepsilon_{ijk} J_{jk} \qquad \Rightarrow \qquad J_1 = J_{23} \text{ etc} \qquad \text{and} \qquad J_{ij} = \varepsilon_{ijk} J_k$$

$$K_i = J_{i0}$$

Now we can re-examine our finite transformation argument

$$-i/2 \, \omega^{\mu\nu} J_{\mu\nu} = -i/2 \, \omega^{ij} J_{ij} - i \, \omega^{j0} J_{j0}$$

where we combine the ω_{0j} type terms into the last term and remove the ½. Then we have

$$-i/2 \omega_{\mu\nu} J^{\mu\nu} = -i/2 \, \omega^{ij} \varepsilon_{ijk} J_k - i \, \omega^{j0} K_j = -i \, r_k J_k - i \, b_j K_j = -i (\mathbf{r} \bullet \mathbf{J} + \mathbf{b} \bullet \mathbf{K})$$

where we used the "little facts" recorded earlier concerning the $\omega^{\mu\nu}$ matrix. Thus we can write our finite Lorentz Transformation in any representation as

$$U(\omega) = \exp(-i/2 \, \omega_{\mu\nu} J^{\mu\nu}) = \exp [-i (\mathbf{r} \bullet \mathbf{J} + \mathbf{b} \bullet \mathbf{K})]$$

and of course we know that the **J** are the rotation generators and the **K** are the boost generators. The J,K form is I think convenient if you want to think about only rotations or only boosts, in which case you set either **b** or **r** to zero.

5. Some details of the vector representation of the Lorentz Group

A. What are the vector representation J and K generators?

Recall that

$$(J_{\mu\nu})_{\alpha\beta} = i\,(\,g_{\mu\alpha}g_{\nu\beta} - g_{\nu\alpha}g_{\mu\beta})$$

Before going another step, we can see that any matrix that is a linear combination of the $(J_{\mu\nu})$ will be antisymmetric in the both-lowered indices, because each of the $J_{\mu\nu}$ matrices has this property. Thus, ahead of time we know that J_i and K_i are antisymmetric matrices.

Let's start setting $\mu\nu = ij$,

$$\Rightarrow\ (J_{ij})_{\alpha\beta} = i\,(\,g_{i\alpha}g_{j\beta} - g_{j\alpha}g_{i\beta})$$

Then we have

$$(J_k)_{\alpha\beta} = \tfrac{1}{2}\,\varepsilon_{kij}\,(J_{ij})_{\alpha\beta} = (i/2)\,\varepsilon_{kij}\,(g_{i\alpha}g_{j\beta} - g_{j\alpha}g_{i\beta}) = i\,\varepsilon_{kij}\,g_{i\alpha}g_{j\beta}$$

since the second term $-\varepsilon_{kij}\,g_{j\alpha}g_{i\beta} = -\varepsilon_{kji}\,g_{i\alpha}g_{j\beta} = +\varepsilon_{kij}\,g_{i\alpha}g_{j\beta}$ = first term. If $\alpha = 0$ or $\beta = 0$, $(J_k)_{\alpha\beta} = 0$ by inspection. Otherwise we get

$$(J_k)_{mn} = i\,\varepsilon_{kij}\,g_{im}g_{jn} = i\,\varepsilon_{kij}\,\delta_{im}\delta_{jn} = i\varepsilon_{kmn}$$

Therefore, we find

$$(J_i)_{jk} = +i\,\varepsilon_{ijk}$$

which we combine to get the full result:

$$(J_i)_{0i} = (J_i)_{i0} = (J_i)_{00} = 0 \qquad (J_i)_{jk} = +i\,\varepsilon_{ijk}$$

And the general result for J can be written as follows

$$(J_a)_{\alpha\beta} = i\,\varepsilon_{abc}g_{b\alpha}g_{c\beta}$$

For example, we have $(J_1)_{23} = +i\,\varepsilon_{123} = +i$, so

$$(J_1)_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} \qquad \text{but} \qquad (J_1)^{\mu}{}_{\nu} = g^{\mu\alpha}(J_1)_{\alpha\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

where in the second form we negate the last three rows.

This says that the 4x4 rotation generators have 0's in the first row and column, then the remaining 3x3 region is the same as that of the normal rotation group generator matrix. As claimed at the start, these the matrices are each antisymmetric.

Technical Note: Notice that we have $(J_i)^j{}_k = -i\,\varepsilon_{ijk}$ which agrees with our "usual" rotation group result. When we deal with the BD metric tensor, we have to remember that this $(J_i)^j{}_k$ with the down-tilted indices is that thing we normally call $(J_i)_{jk}$ when we have the metric tensor (1,1,1).

What about the K generators. Start again with

$$(J_{\mu\nu})_{\alpha\beta} = i\,(g_{\mu\alpha}g_{\nu\beta} - g_{\nu\alpha}g_{\mu\beta}) \qquad \Rightarrow$$

$$(J_{i0})_{\alpha\beta} = i\,(g_{i\alpha}g_{0\beta} - g_{0\alpha}g_{i\beta}) = (K_i)_{\alpha\beta}$$

Thus we have

$$(K_i)_{jk} = i\,(g_{ij}g_{0k} - g_{0j}g_{ik}) = 0 + 0 = 0$$

$$(K_i)_{j0} = i\,(g_{ij}g_{00} - g_{0j}g_{i0}) = i\,(g_{ij}g_{00}) = i\,g_{ij} = -i\,\delta_{ij}$$

$$(K_i)_{0k} = i\,(g_{i0}g_{0k} - g_{00}g_{ik}) = -\,i\,(g_{ik}g_{00}) = -\,i\,g_{ik} = +i\,\delta_{ik}$$

$$(K_i)_{00} = i\,(g_{i0}g_{00} - g_{00}g_{i0}) = 0 + 0 = 0$$

So the boost matrices have i's only in the space part of the first row and first column. The 3x3 region and the main diagonal are all 0. For example: $(K_1)_{j0} = -i\,\delta_{1j}$, so we have

$$(K_1)_{\mu\nu} = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{but} \qquad (K_1)^{\mu}{}_{\nu} = g^{\mu\alpha}(K_1)_{\alpha\nu} = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where once again in the second form we have to change the sign of the last three rows.

B. Powers of the J_i matrix in the vector representation.

While we are here, we should compute powers of the generator matrices, since they will be needed in the next section. For the rotations we can write:

$$(J_i)^{\alpha}{}_{\beta} = i\,\varepsilon_{ijk}g_j{}^{\alpha}g_{k\beta} \qquad \text{j,k are summed 1 to 3}$$

$$[(J_i)^2]^{\alpha}{}_{\beta} = (J_i)^{\alpha}{}_{\mu}\,(J_i)^{\mu}{}_{\beta} = -\,\varepsilon_{ijk}g_j{}^{\alpha}g_{k\mu}\,\varepsilon_{iab}g_a{}^{\mu}g_{b\beta} \qquad \text{// sum on j,k,a,b,}\mu$$

$$= -\,\varepsilon_{ijk}\,g_j{}^{\alpha}\,g_{ka}\,\varepsilon_{iab}\,g_{b\beta} \qquad \text{// sum on j,k,a,b}$$

$$= +\,\varepsilon_{ijk}\,\delta_j{}^{\alpha}\,\delta_{ka}\,\varepsilon_{iab}\,g_{b\beta} \qquad \text{// sum on j,k,a,b}$$

$= + \varepsilon_{ijk}\ \delta_j{}^{\alpha}\ \varepsilon_{ikb}\ g_{b\beta}$ // sum on j,k,b

$= + \varepsilon_{kij}\ \varepsilon_{kbi}\ \delta_j{}^{\alpha}\ g_{b\beta}$ // sum on j,k,b

$= (\delta_{ib}\delta_{ji} - \delta_{ii}\delta_{jb})\ \delta_j{}^{\alpha}\ g_{b\beta}$ // sum on j,b

$= \delta_{ib}\delta_{ji}\ \delta_j{}^{\alpha}\ g_{b\beta} - \delta_{ii}\delta_{jb}\ \delta_j{}^{\alpha}\ g_{b\beta}$ // sum on j,b

$= \delta_{ji}\ \delta_j{}^{\alpha}\ g_{i\beta} - \delta_{ii}\ \delta_j{}^{\alpha}\ g_{j\beta}$ // sum on j

$= \delta_i{}^{\alpha}\ g_{i\beta} - \delta_j{}^{\alpha}\ g_{j\beta}$ // sum on j in 2nd term

$= \delta_i{}^{\alpha}\ g_{\alpha\beta} - \delta_j{}^{\alpha}\ g_{\alpha\beta}$

$= g_{\alpha\beta}\ (\delta_i{}^{\alpha} - \delta_j{}^{\alpha})$ // sum on j in 2nd term

$= -\ \delta_{\alpha\beta}\ (\delta_i{}^{\alpha} - \Sigma_j\delta_j{}^{\alpha})\ = \delta_{\alpha\beta}\ (\Sigma_j\delta_j{}^{\alpha} - \delta_i{}^{\alpha})$

$= \delta_{\alpha\beta}\ (\delta_1{}^{\alpha} + \delta_2{}^{\alpha} + \delta_3{}^{\alpha} - \delta_i{}^{\alpha})$

This is a diagonal 4x4 matrix which has 1's on the diagonal in the two spatial locations ≠ i and two 0's. For example, if i = 1 the result is diag(0,0,1,1). Each equals line above requires some thinking to justify, but I think the result is correct. Our conclusion from the above is this:

$[(J_i)^2]^{\alpha}{}_{\beta}\ = \delta^{\alpha}{}_{\beta}\ (\Sigma_j\delta_j{}^{\alpha} - \delta_i{}^{\alpha})$

Now consider

$[(J_i)^3]^{\alpha}{}_{\beta}\ = (J_i)^{\alpha}{}_{\mu}\ [(J_i)^2]\ ^{\mu}{}_{\beta}\ = i\ \varepsilon_{iab}g_a{}^{\alpha}g_{b\mu} * \delta^{\mu}{}_{\beta}\ (\Sigma_j\delta_j{}^{\mu} - \delta_i{}^{\mu})$ // sum on a,b,μ

$= i\ \varepsilon_{iab}g_a{}^{\alpha}g_{b\beta}\ (\Sigma_j\delta_j{}^{\beta} - \delta_i{}^{\beta})$

$= i\ \varepsilon_{iab}g_a{}^{\alpha}g_{b\beta}\ \Sigma_j\delta_j{}^{\beta} - i\ \varepsilon_{iab}g_a{}^{\alpha}g_{b\beta}\ \delta_i{}^{\beta}$

Consider this painful combination which we claim can be simplified as shown,

$g_{b\beta}\ \Sigma_j\ \delta_j{}^{\beta}\ = g_{b\beta}$

To justify this, if β = 0, both sides are 0. Otherwise if β = k then have $\Sigma_j\ \delta_j{}^{k}\ = 1$ so get $g_{bk} = g_{bk}$. Thus we have

$[(J_i)^3]^{\alpha}{}_{\beta}\ = i\ \varepsilon_{iab}g_a{}^{\alpha}g_{b\beta}\ - i\ \varepsilon_{iab}g_a{}^{\alpha}g_{b\beta}\ \delta_i{}^{\beta}$ // sum on a,b

$= i\ \varepsilon_{iab}g_a{}^{\alpha}\ g_{b\beta}\ (1 - \delta_i{}^{\beta})$

Look now at this second term. If $\beta = 0$, it makes no contribution, otherwise suppose $\beta = k$, then get

$$\text{2nd term} = -\,i\,\varepsilon_{iab} g_a{}^{\alpha}\, g_{bk}\, \delta_i{}^{k} \;=\; +i\,\varepsilon_{iab} g_a{}^{\alpha}\, \delta_{bk}\, \delta_i{}^{k} = +i\,\varepsilon_{iak} g_a{}^{\alpha}\, \delta_i{}^{k} = 0$$

where in the last step we have $\varepsilon_{iak}\,\delta_i{}^{k} = 0$ on symmetry. Thus we finally arrive at the result,

$$[(J_i)^3]^{\alpha}{}_{\beta} = i\,\varepsilon_{iab} g_a{}^{\alpha}\, g_{b\beta} = (J_i)^{\alpha}{}_{\beta} \qquad \text{recall } (J_i)^{\alpha}{}_{\beta} = i\,\varepsilon_{ijk} g_j{}^{\alpha} g_{k\beta}$$

Comments: These calculations are very painful because we are doing non-covariant things, we have some sums going 1 to 3, others 0 to 3, it is very ugly. Perhaps it is easier to realize that the J_i matrix can be written in this form

$$(J_i)^{\alpha}{}_{\beta} \;=\; \begin{pmatrix} 0 & 0 \\ 0 & J_i \end{pmatrix}$$

where inside the inner matrix J_i is the 3x3 submatrix. We know when we take powers of the 4x4 J, nothing happens outside the 3x3 region, everything will remain 0 outside. So we need only ponder the 3x3 region in our power computations. Let's redo the above calculation from this viewpoint:

$$(J_i)^{a}{}_{b} = i\,\varepsilon_{ijk} g_j{}^{a} g_{kb} = -\,i\,\varepsilon_{ijk}\delta_j{}^{a}\delta_{kb} = -i\,\varepsilon_{iab} \qquad (J_1)^2{}_3 = -i\,\varepsilon_{123} = -i \text{ , agrees}$$

Then

$$[(J_i)^2]^{a}{}_{b} = (J_i)^{a}{}_{c}(J_i)^{c}{}_{b} = (-i)^2\,\varepsilon_{iac}\,\varepsilon_{icb} = -\,\varepsilon_{iac}\,\varepsilon_{icb} = -\,\varepsilon_{cia}\,\varepsilon_{cbi} = +\,\varepsilon_{cia}\,\varepsilon_{cib}$$

$$= (\delta_{ii}\delta_{ab} - \delta_{ib}\delta_{ai}) = (\delta_{ii}\delta_{ab} - \delta_{ai}\delta_{ab}) = \delta_{ab}(1 - \delta_{ia})$$

This is a 3x3 diagonal matrix which has two 1's at the $\neq i$ positions, and a 0 at the i position. For example, we have $[(J_1)^2]^{a}{}_{b} = \text{diag}(1\text{-}1,1,1) = \text{diag}(0,1,1)$.

Then

$$[(J_i)^3]^{a}{}_{b} = (J_i)^{a}{}_{c}[(J_i)^2]^{c}{}_{b} = -i\,\varepsilon_{iac} * \delta_{cb}(1 - \delta_{ic}) = -i\,\varepsilon_{iab} = (J_i)^{a}{}_{b}$$

where we know $\varepsilon_{iac}\,\delta_{ic} = 0$ from symmetry. This sure is easier than what I did before.

So now we know all powers of J_i in the 3x3 world as follows:

$$[(J_i)^n]^{j}{}_{k} = (J_i)^{j}{}_{k} = -i\,\varepsilon_{ijk} \qquad \text{if } n = 1,3,5,7..... \qquad \text{odd powers} \qquad (J_i)^n = J_i$$

$$[(J_i)^n]^{j}{}_{k} = \delta_{jk}(1 - \delta_{ij}) = [(J_i)^2]^{j}{}_{k} \qquad \text{if } n = 2,4,6,8.... \qquad \text{even powers} \qquad (J_i)^n = (J_i)^2$$

or more compactly

$$(J_i)^n = J_i \qquad \text{odd powers} \qquad = -i\,\varepsilon_{ijk}$$

$(J_i)^n = (J_i)^2$ even powers $= \delta_{jk}(1 - \delta_{ij})$

$(J_i)^{n+2} = (J_i)^n$ all powers

Let's present these results graphically,

$$(J_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \qquad (J_1)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(J_1)^n = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \quad n = 1,3,5... \qquad (J_2)^n = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \quad n = 1,3,5...$$

$$(J_1)^n = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad n = 2,4,6.... \qquad (J_2)^n = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad n = 2,4,6....$$

For J_2 and J_3 , the only anomoly is that for J_2 the -i occurs in the lower left half of (J_2), but the even powers have the obvious form nevertheless. This is true in effect because $\sigma_2^2 = 1$ but also $(-\sigma_2)^2 = 1$, so in the even powers, you always have +1's on the diagonal. We show this case on the right above.

C. Powers of the K_i matrix in the vector representation.

From above we had

$$(K_i)_{\alpha\beta} = i (g_{i\alpha}g_{0\beta} - g_{0\alpha}g_{i\beta})$$

$$(K_i)^{\alpha}_{\ \beta} = i (g_i{}^{\alpha}g_{0\beta} - g_0{}^{\alpha}g_{i\beta})$$

Then

$$(K_i)^{\alpha}_{\ \mu}(K_i)^{\mu}_{\ \beta} = (-1) (g_i{}^{\alpha}g_{0\mu} - g_0{}^{\alpha}g_{i\mu}) (g_i{}^{\mu}g_{0\beta} - g_0{}^{\mu}g_{i\beta})$$

$$= (-1) (g_i{}^{\alpha}g_{0\mu}\, g_i{}^{\mu}g_{0\beta} + g_0{}^{\alpha}g_{i\mu}\, g_0{}^{\mu}g_{i\beta} - g_i{}^{\alpha}g_{0\mu}\, g_0{}^{\mu}g_{i\beta} - g_0{}^{\alpha}g_{i\mu}\, g_i{}^{\mu}g_{0\beta})$$

$$= (-1) (g_i{}^{\alpha}g_{0\mu}\, \delta_i{}^{\mu}g_{0\beta} + g_0{}^{\alpha}g_{i\mu}\, \delta_0{}^{\mu}g_{i\beta} - g_i{}^{\alpha}g_{0\mu}\, \delta_0{}^{\mu}g_{i\beta} - g_0{}^{\alpha}g_{i\mu}\, \delta_i{}^{\mu}g_{0\beta})$$

$$= (-1) (g_i{}^{\alpha}g_{0i}\, g_{0\beta} + g_0{}^{\alpha}g_{i0}g_{i\beta} - g_i{}^{\alpha}g_{00}g_{i\beta} - g_0{}^{\alpha}g_{ii}g_{0\beta}) \qquad \text{// no sums!}$$

$= (-1)(-g_i{}^{\alpha}g_{00}g_{i\beta} - g_0{}^{\alpha}g_{ii}g_{0\beta}) = g_i{}^{\alpha}g_{00}g_{i\beta} + g_0{}^{\alpha}g_{ii}g_{0\beta} = (g_i{}^{\alpha}g_{i\beta} + g_0{}^{\alpha}g_{ii}g_{0\beta})$

$= (\delta_i{}^{\alpha}g_{i\beta} + \delta_0{}^{\alpha}g_{ii}g_{0\beta}) = (\delta_i{}^{\alpha}g_{i\beta} - \delta_0{}^{\alpha}g_{0\beta})$

Let's try the cube now

$(K_i)^{\alpha}{}_{\mu}(K_i{}^2)^{\mu}{}_{\beta} = i(g_i{}^{\alpha}g_{0\mu} - g_0{}^{\alpha}g_{i\mu})(\delta_i{}^{\mu}g_{i\beta} - \delta_0{}^{\mu}g_{0\beta})$

$= i(\delta_i{}^{\alpha}g_{0\mu} - \delta_0{}^{\alpha}g_{i\mu})(\delta_i{}^{\mu}g_{i\beta} _ \delta_0{}^{\mu}g_{0\beta})$

$= i(\delta_i{}^{\alpha}g_{0\mu}\,\delta_i{}^{\mu}g_{i\beta} - \delta_i{}^{\alpha}g_{0\mu}\,\delta_0{}^{\mu}g_{0\beta} - \delta_0{}^{\alpha}g_{i\mu}\,\delta_i{}^{\mu}g_{i\beta} + \delta_0{}^{\alpha}g_{i\mu}\,\delta_0{}^{\mu}g_{0\beta})$

$= i(\delta_i{}^{\alpha}g_{0i}\,g_{i\beta} - \delta_i{}^{\alpha}g_{00}\,g_{0\beta} + \delta_0{}^{\alpha}g_{i\beta} + \delta_0{}^{\alpha}g_{i0}g_{0\beta})$

$= i(-\delta_i{}^{\alpha}g_{00}\,g_{0\beta} + \delta_0{}^{\alpha}g_{i\beta}) = -i(\delta_i{}^{\alpha}g_{0\beta} - \delta_0{}^{\alpha}g_{i\beta}) = -i(g_i{}^{\alpha}g_{0\beta} - g_0{}^{\alpha}g_{i\beta}) = -(K_i)^{\alpha}{}_{\beta}$

Not too bad. So far then we have:

$(K_i)^{\alpha}{}_{\beta} = i(g_i{}^{\alpha}g_{0\beta} - g_0{}^{\alpha}g_{i\beta})$

$(K_i{}^2)^{\alpha}{}_{\beta} = (\delta_i{}^{\alpha}g_{i\beta} - \delta_0{}^{\alpha}g_{0\beta})$

$(K_i{}^3) = -(K_i)$

$(K_i{}^4) = (K_i{}^3)(K_i) = -(K_i{}^2)$

$(K_i{}^5) = (K_i{}^4)(K_i) = -(K_i{}^3) = +K_i$ etc

so our rule seems to be:

$(K_i{}^n) = (-1)^{(n-1)/2}\ K_i$ odd powers $= (-1)^{(n-1)/2}\ i(g_i{}^{\alpha}g_{0\beta} - g_0{}^{\alpha}g_{i\beta})$

$(K_i{}^n) = (-1)^{n/2+1}\ (K_i)^2$ even powers $= (-1)^{n/2+1}\ (\delta_i{}^{\alpha}g_{i\beta} - \delta_0{}^{\alpha}g_{0\beta})$

$(K_i{}^{n+2}) = -(K_i{}^n)$ all powers!

Here are the first two powers of K_1 , just to see it in the flesh,

$$K_1 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (K_1)^2 = -\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

D. Computation of the finite boost and rotation matrices in the vector representation.

So, how do we compute these objects

$U(\mathbf{r},\mathbf{0}) = \exp(-i\,\mathbf{r}\bullet\mathbf{J})$
$U(\mathbf{0},\mathbf{b}) = \exp(-i\,\mathbf{b}\bullet\mathbf{K})$

In the exponential expansion of either, we will encounter factors like this

$(\mathbf{r}\bullet\mathbf{J})^n = r_i J_i \; r_j J_j \; ....$

which are painful to deal with. The better way to evaluate the expansions shown is to use our knowledge of rotating operators which we have studied now "at great length".

In general, we are interested in the index positions as in $\Lambda^{\mu}{}_{\nu}$, so we will use indices in those positions all the time.

D.1 Rotation and Boost Matrices along axes.

Let's start with the following case

$\exp(-irJ_1) = \cos(rJ_1) - i\sin(rJ_1)$

and we shall use our rules from above

$(J_i)^n = J_i$	odd powers	$= -i\,\varepsilon_{ijk}$
$(J_i)^n = (J_i)^2$	even powers	$= \delta_{jk}(1 - \delta_{ij})$
$(J_i)^{n+2} = (J_i)^n$	all powers	

Then we have

$$\cos(rJ_1) = 1 - (rJ_1)^2/2! + (rJ_1)^4/4! - \ldots = 1 + (J_1)^2\{ - r^2/2! + r^4/4! + \ldots \}$$

$$= 1 + (J_1)^2\{-1 + 1 - r^2/2! + r^4/4! + \ldots \}$$

$$= 1 + (J_1)^2\{ -1 + \cos(r) \}$$

$$\sin(rJ_1) = (rJ_1) - (rJ_1)^3/3! + \ldots = (J_1)\{ r - r^3/3! + \ldots\} = J_1 \sin(r)$$

Thus we get

$$\exp(-irJ_1) = 1 + (J_1)^2\{ -1 + \cos(r) \} - i\,J_1 \sin(r)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + (\cos(r) - 1) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - i \sin(r) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(r)-1 & 0 \\ 0 & 0 & 0 & \cos(r)-1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sin(r) \\ 0 & 0 & +\sin(r) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(r) & -\sin(r) \\ 0 & 0 & \sin(r) & \cos(r) \end{pmatrix}$$

which is our traditional result. And we know what the other cases will be, no need to compute them.

Now let's do a boost matrix:

$$\exp(-ibK_1) = \cos(bK_1) - i \sin(bK_1)$$

and we shall use our rules from above,

$(K_i^n) = (-1)^{(n-1)/2} K_i$ odd powers $= (-1)^{(n-1)/2}$ $i (g_i^\alpha g_{0\beta} - g_0^\alpha g_{i\beta})$

$(K_i^n) = (-1)^{n/2+1} (K_i)^2$ even powers $= (-1)^{n/2+1}$ $(\delta_i^\alpha g_{i\beta} - \delta_0^\alpha g_{0\beta})$

$(K_i^{n+2}) = -(K_i^n)$ all powers!

Then we have

$$\cos(bK_1) = 1 - (bK_1)^2/2! + (bK_1)^4/4! - (bK_1)^6/6!... = 1 - (K_1)^2 \{ + b^2/2! + b^4/4! + b^6/6! ... \}$$

$$= 1 - (K_1)^2 \{-1 + 1 + b^2/2! + b^4/4! + ... \}$$

$$= 1 - (K_1)^2 \{ -1 + \cosh(b) \}$$

$$\sin(bK_1) = (bK_1) - (bK_1)^3/3! + ... = (K_1) \{ b + b^3/3! + ...\} = K_1 \sinh(b)$$

Thus we get

$$\exp(-irJ_1) = 1 - (K_1)^2 \{ -1 + \cosh(b) \} - i K_1 \sinh(r)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + (\cosh(b) - 1) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - i \sinh(b) \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \cosh(b)-1 & 0 & 0 & 0 \\ 0 & \cosh(b)-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sinh(b) & 0 & 0 \\ \sinh(b) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh(b) & \sinh(b) & 0 & 0 \\ \sinh(b) & \cosh(b) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \exp(-ibK_1)$$

And we can see clearly what the other two boost matrices will be.

D.2 Rotations and Boosts along arbitrary directions

Now we want the matrices for *arbitrary directions*. We can do it like this:

$$R \exp(-i\, \mathbf{r}\bullet\mathbf{J}) R^{-1} = \exp(-i\, \mathbf{r} \bullet R\, \mathbf{J}\, R^{-1}) = \exp(-i\, \mathbf{r} \bullet R^{-1}\, \mathbf{J})$$

$$\exp(-i\, R\, \mathbf{r} \bullet\ \mathbf{J}) = \exp(-i\, r\, \hat{\mathbf{z}} \bullet \mathbf{J}) = \exp(-i\, rJ_3)$$

where we select our rotation R such that $R\, \hat{\mathbf{r}} = \hat{\mathbf{z}}$. Here R is the obvious 4x4 extension of my usual 3x3 rotation matrix, and in fact all the R's are the same in this application. We already know that

$$\exp(-i\, rJ_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(r) & -\sin(r) & 0 \\ 0 & \sin(r) & \cos(r) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

so we are left with the problem of finding that matrix R such that $R\, \hat{\mathbf{r}} = \hat{\mathbf{z}}$. But from our "sphericals.doc" paper, we know that in the 3x3 world,

$$\hat{\mathbf{r}} = R_z(\varphi)\, R_y(\theta)\, \hat{\mathbf{z}} = \begin{pmatrix} \cos\varphi\cos\theta & -\sin\varphi & \cos\varphi\sin\theta \\ \sin\varphi\cos\theta & \cos\varphi & \sin\varphi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\varphi\ \sin\theta \\ \sin\varphi\ \sin\theta \\ \cos\theta \end{pmatrix}$$

So the matrix shown here is R^{-1} and we can get $R = (R^{-1})^T$. Let's just call these things R. Then the solution to our problem is this:

$$\exp(-i\,\mathbf{r}\bullet\mathbf{J}) = R^{-1}\exp(-i\,rJ_3)\,R$$

Let's just stick to the 3x3 world since we know how to get from there to the 4x4. Then we have

$$\exp(-i\,\mathbf{r}\bullet\mathbf{J}) =$$

$$\begin{pmatrix} \cos\varphi\cos\theta & -\sin\varphi & \cos\varphi\sin\theta \\ \sin\varphi\cos\theta & \cos\varphi & \sin\varphi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos r & -\sin r & 0 \\ \sin r & \cos r & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\varphi\cos\theta & \sin\varphi\cos\theta & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \\ \cos\varphi\sin\theta & \sin\varphi\sin\theta & \cos\theta \end{pmatrix}$$

which certainly looks like a holy mess. I don't think I have ever written this result down before, but maybe when Maple tells me the answer, I will realize the simpler way to do this calculation. What we are talking about here is this:

$$\exp(-i\,\mathbf{r}\bullet\mathbf{J}) = R_z(\varphi)\,R_y(\theta)\,R_z(r)\,R^{-1}{}_y(\theta)\,R^{-1}{}_z(\varphi)$$

Did this come up in Levitt somewhere? (no!) I probably have a Maple file somewhere with these things pre-entered, and if not, I should make such a Maple file. Did this come up in Goldstein somewhere? (no!) Well, I told Maple to evaluate the above, and after spending time doing some simplify work, the best I could get was this [see "sphere coord matrices.mws" in curvilinear/spherical]

$$\begin{bmatrix} \cos(r)+(\cos(\phi)^2-\cos(\phi)^2\cos(r))\sin(\theta)^2 & -\cos(\theta)\sin(r)+(-\cos(\phi)\sin(\phi)\cos(r)+\sin(\phi)\cos(\phi))\sin(\theta)^2 & \sin(\theta)\sin(r)\sin(\phi)+(-\cos(\phi)\cos(r)+\cos(\phi))\cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(r)+(-\cos(\phi)\sin(\phi)\cos(r)+\sin(\phi)\cos(\phi))\sin(\theta)^2 & \cos(r)+(\cos(\phi)^2\cos(r)+1-\cos(r)-\cos(\phi)^2)\sin(\theta)^2 & -\sin(\theta)\sin(r)\cos(\phi)+(-\sin(\phi)\cos(r)+\sin(\phi))\cos(\theta)\sin(\theta) \\ -\sin(\theta)\sin(r)\sin(\phi)+(-\cos(\phi)\cos(r)+\cos(\phi))\cos(\theta)\sin(\theta) & \sin(\theta)\sin(r)\cos(\phi)+(-\sin(\phi)\cos(r)+\sin(\phi))\cos(\theta)\sin(\theta) & 1+(-1+\cos(r))\sin(\theta)^2 \end{bmatrix}$$

which is not very enlightening! This is the rotation matrix for rotating angle r about an axis $\hat{\mathbf{r}}$ which is located at $\hat{\mathbf{r}} = R_z(\varphi)\,R_y(\theta)\,\hat{\mathbf{z}}$. To verify the above is correct, here are a few test cases:

set $\theta=\pi/2$ and $\varphi = 0$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(r) & -\sin(r) \\ 0 & \sin(r) & \cos(r) \end{bmatrix}$$

set $\theta=\pi/2$ and $\varphi = \pi/2$

$$\begin{bmatrix} \cos(r) & 0 & \sin(r) \\ 0 & 1 & 0 \\ -\sin(r) & 0 & \cos(r) \end{bmatrix}$$

set $\theta = 0$, φ = anything

$$\begin{bmatrix} \cos(r) & -\sin(r) & 0 \\ \sin(r) & \cos(r) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since these are all exactly correct, I think the general result is correct as well.

In a separate document I went off and directly exponentiated exp (– i **r•J**) using $\mathbf{r} = \theta\hat{\mathbf{n}}$ and I obtained the following much simpler form that what Maple is showing from the $R_z(\varphi)\,R_y(\theta)$ method. In fact, the result is this:

$$R_n(\theta) = \exp(-i\theta\mathbf{n}\bullet\mathbf{J}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (1-C_\theta)\begin{pmatrix} 1-n_1^2 & -n_1n_2 & -n_1n_3 \\ -n_1n_2 & 1-n_2^2 & -n_2n_3 \\ -n_1n_3 & -n_2n_3 & 1-n_3^2 \end{pmatrix} + S_\theta\begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$$

Again, this gets embedded in the 4x4 matrix in the obvious manner.

Let's try a similar thing for the boost matrix. I guess the result would be this:

$$\exp(-i\,\mathbf{b}\bullet\mathbf{K}) = R_z(\varphi)\,R_y(\theta)B_z(b)\,R^{-1}{}_y(\theta)\,R^{-1}{}_z(\varphi)$$

but now we have to do everything with 4x4 matrices. Our boost matrix will be this:

$$B_z(b) = \exp(-ibK_3) = \begin{pmatrix} \cosh(b) & 0 & 0 & \sinh(b) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(b) & 0 & 0 & \cosh(b) \end{pmatrix}$$

When I enter the above into Maple, I get this result, (file is LG matrices.mws)

$$\begin{bmatrix} 1+f & \sinh(b)\,nx & \sinh(b)\,ny & \sinh(b)\,nz \\ \sinh(b)\,nx & (-f\,nz^2+f)\cos(\phi)^2+1 & (-f\,nz^2+f)\cos(\phi)\sin(\phi) & f\,nz\,nx \\ \sinh(b)\,ny & (-f\,nz^2+f)\cos(\phi)\sin(\phi) & (-f+f\,nz^2)\cos(\phi)^2+1-f\,nz^2+f & f\,nz\,ny \\ \sinh(b)\,nz & f\,nz\,nx & f\,nz\,ny & 1+f\,nz^2 \end{bmatrix}$$

where I have defined the following

$\mathbf{b} = b\hat{\mathbf{n}}$
nx = sin(θ) cos(φ)
ny = sin(θ) sin(φ)
nz = cos(θ)
f = cosh(b) - 1

I sort of expected the top and left edge of this result, but the rotation part is a bit mysterious. It is interesting that angle φ appears outside the nx type expressions, but θ does not appear! Also, notice that the matrix is symmetric (and real), so we have $B = B^{\dagger}$ as claimed earlier.

To verify the above, I set θ = π/2 and φ = 0 and I do get the $B_x(b)$ matrix I expect. And changing to φ = π/2 gives $B_y(b)$, so I am now more confident that the above matrix is probably correct.

By playing with the above 4x4 matrix, I see now that it can be written in this more compact form, where only the direction cosines appear, very much nicer! But I had to use the above method to get this result, don't know how else I could have gotten here:

$$\exp(-i\,\mathbf{b}\bullet\mathbf{K}) = \begin{pmatrix} \mathrm{ch}(u) & \mathrm{sh}(b)n_1 & \mathrm{sh}(b)n_2 & \mathrm{sh}(b)n_3 \\ \mathrm{sh}(b)n_1 & 1+fn_1^2 & fn_1n_2 & fn_1n_3 \\ \mathrm{sh}(b)n_2 & fn_1n_2 & 1+fn_2^2 & fn_2n_3 \\ \mathrm{sh}(b)n_3 & fn_1n_3 & fn_2n_3 & 1+fn_3^3 \end{pmatrix} \quad \text{where } f = \mathrm{ch}(b) - 1 \qquad \mathbf{b} = b\hat{\mathbf{n}}$$

Interestingly, after doing the above, I found the exact matrix above in my Berkeley notes page 6 of the third binder tab. I obtained it there using the BCH method and it was pretty simple.

Recall that the most general Lorentz transformation is given, in the vector representation, by

$$\Lambda = \exp(-i/2\ \omega_{\mu\nu}J^{\mu\nu}) = \exp\left[-i\left(\mathbf{r}\bullet\mathbf{J} + \mathbf{b}\bullet\mathbf{K}\right)\right] = \exp(\omega) \qquad \text{// last form specific to vector rep}$$

One might ask what this most general 4x4 matrix Λ looks like! We have seen that even the pure rotation and pure boost matrices are pretty ugly, so one can only imagine that this thing will be a horrendous mess. In general **r** and **b** can point in different directions so each would require its own pair of θ,φ angles. My rotation trick won't work for this fact, so I don't even have an approach to solving this problem. perhaps it becomes easier when we deal with the left and right generators below. Maybe there is no closed form for this thing?? Certainly I have never seen it written down. I would assume there is some sort of Chasle's Theorem which says you can write any Λ as perhaps a rotation followed by a boost. In any event, I will keep my eye open for an answer to this question of what the most general LT looks like.

E. A fact about the ω matrix that is a bit confusing

The form of U in *any* representation involves the ω matrix,

$$U = \exp(-i/2\ \omega_{\mu\nu}J^{\mu\nu})$$

In the *vector* representation this becomes just

$$U = \exp(\omega) = \exp\left[-i\left(\mathbf{r}\bullet\mathbf{J} + \mathbf{b}\bullet\mathbf{K}\right)\right]$$

Thus, in the vector representation, we can write

$$\mathbf{r}\bullet\mathbf{J}^{\mu}{}_{\nu} + \mathbf{b}\bullet\mathbf{K}^{\mu}{}_{\nu} = i\omega(\mathbf{r},\mathbf{b})^{\mu}{}_{\nu} = i\begin{pmatrix} 0 & b1 & b2 & b3 \\ b1 & 0 & -r3 & +r2 \\ b2 & +r3 & 0 & -r1 \\ b3 & -r2 & +r1 & 0 \end{pmatrix}$$

If we set for example $\mathbf{r} = \hat{\mathbf{x}}$ and $\mathbf{b} = 0$ the above becomes

$$\mathbf{J_1}^{\mu}{}_{\nu} = i\omega(\hat{\mathbf{x}},\mathbf{0})^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

Thus, the as you insert the six unit vectors into the matrix $i\omega(\mathbf{r},\mathbf{b})^{\mu}{}_{\nu}$, you produce the 6 generators of the vector representation.

What is a little confusing is this fact: When you examine some *other* representation, such as the Dirac representation, you have

$$U = \exp(-i/2\ \omega_{\mu\nu}J^{\mu\nu}) = \exp(-i/4\ \omega_{\mu\nu}\sigma^{\mu\nu})$$

where the generators are $J^{\mu\nu} = i/2\sigma^{\mu\nu}$, but as you examine individual axis-oriented boosts and rotations, you find yourself involved with things like $\omega(\hat{\mathbf{x}},\mathbf{0})^{\mu}{}_{\nu}$ which are the generators of the *vector* representation, not the Dirac representation. In other words, ω is the paramater matrix for *all* representations, and its unit vector values *happen* to equal (when multiplied by i) the vector representation generators. An example of this occurs on BD1 page 21 where we would say that

$$\mathbf{K_1}^{\mu}{}_{\nu} = i\omega(\mathbf{0},\hat{\mathbf{x}})^{\mu}{}_{\nu} = i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \omega(\mathbf{0},\hat{\mathbf{x}})^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is what BD show in (2.18) + (2.19) but they have a minus sign because they deal here with a passive boost instead of an active boost.

6. The L and R generators for the Lorentz Group

We are now back to the general case again. Above we started with generators $J^{\mu\nu}$, and then we recast these into J_i and K_i. Here, we want to recast once more into 6 generators called L_i and R_i. These generators are an important form because with them, the Lie Algebra "decouples". We will need this information from above,

$$[\ J_i, J_j\ ] = +i\ \varepsilon_{ijk}J_k$$
$$[\ J_i, K_j\ ] = +i\ \varepsilon_{ijk}\ K_k = [\ K_i, J_j\ ] \qquad // [\ K_i, J_j\ ] = -\ [J_j, K_i\ ] = -\ i\ \varepsilon_{jik}\ K_k = +\ i\ \varepsilon_{ijk}\ K_k$$
$$[\ K_i, K_j\ ] = -i\ \varepsilon_{ijk}J_k$$

$$J_i = \tfrac{1}{2}\ \varepsilon_{ijk}\ J_{jk} \quad \Rightarrow \quad J_1 = J_{23} \text{ etc} \qquad \text{and} \qquad J_{ij} = \varepsilon_{ijk}J_k$$
$$K_i = J_{i0}$$

Now a few preliminary things using the above: take # 1± i #2 above to get

$$[\ J_i, J_j \pm iK_j\ ] = +i\ \varepsilon_{ijk}\ (J_k \pm iK_k) = \quad \Rightarrow \quad [\ J_i, H^{\pm}{}_j\ ] = +i\ \varepsilon_{ijk}\ H^{\pm}{}_k$$

where we define the following item, which we have just seen commutes like any vector

$$H^{\pm}{}_k \equiv J_k \pm iK_k$$

Let's recheck the aboveL

$$[J_i, J_j \pm iK_j] = [J_i, J_j] \pm i [J_i, K_j] = +i \varepsilon_{ijk} J_k \pm i (+i) \varepsilon_{ijk} K_k = +i \varepsilon_{ijk}(J_k \pm i K_k)$$

Now, what does K_i do with this new thing? detail: $(\pm i) * (-i) = (+i) * (\mp i)$

$$[K_i, J_j \pm iK_j] = [K_i, J_j] \pm i [K_i, K_j] = + i \varepsilon_{ijk} K_k \pm i (-i \varepsilon_{ijk} J_k) = +i \varepsilon_{ijk} (K_k \mp iJ_k)$$

$$= \pm (-i \varepsilon_{ijk})(iJ_k \mp K_k) = \pm (-i \varepsilon_{ijk}) i (J_k \pm i K_k) = \pm \varepsilon_{ijk}(J_k \pm i K_k)$$

so we learn that

$$[J_i, H^{\pm}_j] = +i \varepsilon_{ijk} H^{\pm}_k$$

$$[K_i, H^{\pm}_j] = \pm \varepsilon_{ijk} H^{\pm}_k$$

So let's just define $\mp$

$$L_i = \tfrac{1}{2} (J_i + iK_i) = \tfrac{1}{2} H^{+}_i$$
$$R_i = \tfrac{1}{2} (J_i - iK_i) = \tfrac{1}{2} H^{-}_i$$

And here comes the algebra

$$[L_i, L_j] = \tfrac{1}{4} [J_i + iK_i, H^{+}_j] = \tfrac{1}{4} \{ [J_i, H^{+}_j] + i [K_i, H^{+}_j] \} = \tfrac{1}{4} \{ +i \varepsilon_{ijk} H^{+}_k + i \varepsilon_{ijk} H^{+}_k \} =$$

$$\tfrac{1}{2} i \varepsilon_{ijk} H^{+}_k = i \varepsilon_{ijk} L_k \qquad \text{// correct}$$

$$[R_i, R_j] = \tfrac{1}{4} [J_i - iK_i, H^{-}_j] = \tfrac{1}{4} \{ [J_i, H^{-}_j] - i [K_i, H^{-}_j] \} = \tfrac{1}{4} \{ +i \varepsilon_{ijk} H^{-}_k + i \varepsilon_{ijk} H^{-}_k \} =$$

$$\tfrac{1}{2} i \varepsilon_{ijk} H^{-}_k = i \varepsilon_{ijk} R_k \qquad \text{// correct}$$

$$[L_i, R_j] = \tfrac{1}{4} [J_i + iK_i, H^{-}_j] = \tfrac{1}{4} \{ [J_i, H^{-}_j] + i [K_i, H^{-}_j] \} = \tfrac{1}{4} \{ +i \varepsilon_{ijk} H^{-}_k - i \varepsilon_{ijk} H^{-}_k \}$$

$$= 0 \qquad \text{// correct}$$

So we arrive at this interesting algebra:

$$[L_i, L_j] = i \varepsilon_{ijk} L_k \qquad L_i = \tfrac{1}{2} (J_i + iK_i)$$
$$[R_i, R_j] = i \varepsilon_{ijk} R_k \qquad R_i = \tfrac{1}{2} (J_i - iK_i)$$
$$[L_i, R_j] = 0$$

And we now have 6 generators which partition into two non-interacting "rotation group" algebras. It seems pretty clear that the Casimir's are going to be L^2 and R^2 and the representations are called therefore

(j_1,j_2) $\quad L^2 = j_1(j_1+1)$ $\quad R^2 = j_2(j_2+1)$

We can now invert to find

$J_i = (L_i + R_i)$ $\quad iK_i = L_i - R_i$ $\quad K_i = (-i)(L_i - R_i) = i(R_i - L_i)$
$K_i = i(R_i - L_i)$

Also we can see that

$L^2 = \frac{1}{2}(J_i + iK_i)\,\frac{1}{2}(J_i + iK_i) = \frac{1}{4}(J^2 + iK\bullet J + iJ\bullet K - K^2)$
$R^2 = \frac{1}{2}(J_i - iK_i)\,\frac{1}{2}(J_i - iK_i) = \frac{1}{4}(J^2 - iK\bullet J - iJ\bullet K - K^2)$

Add and subtract to get

$L^2 + R^2 = \frac{1}{2}(J^2 - K^2)$
$L^2 - R^2 = \frac{1}{2}(i\mathbf{K}\bullet\mathbf{J} + i\,\mathbf{J}\bullet\mathbf{K}) = (i/2)(\mathbf{K}\bullet\mathbf{J} + \mathbf{J}\bullet\mathbf{K})$

But $\mathbf{K}\bullet\mathbf{J} = K_1J_1 + \ldots = J_1K_1 \ldots$ so we know that

$\mathbf{K}\bullet\mathbf{J} = \mathbf{J}\bullet\mathbf{K}$

So we then have

$L^2 + R^2 = \frac{1}{2}(J^2 - K^2) = j_1(j_1+1) + j_2(j_2+1)$
$L^2 - R^2 = i\,\mathbf{J}\bullet\mathbf{K} = j_1(j_1+1) - j_2(j_2+1)$

In any event, we can see that the two invariants of the J,K world are $(J^2 - K^2)$ and $\mathbf{J}\bullet\mathbf{K}$. Each of these things must commute with all 6 generators. Let's just verify that here

$[J_i, J_j] = +i\,\varepsilon_{ijk}J_k$
$[J_i, K_j] = +i\,\varepsilon_{ijk}K_k = [K_i, J_j]$
$[K_i, K_j] = -i\,\varepsilon_{ijk}J_k$

So we have

$[J_i, \mathbf{J}\bullet\mathbf{K}] = [J_i, J_j K_j] = [J_i, J_j]K_j + J_j[J_i, K_j] = +i\,\varepsilon_{ijk}J_k K_j + J_j(+i\,\varepsilon_{ijk})K_k$
$= i\varepsilon_{ijk}(J_k K_j + J_j K_k) = 0$ by symmetry

$[K_i, \mathbf{J}\bullet\mathbf{K}] = [K_i, J_j K_j] = [K_i, J_j]K_j + J_j[K_i, K_j] = +i\,\varepsilon_{ijk}K_k K_j + J_j(-i\,\varepsilon_{ijk})J_k$

$= i\varepsilon_{ijk}(K_k K_j - J_j J_k) = 0+0 = 0$ by symmetry

I will skip showing this for $J^2 - K^2$, I have no doubt it verifies.

7. Question: Why do we identify (½ , ½) with the 4-vector representation?

See Section 17 below for a detailed answer to this question, but here are some older notes just to keep.

To answer this question, I first perused my relevant books and found nothing much. Then I did a very long web search, found a few things noted below, but not my answer. Then, amazingly enough, I dug into my 30 year old box of Berkeley notes and found the answer! And I see that those notes have lots of LG information, and I also have Xerox of portions of many relevant books!

I don't see the connection at all right now. Moore does not comment on this. None of my books on group theory talks about this. A Google on-line book has some stuff

http://books.google.com/books?id=ZtthVxxc3SkC&pg=PA164&lpg=PA164&dq=lorentz+group+representations+1/2,1/2&source=web&ots=QfgoSmjgoc&sig=yIh8KJPFro6aH0qDx20ovzxG_HY&hl=en&sa=X&oi=book_result&resnum=4&ct=result#PPA163,M1

For example, we see this:

Since the highest weight (s,t) uniquely identifies an irreducible representation of $so(1,3)$, we shall refer to the representation $D_{s,t}$ as (s,t). Recall that the dimension of (s,t) is $(2s+1)(2t+1)$. The vector representation has dimension four. The only four-dimensional irreducible representations of $so(1,3)$ have weights $(3/2,0)$, $(1/2,1/2)$, and $(0,3/2)$. Since the vector representation is real (equivalent to its complex conjugate), it must be the $(1/2,1/2)$ representation.

From Theorem 6.8.3, the tensor product of two vector representations of $so(1,3)$ decomposes into a sum

$$(1/2,1/2)\otimes(1/2,1/2)=(0,0)+(1,0)+(0,1)+(1,1), \qquad (6.10.1)$$

Here is another Google book which has a LOT of stuff, but focus is on gen rel.

http://books.google.com/books?hl=en&id=GmGal5SEyqAC&dq=lorentz+group+representations+pdf&printsec=frontcover&source=web&ots=tb6A2Wvff_&sig=pu3CpG-Io3BS5eMgswJ_L91X5Cc&sa=X&oi=book_result&resnum=8&ct=result#PPA19,M1

Weinberg's book on page 58 has a small chunk on LG reps, and he says on page 60 that it is a "straightforward calculation" to show that the (1/2,1/2) is Λ", but he does not give this calculation. He refers to the vector generators when he says this.

I found Wigner's famous 1939 paper on line (on LG reps) and saved it. He makes this claim which reminds us of the Euler angles for a general rotation and of Chasle's theorem:

this, the two coördinate systems can differ only in a rotation. This means that every homogeneous Lorentz transformation can be decomposed in the following way[17]

(21) $$\Lambda = RZS$$

where R and S are rotations, and Z is a simple z-boost. His paper is quite long, and I think the answer to my question lies in this structure I dimly remember, where you embed your 4-vector components into a 2x2 matrix. Remember that SO(3,1) is like SL(2,C).

This has been a tough search, I am finding not much.

I *suspect* something like this: [a good suspicion!]

$$\Lambda_{\mu\nu} = \tfrac{1}{2}\, tr(\sigma_\mu \Lambda \sigma_\nu \Lambda^\dagger)$$

which I am just making up in analogy with the rotation item on my rotation matrix page. I don't know the SL(2,C) 2x2 representations. [yes I do!] I see that my UCB box has lots of stuff on group theory, glad I kept it. One folder is called "UIR's of SO(3,1) = SL(2,C). In on section I have some notes on a book of Gelfand, Graev & and Vilenkin (Volume 5 of "Generalized Functions" 1966. I see now what I was really looking for:

Some pages later I see this interesting claim

$$F = -1/2\, M_{\mu\nu}M^{\mu\nu} \qquad \text{and } G = \tfrac{1}{4}\, \varepsilon^{\mu\nu\rho\sigma} M_{\mu\nu}M_{\rho\sigma}$$

are the two Casimirs of the LG, shown in covariant notation. Wonder how related to the K and J ones?

My binder has some good (if tiny print) Liubarskii notes, here is one section:

The two basic spinor reps he calls $\tau_{1/2,0}$ and $\tau_{0,1/2}$. Using these, you can create complicated " spinor tensors" with first and second kind spinor indices, fine. But now what about the $\tau_{1/2,1/2}$ representation? He points out that from a rotation point of view only, you can do $\tfrac{1}{2} \otimes \tfrac{1}{2} = 1 \oplus 0$ reduction (ie, $\tfrac{1}{2} \otimes \tfrac{1}{2}$ is not irreducible if you only consider rotations). This corresponds to the fact that the 4-vector is made up of x° and $\mathbf{x}$ which belong to these two rotation group representations! That is the big point I was missing. But since boosts link these two reps, when boosts are included you find that the rep $\tfrac{1}{2} \otimes \tfrac{1}{2}$ is irreducible for the LG as a whole.

This really is a direct product deal, and you can take your basis as $|1/2 ,m;1/2,m'\rangle = |m,m'\rangle$. On page 300 Liu talks about basis elements of the form $e_{\pm 1/2,\pm 1/2}$ which are of course $e_{mm'}$ and that is just what I am talking about. He calls this the "canonical basis". But then he talks about $e_0 = (1,0,0,0)$ etc as the "natural basis". He then finds out how these two bases are related!

How exactly does he do this? He uses some strange letters I = rotation generators and J = boost generators, and he shows J_1 on page 289. He uses a nice little notation e_{01} to indicate a 1 in that position in the matrix, so he can then just write $J_1 = -e_{01}-e_{10}$, something I did not think of doing, see 66.1. His *decoupled* generators are called A and B on that same page.

He starts by figuring out what his I and J generators do to the natural basis elements like (1,0,0,0). Then he finds for example that (note that $A_0 = A_3$ for some reason)

$$A_3 (e_2 - ie_1) = -\tfrac{1}{2}\, i (e_2 - ie_1) \qquad B_3 (e_2 - ie_1) = -\tfrac{1}{2}\, i (e_2 - ie_1)$$

so he concludes (ignoring minus signs) that $e_{1/2,1/2} = (1/\sqrt{2})\ (e_2 - ie_1)$. In this way he gets all the relations and then inverts. He then finds a matrix ρ which makes the connection

$e_i = \rho_{i,mm'}\ e_{mm'}$ as shown top page 301.

So this matrix IS the basis change matrix which takes you from the 4x4 world of |m,m'> to the 4x4 world of x^μ !!! So this IS the connection between the (1/2,1/2) rep of the LG and our vector rep, and that is what I have been seeking.

Main points of the answer to the question: How is (1/2,1/2) rep related to Λ (Liubarskii)

(1) He points out that from a rotation point of view only, you can do ½ ⊗ ½ = 1 ⊕ 0 reduction (ie, ½ ⊗ ½ is not irreducible if you only consider rotations). This *corresponds to the fact* that the 4-vector is made up of x° and **x** which belong to these two rotation group representations! That is the big point I was missing. But since boosts link these two reps, when boosts are included you find that the rep ½ ⊗ ½ is irreducible for the LG as a whole. The other 4x4 reps of LG such as (3/2,0) don't have the right rotation group decomposition so cannot be the ones that go with Λ !

(2) The 4-vector basis with elements like (1,0,0,0) is related to the A,B decoupled basis |1/2 ,m;1/2,m'> by a matrix which Lyubarskii directly calculates!

(3) Probably when you study the $D^{(1/2,1/2)}_{mn,m'n'}(a)$ matrix elements and shuffle them by the basis change matrix just mentioned, you get Λ(a) from the formula $\Lambda^\mu{}_\nu = ½\ tr(\sigma_\mu a \sigma_\nu a^\dagger)$. See later section.

I have much more information on this subject in Section 17 below.

8L. The Left Spinor Representation and the Weyl Left Handed Spinor

We now have this decoupled algebra and its relation to the J,K generators:

$[L_i, L_j] = i\,\varepsilon_{ijk}\,L_k$ $\quad L_i = ½\,(J_i + iK_i)$ $\quad J_i = (L_i + R_i)$
$[R_i, R_j] = i\,\varepsilon_{ijk}\,R_k$ $\quad R_i = ½\,(J_i - iK_i)$ $\quad K_i = i(R_i - L_i)$
$[L_i, R_j] = 0$

$L^2 + R^2 = ½\,(J^2 - K^2)$ $\quad = j_1(j_1+1) + j_2(j_2+1)$
$L^2 - R^2 = i\,\mathbf{J}\bullet\mathbf{K}$ $\quad = j_1(j_1+1) - j_2(j_2+1)$

In general, using the L/R algebra, we know we can enumerate all FDR's of the LG as j1 ⊗ j2 where we are implying L ⊗ R. We know that Pauli σ/2 form the 2D lowest non-trivial rep with j = ½, so we are now going to look at the ½ ⊗ 0 representation which we will call "left spinor". That is to say:

$L_i = \sigma_i/2$
$R_i = 0$

There is hardly anything to do! We can compute the J and K like so:

$J_i = (L_i + R_i) = L_i = \sigma_i/2$
$K_i = i(R_i - L_i) = -iL_i = -i\sigma_i/2$

The implication here is that $R_i = 0$. Such a generator assures that $e^{-ia\bullet R} = 1$ so in the "0" representation, we represent all group elements by "1". Our notation ½ ⊗ 0 is showing j values, and j=0 means "1" as the group representation. Our 2x2 finite group elements (matrices) are now

$$U(\omega) = \exp(-i/2\ \omega_{\mu\nu}J^{\mu\nu}) = \exp[-i(\mathbf{r}\bullet\mathbf{J} + \mathbf{b}\bullet\mathbf{K})] = \exp[-(i/2)(\mathbf{r}\bullet\boldsymbol{\sigma} - i\,\mathbf{b}\bullet\boldsymbol{\sigma})]$$

$$= \exp[-(i/2)(\mathbf{r} - i\,\mathbf{b})\bullet\boldsymbol{\sigma})] = \exp[-(i/2)\,\boldsymbol{\alpha}\bullet\boldsymbol{\sigma})] \equiv U(\boldsymbol{\alpha}) \qquad \boldsymbol{\alpha} \equiv (\mathbf{r} - i\,\mathbf{b})$$

where we have defined a *complex* parameter vector **α** as shown. Another notation for the above would be

$$U(\boldsymbol{\alpha}) = D^{(1/2\ \otimes\ 0)}(\boldsymbol{\alpha}) = D^{(1/2)}(\boldsymbol{\alpha}) \qquad D = \text{Darstellung} < G \text{ "representation"}$$

The spinors ψ_L you act on with this representation are called Weyl Left Handed Spinors. And you would write transformations as follows

$\psi_L'^a(x') = U(\boldsymbol{\alpha})^a{}_b\psi_L{}^b(x)$ $\quad x' = \Lambda x$ $\quad \psi$ = classical field amplitude

$U(\omega)\,\psi_L(x)\,U^{-1}(\omega) = [U(\boldsymbol{\alpha})]^{-1}\,\psi_L(x')$ $\quad \psi$ = field operator

$U(\omega)\,[\psi_L]^a(x)\,U^{-1}(\omega) = [\,U^{-1}(\boldsymbol{\alpha})]^a{}_b\ [\psi_L]^b(x')$ $\quad \psi$ = field operator

$U(-\omega)\,\psi_L(x)\,U^{-1}(-\omega) = U(\boldsymbol{\alpha})\,\psi_L(x')$ $\quad \psi$ = field operator

For a derivation and dicussion of these equations, see a separate and very detailed document entitled "explanation of the rotation of a vector operator.doc". The unitary operators U appearing in these last two equations are not 2x2 matrices, they are infinite dimensional unitary operators which act in a covariantly normalized QM Hilbert Space. Think of U(ω) as U(**a**,**b**) if you like.

8R. The Right Spinor Representation and the Weyl Right Handed Spinor

Here we are talking $0 \otimes ½$ so that now

$L_i = 0$
$R_i = \sigma_i/2$

There is hardly anything to do! We can compute the J and K like so:

$J_i = (L_i + R_i) = R_i = \sigma_i/2$
$K_i = i(R_i - L_i) = +iR_i = +i\sigma_i/2$

$U(\omega) = \exp(-i/2\ \omega_{\mu\nu}J^{\mu\nu}) = \exp[-i(\mathbf{r}\bullet\mathbf{J} + \mathbf{b}\bullet\mathbf{K})] = \exp[-(i/2)(\mathbf{r}\bullet\boldsymbol{\sigma} + i\,\mathbf{b}\bullet\boldsymbol{\sigma})]$

$= \exp[-(i/2)(\mathbf{r} + i\,\mathbf{b})\bullet\boldsymbol{\sigma})] = \exp[-(i/2)\,\boldsymbol{\alpha}^{*}\bullet\boldsymbol{\sigma})] \equiv U(\boldsymbol{\alpha}^{*})$ $\qquad \boldsymbol{\alpha} \equiv (\mathbf{r} - i\,\mathbf{b})$

where we have not changed our defintion of $\boldsymbol{\alpha}$ and overbar means complex conjugation.

$U(\bar{\boldsymbol{\alpha}}) = D^{(0\ \otimes\ 1/2)}(\boldsymbol{\alpha}) = D^{(1/2)}(\bar{\boldsymbol{\alpha}})$

$\psi_{R}^{\prime\dot{a}}(x') = U(\boldsymbol{\alpha}^{*})^{\dot{a}}{}_{\dot{b}}\ \psi_{R}^{\dot{b}}(x)$ $\quad x' = \Lambda x$ $\qquad\qquad$ ψ = classical field amplitude

$U(\omega)\,\psi_{R}(x)\,U^{-1}(\omega) = [U(\boldsymbol{\alpha}^{*})]^{-1}\,\psi_{R}(x')$ $\qquad\qquad$ ψ = field operator

$U(-\omega)\,\psi_{R}(x)\,U^{-1}(-\omega) = U(\boldsymbol{\alpha}^{*})\,\psi_{R}(x')$ $\qquad\qquad$ ψ = field operator

Here, the dots tell us which representation we are talking about. You might have a fancy "spinor tensor" with a mix of the two types, and the dots tell you which ones are which. When you make "vector tensors", there is only one kind of index. Both tensor and spinor indices can be "up or down". In passing, I am wondering what that "spinor metric tensor" is.

Notice that the $U(\omega) = U(\mathbf{a},\mathbf{b})$ appearing in the last equation here is exactly the same as that appearing in the corresponding Left spinor transformation equation. What is different is the type of spinor, and the 2x2 matrix on the RHS which does the transformation.

Comments: You might wonder why these two representations are *different*. It would seem that you could say something like $U_{1/2\ \otimes\ 0}(\mathbf{a},\mathbf{b}) = U_{\ 0\ \otimes\ 1/2}(\mathbf{a},-\mathbf{b})$. So this goes on my questions list for now.

9. How to convert a Left into a Right Spinor

Now consider the notion of transforming a spinor from one type to the other as follows:

$\psi_{L}' = -\varepsilon\,\psi_{R}^{*}$ $\qquad \varepsilon = i\,\sigma_{2}$ $\qquad \psi_{R}' = +\varepsilon\,\psi_{L}^{*}$

Why would this be true? First, these preliminaries,

$\varepsilon^{-1} = (i\,\sigma_{2})^{-1} = (i\,)^{-1}\sigma_{2}^{-1} = (-i)\,\sigma_{2} = -\varepsilon$
$\varepsilon^{\dagger} = (-i)\,\sigma_{2}^{\dagger} = -i\,\sigma_{2} = -\varepsilon$
$\sigma_{2}\,\boldsymbol{\sigma}\,\sigma_{2} = -\boldsymbol{\sigma}^{*}$ $\quad\Rightarrow\quad$ $\sigma_{2}\,\boldsymbol{\sigma} = -\boldsymbol{\sigma}^{*}\,\sigma_{2}$
$\varepsilon\,\boldsymbol{\sigma}\,\varepsilon = i\,\sigma_{2}\,\boldsymbol{\sigma}\,i\,\sigma_{2} = -\sigma_{2}\,\boldsymbol{\sigma}\,\sigma_{2} = +\boldsymbol{\sigma}^{*}$ $\quad\Rightarrow\quad$ $\varepsilon\,\boldsymbol{\sigma}\,\varepsilon = \boldsymbol{\sigma}^{*}$

Apply $\varepsilon^{-1} = -\varepsilon$ from the right to get

$\varepsilon\,\boldsymbol{\sigma} = -\boldsymbol{\sigma}^{*}\varepsilon$

Now here is how a Left spinor field operator transforms, where we use Moore's trick of going to -ω on the LHS just to get rid of the inverse complication on the RHS.

$$U(-\omega)\,\psi_L(x)\,U^{-1}(-\omega) = U(\boldsymbol{\alpha})\,\psi_L(x')$$

********* Extra information needed:

At this point I cannot continue because I need to know what $U(\omega)^*$ is, where * means CC. Suppose I knew somehow that $U(\omega)$ was real. For the moment, I will just assume this. Then I could complex conjugate the above line and say

$$U(-\omega)\,\psi_L^*(x)\,U^{-1}(-\omega) = U(\boldsymbol{\alpha})\,\psi_L^*(x')$$

Now I can apply ε from the left to get (it acts only on spinor indices, not HS indices)

$$U(-\omega)\,\varepsilon\,\psi_L^*(x)\,U^{-1}(-\omega) = \varepsilon\,U(\boldsymbol{\alpha})^*\,\psi_L^*(x')$$

But we know that $\varepsilon\,\boldsymbol{\sigma}^* = -\boldsymbol{\sigma}\,\varepsilon$ so we can say

$$\varepsilon\,U(\boldsymbol{\alpha})^* = \varepsilon\exp[-(i/2)\,\boldsymbol{\alpha}\bullet\boldsymbol{\sigma})]^* = \varepsilon\exp[+(i/2)\,\boldsymbol{\alpha}^*\bullet\boldsymbol{\sigma}^*)] = \exp[-(i/2)\,\boldsymbol{\alpha}^*\bullet\boldsymbol{\sigma})]\,\varepsilon = U(\boldsymbol{\alpha}^*)\,\varepsilon$$

Thus we conclude that

$$U(-\omega)\,\{\varepsilon\,\psi_L^*(x)\}\,U^{-1}(-\omega) = U(\boldsymbol{\alpha}^*)\,\{\varepsilon\,\psi_L^*(x)\}$$

and we conclude that $\varepsilon\,\psi_L^*(x)$ transforms like a ψ_R type spinor.

Now let's do it the other way. Start with some right spinor and repeat all the steps as above,

$$U(-\omega)\,\psi_R(x)\,U^{-1}(-\omega) = U(\boldsymbol{\alpha}^*)\,\psi_R(x')$$

$$U(-\omega)\,\psi_R^*(x)\,U^{-1}(-\omega) = U(\boldsymbol{\alpha}^*)^*\,\psi_R^*(x')$$

$$U(-\omega)\,\varepsilon\,\psi_R^*(x)\,U^{-1}(-\omega) = \varepsilon\,U(\boldsymbol{\alpha}^*)^*\,\psi_R^*(x')$$

$$\varepsilon\,U(\boldsymbol{\alpha}^*)^* = \varepsilon\exp[-(i/2)\,\boldsymbol{\alpha}^*\bullet\boldsymbol{\sigma})]^* = \varepsilon\exp[+(i/2)\,\boldsymbol{\alpha}\bullet\boldsymbol{\sigma}^*)] = \exp[-(i/2)\,\boldsymbol{\alpha}\bullet\boldsymbol{\sigma})]\,\varepsilon = U(\boldsymbol{\alpha})\,\varepsilon$$

$$U(-\omega)\,\{\varepsilon\,\psi_R^*(x)\}\,U^{-1}(-\omega) = U(\boldsymbol{\alpha})\,\{\varepsilon\,\psi_R^*(x')\}$$

and we conclude that $\varepsilon\,\psi_R^*(x)$ transforms as a ψ_L type spinor. I don't see the purpose of Moore's minus sign here, because both signs would be true. That is to say, the above implies

$$U(-\omega)\,\{-\varepsilon\,\psi_R^*(x)\}\,U^{-1}(-\omega) = U(\boldsymbol{\alpha})\,\{-\varepsilon\,\psi_R^*(x')\}$$

so $\pm\,\varepsilon\,\psi_R^*(x)$ *both* transform as ψ_L spinors.

10. Majorana and Dirac Spinors.

Moore at this point comments that upper and lower placement of spinor indices is NOT based on contra or covariant, but on whether or not you apply ε to something! This seems confusing, but I still don't know what the spinor metric tensor is or even means. So he claims this. I don't understand this at all, but I won't be reading string theory soon, so OK. This would be the "**Weyl notation**" .

The 4-component **Majorana Spinor** contains redundant information about a 2-component left spinor.

$\psi_M = (\psi_L, \varepsilon\psi_L^*)$ // column vector

The 4-component Dirac Spinor is not the same, but transforms the same, and Moore writes this as

$\psi_{Dirac} = (\mathcal{E}_L, E_R)$ // column vector

I don't understand his discussion in terms of physics, but I get the point that both these objects tranform as a spinor of type (L,R). Right now I am interested in the matrices, which is why I started this long document in the first place. Here is the basic idea now:

$\psi_{Dirac} = \begin{pmatrix} \mathcal{E}_L \\ E_R \end{pmatrix}$ transforms according to $(½ \otimes 0) \oplus (0 \otimes ½)$

Finally this famous notation makes some sense to me! And right away, I know all the matrices, so let's see if they come out the way I think they should! Collecting info from above,

$L_i = \sigma_i/2$	$J_i = \sigma_i/2$	$½ \otimes 0$
$R_i = 0$	$K_i = -i\sigma_i/2$	
$L_i = 0$	$J_i = \sigma_i/2$	$0 \otimes ½$
$R_i = \sigma_i/2$	$K_i = +i\sigma_i/2$	

Therefore we have these generators which be be in block diagonal form

$J_i = \sigma_i/2 \oplus \sigma_i/2 = ½ \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} = ½ \Sigma_i = ½ \varepsilon_{ijk} J_{jk} = J_{jk}$ cyclic

$K_i = -i\sigma_i/2 \oplus i\sigma_i/2 = ½ \begin{pmatrix} -i\sigma_i & 0 \\ 0 & i\sigma_i \end{pmatrix} = J_{i0}$

BUT, as we learn now, there is an *extra shuffle* before we get to the true Dirac spinors of BD.

11. Extra Shuffle needed on the (½ ⊗ 0) ⊕ (0 ⊗ ½) to get to the true Dirac Representation

What are the covariant generators $J^{\mu\nu}$ for this direct sum representation? Is it possible that we have

$J^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu}$? // as BD1 page22 seems to suggestp

If so, then we have

$J_{\mu\nu} = \frac{1}{2}\sigma_{\mu\nu}$ =>

$J'_i = J_{jk} = \frac{1}{2}\sigma_{jk} = i\,\frac{1}{2}\gamma_j\gamma_k = +\frac{1}{2}\,i\,\gamma^j\gamma^k = \frac{1}{2}\Sigma_i = J_i$ // from my sheet, agrees!

$$K'_i = J_{i0} = \tfrac{1}{2}\sigma_{i0} = \tfrac{1}{2}\,i\,\gamma_i\gamma_0 = -i\,\tfrac{1}{2}\gamma^i\gamma^0 = \tfrac{1}{2}(-i)\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \tfrac{1}{2}(-i)\begin{pmatrix} 0 & -\sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

$$= \tfrac{1}{2}\,i\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} = \tfrac{1}{2}\,i\,\alpha_i \neq K_i$$ // does NOT agree

So things seem right for J_i, but not for K_i ! The $\sigma_{\mu\nu}$ idea gives an off-diagonal K'_i, so this throws a cloud on my assumed transformation properties of the Dirac spinor!

Question: What happens to our Lie Algebra for J and K if I do a shuffle on the K's? Right now we have this conflict:

$$K_i = \tfrac{1}{2}\begin{pmatrix} -i\sigma_i & 0 \\ 0 & i\sigma_i \end{pmatrix} = \tfrac{1}{2}\,i\gamma_5\gamma^i = \tfrac{1}{2}\,i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

$$K_i' = \tfrac{1}{2}\,i\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} = \tfrac{1}{2}\,i\,\alpha_i$$

so again we have

$K_i = \frac{1}{2}\,i\gamma_5\gamma^i$

$K_i' = \frac{1}{2}\,i\,\alpha_i = \frac{1}{2}\,i\,\gamma_0\gamma^i$

Are these in some way equivalent? I think we would have to show that the Lie Algebra stayed the same when you made this replacement. That just seems unlikely to me [but it is true!]. We have

$\gamma_0 K_i' = \frac{1}{2}\,i\,\gamma^i = \gamma_5 K_i$ => $K_i' = \gamma_0\gamma_5\,K_i = -\gamma_6\,K_i$ $\gamma_6 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Recalling that $[K_i, K_j] = -i\,\varepsilon_{ijk}J_k$, let's consider

$[K'_i, K'_j] = [-\gamma_6 K_i, -\gamma_6 K_j] = [\gamma_6 K_i, \gamma_6 K_j] = = \gamma_6 K_i\,\gamma_6 K_j - \gamma_6 K_j\,\gamma_6 K_i$

Now

$$\gamma_6 K_i \gamma_6 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tfrac{1}{2} \begin{pmatrix} -i\sigma_i & 0 \\ 0 & i\sigma_i \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \tfrac{1}{2} \begin{pmatrix} 0 & -i\sigma_i \\ -i\sigma_i & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \tfrac{1}{2} \begin{pmatrix} -i\sigma_i & 0 \\ 0 & i\sigma_i \end{pmatrix} = K_i$$

Thus we have

$$[\, K'_i, K'_j] = \gamma_6 K_i \gamma_6 K_j - \gamma_6 K_j \gamma_6 K_i = K_i K_j - K_j K_i = [K_i, K_j]$$

so this is very promising. Then we know that $[\, K'_i, K'_j] = -i\, \varepsilon_{ijk} J_k$ and this part of the Lie Algebra is in fact satisfied by the K' generators. Now let's look at the other Lie part involving K:

Recalling $[\, J_i, K_j] = +i\, \varepsilon_{ijk} K_k$, consider

$$[\, J_i, K'_j] = [\, J_i, -\gamma_6 K_j] = -J_i \gamma_6 K_j + \gamma_6 K_j J_i$$

We need to get γ_6 in the first term to the left of J_i so we can "do" the commutator. Recall $\tfrac{1}{2}\Sigma_i = J_i$ = a diagonal matrix. So use this fact along with $\gamma_6^2 = -1$ to get

$$\gamma_6 \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \gamma_6 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -d \\ a & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -d & 0 \\ 0 & -a \end{pmatrix}$$

$$\Rightarrow \gamma_6 \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \gamma_6 = \begin{pmatrix} -a & 0 \\ 0 & -a \end{pmatrix} = -\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \qquad \Rightarrow \qquad \gamma_6 \Sigma_i \gamma_6 = -\Sigma_i$$

$$\Rightarrow \gamma_6 J_i \gamma_6 = -J_i \qquad \Rightarrow \qquad -\gamma_6 J_i = -J_i \gamma_6 \qquad \Rightarrow \gamma_6 J_i = J_i \gamma_6$$

which says γ_6 passes through J_i on a free ticket. Thus we have

$$[\, J_i, K'_j] = [\, J_i, -\gamma_6 K_j] = -J_i \gamma_6 K_j + \gamma_6 K_j J_i = -\gamma_6 J_i K_j + \gamma_6 K_j J_i$$

$$= -\gamma_6 [\, J_i, K_j] = -\gamma_6 (+i) \varepsilon_{ijk} K_k = i\, \varepsilon_{ijk} (-\gamma_6 K_k) = i\, \varepsilon_{ijk} K'_k$$

so we have now confirmed that $[\, J_i, K'_j] = i\, \varepsilon_{ijk} K'_k$.

Therefore, we can take our generators to be J_i and K_i' and we still satisfy the J,K Lie Alegebra, so this must define a LG representation. We started with $(\tfrac{1}{2} \otimes 0) \oplus (0 \otimes \tfrac{1}{2})$, and we did a shuffle without changing the J's. From a rotation group point of view, we have $\tfrac{1}{2} \oplus \tfrac{1}{2}$ both before and after the shuffle because the J generators are the Σ matrices. So the 4 dimensional representation we end up with cannot be the $(\tfrac{1}{2}, \tfrac{1}{2})$ representation which is $\tfrac{1}{2} \otimes \tfrac{1}{2} = 1 \oplus 0$ from the rotational viewpoint. So I can only conclude that our J,K' 4 dimensional representation is something equivalant to $(\tfrac{1}{2} \otimes 0) \oplus (0 \otimes \tfrac{1}{2})$.

Summary of what we did here:

(1) started with the pure $(\tfrac{1}{2} \otimes 0) \oplus (0 \otimes \tfrac{1}{2})$ where we had

$$J_i = \sigma_i/2 \oplus \sigma_i/2 = \tfrac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} = \tfrac{1}{2}\Sigma_i = \tfrac{1}{2}\varepsilon_{ijk} J_{jk} = J_{jk} \text{ cyclic}$$

$$K_i = -i\sigma_i/2 \oplus i\sigma_i/2 = \tfrac{1}{2}\begin{pmatrix} -i\sigma_i & 0 \\ 0 & i\sigma_i \end{pmatrix} = J_{i0}$$

(2) We ended up with a *shuffled version* of (½ ⊗ 0) ⊕ (0 ⊗ ½) wherein we have

$$J_i' = J_i = \tfrac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} = \tfrac{1}{2}\Sigma_i = \tfrac{1}{2}\sigma_{jk} = \tfrac{1}{2}\sigma^{jk} \quad \text{where ijk = cyclic}$$

$$K_i' = -\gamma_6 K_i = \tfrac{1}{2} i \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} = \tfrac{1}{2} i \alpha_i = \tfrac{1}{2} i \gamma_0\gamma^i = \tfrac{1}{2}\sigma^{0i} = \tfrac{1}{2}\sigma_{i0}$$

and we showed that J, K' satisfy the same Lie Algebra as J,K.

(3) Therefore, using this "shuffled (½ ⊗ 0) ⊕ (0 ⊗ ½)" representation, we end up with this nice fact:

$$J^{\mu\nu} = \tfrac{1}{2}\sigma^{\mu\nu} = i/4\,[\gamma^\mu, \gamma^\nu] \quad \text{" Dirac representation"}$$

so our transformation would then be

$$\exp(-i/2\,\omega_{\mu\nu}J^{\mu\nu}) = \exp(-i/4\,\omega_{\mu\nu}\sigma^{\mu\nu}) \quad \text{// which I think agrees with BD1 page 22 for example}$$

By the way, Moore is silent on this "extra shuffle" business. He just rolls along as if it never happened. I am tempted to send him an email.

12. Showing that γ^μ transforms as a 4-vector under a Dirac representation LG transformation.

Now we are going to put some more pieces together. We know that

$$J^{\mu\nu} = \tfrac{1}{2}\sigma^{\mu\nu} = i/4\,[\gamma^\mu, \gamma^\nu]$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g_{\mu\nu} \quad \text{the "Clifford algebra" fact}$$

Quoting a result from my sheet (and I think I did this in detail in the BD notes), we find that

$$[J^{\mu\nu}, \gamma^\alpha] = \tfrac{1}{2}[\sigma^{\mu\nu}, \gamma^\alpha] = -i\,(g^{\alpha\mu}\gamma^\nu - g^{\alpha\nu}\gamma^\mu)$$

which I notice is the same as the relationship in the Poincare Group between $[J^{\mu\nu}, P^\alpha]$, which suggests to me that γ^α is a momentum operator in this Dirac space.

Now consider:

$$\exp(+i/2\,\omega_{\mu\nu}J^{\mu\nu})\,\gamma^\alpha\,\exp(-i/2\,\omega_{\mu\nu}J^{\mu\nu}) = U^\dagger\,\gamma^\alpha\,U$$

We can probably do a BCH thing on this.

$$e^{-A} B e^{A} = B + [B,A]/1! + [prev,A]/2 + [prev,A]/3 + ... \qquad A = -i/2\ \omega_{\mu\nu}J^{\mu\nu} \qquad B = \gamma^{\alpha}$$

Here is something that helps, where we fold the second term due to AS of the $\omega_{\mu\nu}$,

$$[\gamma^{\alpha}, \omega_{\mu\nu}J^{\mu\nu}] = \omega_{\mu\nu}[\gamma^{\alpha}, J^{\mu\nu}] = \omega_{\mu\nu}\, i\,(g^{\alpha\mu}\gamma^{\nu} - g^{\alpha\nu}\gamma^{\mu}) = 2i\ \omega_{\mu\nu}\, g^{\alpha\mu}\gamma^{\nu} = 2i\ \omega^{\alpha}{}_{\nu}\gamma^{\nu}$$

Then we start on the BCH

$$[B,A] = -i/2\ [\gamma^{\alpha}, \omega_{\mu\nu}\, J^{\mu\nu}] = -i/2 * 2i\ \omega^{\alpha}{}_{\nu}\gamma^{\nu} = \omega^{\alpha}{}_{\nu}\gamma^{\nu}$$

Then the next BCH term is

$$[prev,A]/2 = [\omega^{\alpha}{}_{\nu}\,\gamma^{\nu}, -i/2\ \omega_{ab}J^{ab}\ ]\ /2 = \omega^{\alpha}{}_{\nu}\ (-i/4)\ [\gamma^{\nu}, \omega_{ab}J^{ab}]$$

$$= (-i/4)\ \omega^{\alpha}{}_{\nu}\ 2i\ \omega^{\nu}{}_{b}\gamma^{b} = \tfrac{1}{2}\ \omega^{\alpha}{}_{\nu}\omega^{\nu}{}_{b}\gamma^{b} = (1/2!)\ (\omega^{2})^{\alpha}{}_{\nu}\gamma^{\nu}$$

which looks promising. Next term is

$$[prev,A]/3 = (1/3!)\ [\omega^{\alpha}{}_{c}\omega^{c}{}_{b}\gamma^{b}\ , -i/2\ \omega_{\mu\nu}J^{\mu\nu}] = (-i/2)(1/3!)\ \omega^{\alpha}{}_{c}\omega^{c}{}_{b}\ [\gamma^{b}, \omega_{\mu\nu}J^{\mu\nu}]$$

$$= (-i/2)(1/3!)\ \omega^{\alpha}{}_{c}\omega^{c}{}_{b}\ 2i\ \omega^{b}{}_{\nu}\gamma^{\nu} = (1/3!)\ \omega^{\alpha}{}_{c}\omega^{c}{}_{b}\omega^{b}{}_{\nu} = (1/3!)\ (\omega^{3})^{\alpha}{}_{\nu}\,\gamma^{\nu}$$

I think we can see the series that is forming.

$$e^{-A} B e^{A} = \gamma^{\alpha} + \omega^{\alpha}{}_{\nu}\gamma^{\nu} + (1/2!)\ (\omega^{2})^{\alpha}{}_{\nu}\gamma^{\nu} + (1/3!)\ (\omega^{3})^{\alpha}{}_{\nu}\,\gamma^{\nu} + ...$$

$$= \exp(\omega)^{\alpha}{}_{\nu}\gamma^{\nu}$$

But we saw above that $\exp(\omega)^{\alpha}{}_{\nu} = \Lambda^{\alpha}{}_{\nu}$ which is the vector representation finite matrix! Thus we have shown the following:

$$\exp(+i/2\ \omega_{\mu\nu}J^{\mu\nu})\ \gamma^{\alpha}\ \exp(-i/2\ \omega_{\mu\nu}J^{\mu\nu}) = U^{-1}\ \gamma^{\alpha}\ \ U = \Lambda^{\alpha}{}_{\nu}\ \gamma^{\nu} \qquad J^{\mu\nu} = \tfrac{1}{2}\ \sigma^{\mu\nu} \qquad = i/4\ [\ \gamma^{\mu}, \gamma^{\nu}]$$

just as Moore states. Notice that here U is the 4x4 Dirac Representation transformation, not the infinite HS one, and not any other one.

Given this fact, if is easy so show another fact which is this:

$$U^{-1}\ \gamma^{\alpha}\ U = \Lambda^{\alpha}{}_{\nu}\ \gamma^{\nu} \qquad => \qquad U^{-1}\not{a}'\ U = \not{a} \qquad \text{for any 4-vector } a^{\mu}$$

The proof goes like this:

$$U^{-1}\not{a}'\,U = U^{-1}a'_\mu\gamma^\mu\,U = (a'_\mu)\,(U^{-1}\gamma^\mu\,U) = (\Lambda_\mu{}^\beta a_\beta)(\Lambda^\mu{}_\nu\,\gamma^\nu)$$

$$= (\Lambda_\mu{}^\beta\,\Lambda^\mu{}_\nu)\,a_\beta\,\gamma^\nu = \delta^\beta{}_\nu\,a_\beta\,\gamma^\nu = a_\beta\,\gamma^\beta = \not{a}$$

where we use our *general orthogonality rule* for *any* coordinate transformation $\delta_b{}^c = T_a{}^c\,T^a{}_b$ (see curvilinear folder notes for more details on this, can sum on both first or both second indices).

Our γ^α result above is quite general. Suppose we start with this commutation relation

$$[J^{\mu\nu},\, p^\alpha] = -i\,(g^{\alpha\mu}p^\nu - g^{\alpha\nu}p^\mu)$$

where J and p are in some arbitrary space (no longer just Dirac space). Then we still get

$$[p^\alpha,\, \omega_{\mu\nu}J^{\mu\nu}] = \omega_{\mu\nu}[p^\alpha,\, J^{\mu\nu}] = \omega_{\mu\nu}\,i\,(g^{\alpha\mu}p^\nu - g^{\alpha\nu}p^\mu) = 2i\,\omega_{\mu\nu}\,g^{\alpha\mu}p^\nu = 2i\,\omega^\alpha{}_\nu p^\nu$$

and everything goes forward with $\gamma \rightarrow p$, and we conclude that

$$\Rightarrow\ \exp(+i/2\,\omega_{\mu\nu}J^{\mu\nu})\,p^\alpha\,\exp(-i/2\,\omega_{\mu\nu}J^{\mu\nu}) = U^{-1}\,p^\alpha\,U = \exp(\omega)^\alpha{}_\nu\,p^\nu = \Lambda^\alpha{}_\nu\,p^\nu$$

Thus, it is *the commutator rule itself* which is saying that p^α transforms like a 4-vector. This is in accordance from my general finding in "explanation of the rotation of a vector operator"

$$(A')^\mu \equiv \Lambda^{-1}\,A^\mu\,\Lambda = (\boldsymbol{\Lambda})^\mu{}_\nu\,A^\nu$$

where here I put a single prime on the LHS.

Theorem: This commutation rule in ANY space implies that object p^α transforms as a 4-vector

$$[J^{\mu\nu},\, p^\alpha] = -i\,(g^{\alpha\mu}p^\nu - g^{\alpha\nu}p^\mu) \Rightarrow$$

$$p'^\alpha = \exp(+i/2\,\omega_{\mu\nu}J^{\mu\nu})\,p^\alpha\,\exp(-i/2\,\omega_{\mu\nu}J^{\mu\nu}) = U^\dagger\,p^\alpha\,U = \exp(\omega)^\alpha{}_\nu\,p^\nu = \Lambda^\alpha{}_\nu\,p^\nu$$

Example 1: p^α and $J^{\mu\nu}$ are operators in the QM HS where p = momentum

Example 2: γ^μ and $J^{\mu\nu}$ are operators in the Dirac Representation

Armed with the above, it is trivial to show the "covariance" of the Dirac equation which is this in "the primed frame" (where the 4x4 matrix aspect of the equation is implied),

$$(\hat{\not{p}}' - m)\psi'(x') = 0$$

where $\hat{p}^\mu \equiv i\partial^\mu$, the differential operator. Recalling that for a "classical Dirac field amplitude" we have

$$\psi'(x') = U\psi(x) \qquad \text{where } U(\omega) = \exp(-i/2\,\omega_{\mu\nu}J^{\mu\nu}) = \exp(-i/4\,\omega_{\mu\nu}\sigma^{\mu\nu}) = \text{4x4 Dirac LT}$$

then we can apply U^{-1} from the left to get

$$U^{-1}(\hat{p}' - m)\, U\psi(x) = 0 \;\Rightarrow\; (U^{-1}\,\hat{p}'\,U - m)\,\psi(x) = 0 \;\Rightarrow\; (\hat{p} - m)\,\psi(x) = 0 \qquad \text{QED}$$

BD1 spend a whole messy section on this subject, but when you "do the group theory" as we have done in this document, the result is really totally obvious. This confirms that the choice $J^{\mu\nu} = i/2\ \sigma^{\mu\nu}$ is the correct choice for making the Dirac equation covariant, which in turn means that the 4-spinor ψ transforms according to this "Dirac representation", which in turn justifies the "shuffle" we did above where we had to change from K_i to K'_i to get a slightly shuffled version of the $(½ \otimes 0) \oplus (0 \otimes ½)$ representation of the LG.

13. Derivation of the form of the vector representation generators.

I think now would be a great time to derive an omitted fact which is this:

" $(J^{\mu\nu})^{\alpha}{}_{\beta} = i\,(\, g^{\mu\alpha}g^{\nu}{}_{\beta} - g^{\nu\alpha}g^{\mu}{}_{\beta})$ is the correct vector representation generator matrix set "

In other words, we want to show that

$$\exp(-i/2\ \omega_{\mu\nu}J^{\mu\nu}) = \exp(\omega)$$

is in fact our finite LT we always call Λ . We know the line above trivially from earlier, but I repeat that here:

$$(-i/2)\omega_{\mu\nu}[\, J^{\mu\nu}\,]^{\alpha\beta} = (-i/2)\omega_{\mu\nu}\, i\,(\, g^{\mu\alpha}g^{\nu\beta} - g^{\nu\alpha}g^{\mu\beta}) = \omega_{\mu\nu}\, g^{\mu\alpha}g^{\nu\beta} = \omega^{\alpha\beta}$$

$$\Rightarrow \qquad -i/2\ \omega_{\mu\nu}J^{\mu\nu} = \omega$$

But we know what this matrix looks like, namely (tilt-down indices), think of this as ω's definition:

$$\omega = \begin{pmatrix} 0 & b1 & b2 & b3 \\ b1 & 0 & -r3 & +r2 \\ b2 & +r3 & 0 & -r1 \\ b3 & -r2 & +r1 & 0 \end{pmatrix} = -i\,(\mathbf{r}\bullet\mathbf{J} + \mathbf{b}\bullet\mathbf{K}) = \text{"the matrix in the exponent"}$$

But this tells us that

$$\mathbf{r}\bullet\mathbf{J} + \mathbf{b}\bullet\mathbf{K} = r\ \hat{\mathbf{r}}\bullet\mathbf{J} + b\ \hat{\mathbf{b}}\bullet\mathbf{K} = i \begin{pmatrix} 0 & b1 & b2 & b3 \\ b1 & 0 & -r3 & +r2 \\ b2 & +r3 & 0 & -r1 \\ b3 & -r2 & +r1 & 0 \end{pmatrix}$$

If we then look at each of the 6 generators one at a time, we get "the right thing" which we know makes the right finite matrix. If it works for each generator of the six generators solo, then it must be right. For example, we can set

$$\mathbf{r} = r\,\hat{\mathbf{x}} \quad \mathbf{b} = 0$$

to get

$$\mathbf{r}\bullet\mathbf{J} + \mathbf{b}\bullet\mathbf{K} = rJ_1 = r\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} = \text{r x the correct matrix for making } R_1(r)$$

I don't see any benefit to the battle of trying to directly exponentiate the full 4x4 matrix ω. Even if I could do it, I wouldn't know whether the result was right or not because I have never seen the result and I know it will be a huge mess anyway.

So what I just showed above was that

$$(J^{\mu\nu})^{\alpha}{}_{\beta} = i\,(\,g^{\mu\alpha}g^{\nu}{}_{\beta} - g^{\nu\alpha}g^{\mu}{}_{\beta})$$

produces exactly the right 6 generators for J_i and K_i for the vector representation of the LG Lie Algebra. We know these J_i and K_i generators are correct because they generate the correct finite rotation and boost matrices. Thus, the form shown above must be right. That is my proof.

Comments on this form: As a side note, consider

$$(J^{\mu\nu})^{\alpha\beta} = i\,(\,g^{\mu\alpha}g^{\nu\beta} - g^{\nu\alpha}g^{\mu\beta})$$

We showed early on that $J^{\mu\nu}$ must be antisymmetric in its labels, and the above form has this property. Also, notice that the RHS is antisymmetric in the indices (when we are in both up or both down form), so the matrix is also Hermitian.

$$(J^{\mu\nu})^{\alpha\beta} = -(J^{\mu\nu})^{\beta\alpha} = -(J^{\mu\nu\ T})^{\alpha\beta} = +(J^{\mu\nu\ T*})^{\alpha\beta} = +\ (J^{\mu\nu\ \dagger})^{\alpha\beta} \qquad \text{so } (J^{\mu\nu}) = (J^{\mu\nu})^{\dagger}$$

This is *not true* in the tilt-down notation:

$$(J^{\mu\nu})^{\alpha}{}_{\beta} = i\,(\,g^{\mu\alpha}g^{\nu}{}_{\beta} - g^{\nu\alpha}g^{\mu}{}_{\beta}) \neq -(J^{\mu\nu})^{\beta}{}_{\alpha}$$

You have to keep in mind that a matrix is a *different matrix* for each index format -- it is a different tensor. For example, $(J^{\mu\nu})^{\alpha}{}_{\beta}$ is a mixed down-tilt tensor, whereas $(J^{\mu\nu})^{\alpha\beta}$ is a pure contravariant tensor. So a rule like $A = A^{\dagger}$ might be true for some index formats and not others, as is the case here. With both indices up or down, K_i and J_i will be Hermitian since they are just particular $J^{\mu\nu}$matrices. But in down-tilt form, it turns out that J_i is still Hermitian, but K_i is not. The significance of the tilt-down format is that this is the form our finite matrix must have if we are doing things like $x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu}$, so the tilt-down is always of "special interest" to us.

So my comment was going to be that if you want to *construct* a covariant tensor $(J^{\mu\nu})^{\alpha\beta}$, the only objects you have to work with are $g^{\mu\nu}$ and perhaps $\varepsilon^{\alpha\beta\delta\gamma}$ and you have to have antisymmetry in the labels $\mu\nu$ for $(J^{\mu\nu})^{\alpha\beta}$, so I think just those facts severely limit what forms $(J^{\mu\nu})^{\alpha\beta}$ could possibly have. If we add that $(J^{\mu\nu})^{\alpha\beta}$ must be Hermitian, we have to add $\pm$ i (active/passive). One argument for why $(J^{\mu\nu})^{\alpha\beta}$ should be Hermitian is that we know that the rotation generators ARE Hermitian, and somehow this forces all the $J^{\mu\nu}$ to be Hermitian to maintain a covariant form.

Here we want to show that the rotation generators are Hermitian in the down-tilt format, although this is not true for all the $(J^{\mu\nu})^{\alpha}{}_{\beta}$:

$(J^{ij})^{0}{}_{\beta} = i\,(g^{i0}g^{j}{}_{\beta} - g^{j0}g^{i}{}_{\beta}) = 0$ for $\beta = 0,1,2,3$

$(J^{ij})^{\alpha}{}_{0} = i\,(g^{i\alpha}g^{j}{}_{0} - g^{j\alpha}g^{i}{}_{0}) = 0$ for $\alpha = 0,1,2,3$

Thus we have shown that

$(J^{ij})^{0}{}_{\beta} = -\,(J^{ij})^{\beta}{}_{0}$ for $\beta = 0,1,2,3$ // both sides are zero

Now all the other matrix elements have this form: (where a,b = 1,2,3)

$(J^{ij})^{a}{}_{b} = i\,(g^{ia}g^{j}{}_{b} - g^{ja}g^{i}{}_{b}) = -\,i\,(\delta_{ia}\delta_{jb} - \delta_{ja}\delta_{ib}) = -\,(J^{ij})^{b}{}_{a}$

Thus we have shown that for all 16 matrix elements, we have

$(J^{ij})^{\alpha}{}_{\beta} = -\,(J^{ij})^{\beta}{}_{\alpha}$

so this shows that the J^{ij} (ie, the rotation generators) are Hermitian in tilt-down notation as well as in both up or both down notation! This is not true for $(J^{i0})^{\alpha}{}_{\beta}$ which is boost. So this firms up our notion that our knowledge that the 3D rotation generators must be Hermitian together with our knowledge from the Lie Algebra that $J^{\mu\nu}$ is antisymmetric in the labels plus our knowledge of available tensors pretty much forces us to the form shown for $(J^{\mu\nu})^{\alpha\beta}$ and in this format we have $J^{\mu\nu}$ Hermitian.

I am now going to **review Moore's entire appendix** and just add comments here:

- He shows that Λ preserves 4-distance and this puts the usual constraint on Λ
- doing the usual infinitesimal, he shows that we have in that limit the $\Lambda = e^{\omega}$ form and that ω must be antisymmetric and therefore has only 6 elements, so LG must be a 6 real parameter group.
- In the original appendix he uses the BD metric, but this is changed in the corrected one.
- his 3.1.10 confirms that $\Lambda = e^{\omega}$ in the vector rep
- he comments on the disconnected pieces of O(3,1) and SO(3,1) has $\det\Lambda = 1$
- he claims we can deduce the full Poincare algebra by considering various vector transformation forms
- he defines J and K as I show above and gives the Lie Algebra in them
- he springs the general field operator transformation formula with no explanation
- he then talks about other representations of the LG, not just the vector one
- he goes into the decoupled L and R generators and gives their Lie Alebra

- he says (1/2,1/2) is the vector, but gives not evidence for why that is true
- he does the (1/2,0) and (0,1/2) reps, spinors left and right types, and $\psi_R = i\,\sigma_2\psi_L^*$
- he incorrectly states that the Dirac rep has Majorana transformation, ignoring my "shuffle"
- he shows that $\bar\psi\psi$ transforms as a scalar because $\bar\psi$ transforms with D^{-1} relative to ψ
- he then defines some new generators he calls $\mathcal{J}^{\mu\nu}$ which he identifies with ½ $\sigma^{\mu\nu}$ and says nothing about how these might be related to the $J^{\mu\nu}$ but he does claim that using the $\mathcal{J}^{\mu\nu}$ you find γ^μ xfrming as 4-vector
- from this fact, he then shows the way the 16 Dirac theory $\bar\psi\Gamma\psi$ objects transform

So it is an interesting appendix, omitting some things, including others. I wonder how technical his book might be? I wonder if everything I have done above is contained in a single LG book?

14. What exactly is this U transformation thing?

We write $U(\omega) = \exp(-i/2\ \omega_{\mu\nu}J^{\mu\nu})$ so U exists in whatever space the $J^{\mu\nu}$ exist in. In the vector rep, U is called Λ, and in the Dirac rep it is called S by BD and I guess D_{ab} by Moore. In QFT, however, we have to think of U as some infinite dimensional representation which is unitary. One theorem about the LG I remember is that the only unitary representations are infinite dimensional.

What examples of this do we already know about? Elsewhere I talk about states $|\mathbf{p}\rangle$ forming an infinite dimensional basis for a QM HS which has transformation operators R where we are just thinking 3D and SO(3). So here we have an example of an infinite dimensional representation of O(3) which I suspect is unitary. It happens that for SO(3), the FDR's are also unitary, which is not true for the LG.

So how would you show that R was unitary? You would need to show that J was Hermitian. In this case, J is L and we here is a little derivation. As a preliminary, we need to show that P is Hermitian

$$(P_k)_{p'p} = \langle p'|\,\hat p_k|p\rangle = \langle p'|\,p_k|p\rangle = p_k\langle p'|p\rangle = p'_k\delta(p-p')$$

$$= p'_k\langle p|p'\rangle = \langle p|\,p'_k\,|p'\rangle = \langle p|\,\hat p_k\,|p'\rangle = (P_k)_{pp'}$$

So in the momentum basis, P_k as an infinite dimensional matrix is real and symmetric, hence Hermitian. Now for the main act, where each step needs a few words mumbled,

$$(\hat L_i)_{x'x} = \langle x'|\,(\hat r \times \hat p)_i\,|x\rangle = \varepsilon_{ijk}\langle x'|\,\hat r_j\hat p_k\,|x\rangle = \varepsilon_{ijk}\langle x'|\,x'_j\hat p_k\,|x\rangle$$

$$= \varepsilon_{ijk}\,x'_j\langle x'|\,\hat p_k\,|x\rangle = \varepsilon_{ijk}\,x'_j\langle x'|\,\hat p_k\,|p\rangle\langle p|\,x\rangle = \varepsilon_{ijk}\,x'_j\langle x'|\,p_k\,|p\rangle\langle p|\,x\rangle$$

$$= \varepsilon_{ijk}\,x'_j\,\{\ \langle x|p\rangle\langle p|\,p_k\,|x'\rangle\ \}^* = \varepsilon_{ijk}\,x'_j\,\{\ \langle x|p\rangle\langle p|\,\hat p_k\,|x'\rangle\ \}^* \text{(since } \hat p_k \text{ Herm)}$$

$$= \varepsilon_{ijk}\,x'_j\,\{\ \langle x|\ \hat p_k\,|x'\rangle\ \}^* = \varepsilon_{ijk}\,\{\ \langle x|\ \hat p_k\,x'_j\,|x'\rangle\ \}^* = \varepsilon_{ijk}\,\{\ \langle x|\ \hat p_k\,\hat x_j\,|x'\rangle\ \}^*$$

$$= \varepsilon_{ijk}\ \{\ \langle x|\,\hat x_j\,\hat p_k\,|x'\rangle\ \}^* \qquad (\text{because } \hat x_j\,\hat p_k \text{ commute with different indices as per } \varepsilon_{ijk})$$

$$= \{\ \langle x|\,\varepsilon_{ijk}\,\hat x_j\,\hat p_k\,|x'\rangle\ \}^* = \{\ \langle x|\ (\hat r \times \hat p)_i\,|x'\rangle\ \}^* = (\hat L_i)_{xx'}^* \qquad \Rightarrow \hat{\mathbf{L}} = \hat{\mathbf{L}}^\dagger$$

So this was an example of generators Hermitian and this transformations R = exp(-i**n**•**L**) being unitary, all in an infinite dimensional space. Notice that I avoided using any differential operators above.

If we go to the more general case of states $|p^{\mu}>$ with covariant normalization, we find that all the generators $J^{\mu\nu}{}_{pp'}$ are Hermitian, and again U will be unitary. In this case, U is unitary in its infinite dimensional representation, but none of the FDR's are unitary because this is the LG now.

Finally, this will be true in the QFT Hilbert Space, though this is a more complicated many particle space.

I have still some "required information" at the ****** above about why U(ω) should be real in QFT, but since I have not yet studied QFT in the current era, that will just have to pend for a while longer.

15. Sandwich Rules for the LG

I show these in the first Berkeley binder section, and I think the J and K are the same as I have been using. The Lie algebra is the same and the sandwich theorems just follow from the Lie Algebra. I just quote the results here and they should be true in any representation

$$\exp(-iuJ_i)\, J_j \exp(+iuJ_i) = J_j\cos(u) + \varepsilon_{ijk}\, J_k \sin(u)$$

$$\exp(-iuJ_i)\, K_j \exp(+iuJ_i) = K_j\cos(u) + \varepsilon_{ijk}\, K_k \sin(u)$$

$$\exp(-iuK_i)\, K_j \exp(+iuK_i) = K_j\mathrm{ch}(u) - \varepsilon_{ijk}\, K_k\, \mathrm{sh}(u)$$

$$\exp(-iuK_i)\, J_j \exp(+iuK_i) = J_j\mathrm{ch}(u) + \varepsilon_{ijk}\, K_k\, \mathrm{sh}(u)$$

The first two are "the same" because both J and K transform like rotational vectors, and the first agrees with my recent Levitt work. The all-K one has a minus sign due to the minus in the Lie Algebra for the K's. In my notes I derive all these things from BCH. If you use cyclic order notation, you can leave out the ε_{ijk} factors. I don't know exactly where these things might be useful, but I am sure it will come up somewhere!

16. A few small comments on SL(2,C) .

1. In the SL(2,C) world one can decompose g = hu, where $h = \sqrt{gg^{\dagger}}$ where h is a boost and u is a rotation, so I think this carries over to SO(3,1) and in this way we could write ANY Lorentz transformation as a rotation followed by a boost. I am not quite sure this is true, but it sounds on the right track.

2. On page "4" of my notes, I show the following SL(2,C) information

$J = \frac{1}{2}\sigma \qquad K = \frac{1}{2} i\sigma$ // agrees with the Weyl Right spinor representation $0 \otimes \frac{1}{2}$

$$\exp(-i\varphi J_i) = \cos(\varphi/2) - i\sin(\varphi/2)\,\sigma_i$$
$$\exp(-iuK_i) = \mathrm{ch}(u/2) + i\,\mathrm{sh}(u/2)\,\sigma_i$$

and then I go on to write out the 6 2x2 matrices for the finite boosts and rotations, which I won't copy here. In some nearby notes, I make this claim for exponentiating Pauli matrices

$$\exp(\boldsymbol{\alpha}\bullet\boldsymbol{\sigma}) = \mathrm{ch}(\alpha) + (1/\alpha)\,\mathrm{sh}(\alpha)\,(\boldsymbol{\alpha}\bullet\boldsymbol{\sigma}) \qquad \text{where } \alpha = \sqrt{\alpha_1{}^2 + \alpha_2{}^2 + \alpha_3{}^2}$$

$$\exp(i\boldsymbol{\beta}\bullet\boldsymbol{\sigma}) = \cos(\beta) + i\,(1/\beta)\sin(\beta)\,(\boldsymbol{\beta}\bullet\boldsymbol{\sigma}) \qquad \text{where } \beta = \sqrt{\beta_1{}^2 + \beta_2{}^2 + \beta_3{}^2}$$

where these expansions are true for complex α or β.

17. Spinor Representations of the Lorentz Group

The FDR's of the Lorentz group can obviously be written j1 ⊗ j2 based on the decoupled Lie Algebra. That is to say, this is a *direct product* of two normal O(3) representations. There are various spinor world conventions, but here is one.

spinor of the first kind (undotted indices) $\qquad \xi'^{a} = \Lambda^{a}{}_{b}\xi^{b} \qquad \Lambda^{a}{}_{b} = D^{(1/2,0)}(g)_{ab} = g$

Here, we use Λ in analogy with our vector representation notation, but this is a 2x2 matrix. We also use the contravariant index location, again in analogy with the vector case. Tradition uses the odd character ξ (xi = "zeye") to represent a 2D spinor. So we are associating this first kind spinor with the ½ ⊗ 0 representation of the LG which we call "Left". For g we can write

$$g = g(\mathbf{r},\mathbf{b}) = \exp[-i(\mathbf{r}\bullet\mathbf{J} + \mathbf{b}\bullet\mathbf{K})] \qquad \text{where } \mathbf{J} = 1/2\boldsymbol{\sigma} \text{ and } \mathbf{K} = -\,1/2i\boldsymbol{\sigma} \qquad \text{" Weyl Left"}$$

exactly as we have above for the arbitrary representation. Notice that $g^* = \bar{g}$ in effect changes us to $\mathbf{K} = +\,1/2i\boldsymbol{\sigma}$ which we know is the "Right" representation. That leads us to introduce

spinor of the 2nd kind (dotted indices) $\qquad \xi'^{\dot{a}} = \Lambda^{\dot{a}}{}_{\dot{b}}\xi^{\dot{b}} \qquad \Lambda^{\dot{a}}{}_{\dot{b}} = D^{(0,1/2)}(g)_{ab} = \bar{g}$

Again, spinor indices are up, but are now dotted. The letter "g" refers to the 2x2 complex matrix which is our actual matrix $\Lambda = g$ and then $\dot{\Lambda} = \bar{g}$ so we can say that $\dot{\Lambda} = (\Lambda)$ which is pretty convenient. It is a convention on which g you put your overbar. Lest there be confusion, overbar means complex conjugation, but people use overbar instead of * because you get more compact formulas which are already widening out due to various other symbols (see below).

These two representations are called **conjugates** of each other because the two matrices are related in exactly that way. The two matrices form *different* group representations because there is no matrix S which can relate them as $SgS^{-1} = g^*$. I have not tried to prove this, but it is surely true and it is why the $\tau_{1/2,0}$ and $\tau_{0,1/2}$ representations are really different. There is such an S if you only worry about rotations, but not if you include boosts.

Now consider the following mixed "spinor tensor", and the reader is reminded that the purpose of the dots is so you can tell which spinor that index belongs to (not needed in tensors!)

$$X'^{a\dot{b}} = \Lambda^{a}{}_{a'}\Lambda^{\dot{b}}{}_{\dot{b}'}\, X^{a'\dot{b}'}$$

where we can regard

$$\Lambda^{a}{}_{a'}\Lambda^{\dot{b}}{}_{\dot{b}'} = [D^{(1/2,1/2)}(g)]^{a}{}_{a'}{}^{\dot{b}}{}_{\dot{b}'} = [D^{(1/2,0)}(g)]^{a}{}_{a'}\,[D^{(0,1/2)}(g)]^{\dot{b}}{}_{\dot{b}'} = g_{aa'}\,\bar{g}_{bb'}$$

where we don't put up-down indices on the final actual matrix g or $\bar{g}$, these are just 2x2 matrices.

This spinor tensor $X^{a\dot{b}}$ transforms according to the (1/2,1/2) representation of the LG, no question about it I think. If we always keep in mind that the indices are "mixed" in this manner, we can just call this thing X, which is a 2x2 matrix as well. It has four matrix elements, which is why this is a 4 dimensional representation. We are used to having a column vector with 4 components as "the object that transforms" in the 4-vector world, but here the "object which transforms" is a 2x2 matrix. It is a thing of rank 2, like a rank 2 tensor, but is a spinor rank 2 tensor.

To clarify this point, go back to 3-vectors and regular rotations. We know we can construct a rank 2 tensor from two 3- vectors as follows: (called an outer product)

$$M_{ij} = P_i Q_j \quad \Rightarrow \quad M'_{ij} = R_{ii'}R_{jj'}M_{i'j'}, \quad \text{where} \quad P'_i = R_{ii'}P_{i'} \text{ and same for Q}$$

So this M_{ij} object could transform according to the $1 \otimes 1$ representation of the rotation group. So our X object is the exact same idea, it is an outer product of two j = ½ representations so ½ ⊗ ½, but here two representations are different and the direct product is not "reducible" as it is with the rotation group. We use the dots to remind ourselves that the two reps are different.

Side comment: We know we can think of the ½ ⊗ ½ basis as $|1/2,m_1\rangle \otimes |1/2,m_2\rangle \equiv |m_1,m_2\rangle$ so the four basis elements are then $|\pm 1/2, \pm 1/2\rangle$. One might think of this as $|\, a,\dot{b}\,\rangle$ in terms of index labels. The matrix element of a LG group element G is then $\langle a,\dot{b} \mid G \mid a',\dot{b}'\rangle$ = the D stuff shown above. This is one of those operators that can be represented as G ⊗ G so things factorize to $\langle a \mid G \mid a'\rangle\langle \dot{b} \mid G \mid \dot{b}'\rangle$, which is I think what we mean by a a "direct product" representation.

Now let's return to how X transforms.

$$X'^{a\dot{b}} = \Lambda^{a}{}_{a'}\Lambda^{\dot{b}}{}_{\dot{b}'}\, X^{a'\dot{b}'} = g_{aa'}\,\bar{g}_{bb'}\, X^{a'\dot{b}'} = g_{aa'}\, X^{a'\dot{b}'}\, g^{\dagger}{}_{bb'}$$

where we do the usual trick of transposing indices on the second g to get "matrix index order", and so this transpose changes ovebar to † . Then, again remembering the mixed sense of X, we can write

$$X' = gXg^{\dagger}$$

At this point we must digress and comment a little on g which is an element of SL(2,C) which we know maps to SO(3,1) = LG except for the usual 2:1 detail. This 2x2 matrix has 8 real numbers, but det(g) = 1 is what the S = special means, so this is two equations on our 8 numbers, and we are left with 6 free real parameters which are those of the LG in effect. Now notice therefore that the above equation implies that

$$\det(X') = \det(gXg^\dagger) = \det(g)\det(X)\det(g^\dagger) = \det(X)\,\det(g^T)^* = \det(X)\,\det(g)^* = \det(X)$$

Suppose we were to write X as a linear combination of the Pauli matrices with real coefficients x^μ

$$X = x^\mu\sigma_\mu = \begin{pmatrix} x^0+x^3 & x^1-ix^2 \\ x^1+ix^2 & x^0-x^3 \end{pmatrix} \qquad // \ \sigma_\mu \text{ are all Hermitian, so X too is Hermitian (real } x^\mu)$$

The explicit matrix form then just results from staring at the Pauli matrices. Notice that

$$\det X = (x^0+x^3)(x^0-x^3) - (x^1+ix^2)(x^1-ix^2) = (x^0)^2 - (x^3)^2 - (x^3)^2 - (x^2)^2 = x_\mu x^\mu$$

Side Comment: If we represent $g = a^\mu\sigma_\mu = a^0 + \mathbf{a}\bullet\boldsymbol{\sigma}$ then we can draw the above matrix with the a^μ elements installed, and we can see that $\det(g) = a_\mu a^\mu$, so our condition on the a^μ is that $a_\mu a^\mu = 1$ because SL(2,C) requires det(g) = 1, "special". As noted earlier, this $a_\mu a^\mu = 1$ acts as two real conditions on the 8 numbers a^μ and we are left with our expected 6 real parameters.

We then have the following transformation on our object X written in this way,.

$$x'^\mu\sigma_\mu = g\,x^\alpha\sigma_\alpha\,g^\dagger \qquad \text{or} \qquad X' = gXg^\dagger$$

and we know that this transformation *preserves the Lorentz length* of our candidate 4-vector

$$x'^\mu x'_\mu = \det(X') = \det(X) = x_\mu x^\mu$$

and we know that this transformation has 6 real parameters and we know that $X = X^{a\dot{b}}$ transforms according to the (1/2,1/2) rep of the LG, *so we suspect* that if we could write our transformation in the form x' = Yx, the Y would be our old friend Λ for the 4-vector representation! The 4-vector representation and the (1/2,1/2) canonical represention are homomorphic to each other.

To this end, we right-multiply both sides by σ_ν to get

$$x'^\mu\sigma_\mu\,\sigma_\nu = g\,x^\alpha\sigma_\alpha\,g^\dagger\sigma_\nu = (g\,\sigma_\alpha\,g^\dagger\sigma_\nu)\,x^\alpha \qquad // \text{ since } x^\mu \text{ is just a number}$$

Now take the trace of both sides and use the fact that $\mathrm{tr}(\sigma_\mu\,\sigma_\nu) = 2\,\delta_{\mu,\nu}$ and we get

$$x'^\nu = \tfrac{1}{2}\,\mathrm{tr}(g\,\sigma_\alpha\,g^\dagger\sigma_\nu)\,x^\alpha \qquad \Rightarrow \qquad \Lambda^\nu{}_\alpha = \tfrac{1}{2}\,\mathrm{tr}(g\,\sigma_\alpha\,g^\dagger\sigma_\nu) = \tfrac{1}{2}\,\mathrm{tr}(\sigma_\nu\,g\,\sigma_\alpha\,g^\dagger)$$

Of course we can write these four matrices in any cyclic order. So here then is the big result

$$\Lambda^\mu{}_\nu = \tfrac{1}{2}\,\mathrm{tr}(\sigma_\mu\,g\,\sigma_\nu\,g^\dagger)$$

which tells us about the (½ , ½) vector representation for 4-vectors. My UCB notes show how you can invert the above equation to get g = f(Λ), but it is a bit ugly.

Now, it turns out that the Pauli matrix form a complete basis for the (1/2,1/2) representation, so our expression $x^\mu\sigma_\mu$ with its 4 parameters is the most general object that you can transform. Here is an interesting completeness formula I see in my notes:

$$\Sigma_\mu (\sigma^\mu)_{ab}(\sigma^\mu)_{cd} = 2\,\delta_{a,d}\,\delta_{b,c}$$

Just for fun, we can ask about det(Λ). We have

$$\det(\Lambda) = \varepsilon_{\alpha\beta\mu\nu}\Lambda^0{}_\alpha\Lambda^1{}_\beta\Lambda^2{}_\mu\Lambda^3{}_\nu$$

$$= 1/16\ \varepsilon_{\alpha\beta\mu\nu}\ \mathrm{tr}(\sigma_0\, g\, \sigma_\alpha\, g^\dagger)\ \mathrm{tr}(\sigma_1\, g\, \sigma_\beta\, g^\dagger)\ \mathrm{tr}(\sigma_2\, g\, \sigma_\mu\, g^\dagger)\ \mathrm{tr}(\sigma_3\, g\, \sigma_\nu\, g^\dagger)$$

There is some way to show this is = 1, but I don't see it offhand! I know it uses the above completeness somehow. Of course we know, on the other hand, that (here $g = g_{\mu\beta} \neq$ g element of SL(2C) !)

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad x'^\mu x'_\mu = \Lambda^\mu{}_\nu x^\nu\ \Lambda_\mu{}^\alpha x_\alpha = (\ \Lambda^\mu{}_\nu\Lambda_\mu{}^\alpha)\ x^\nu x_\alpha = x^\nu x_\nu$$

$$\Rightarrow (\ \Lambda^\mu{}_\nu\Lambda_\mu{}^\alpha) = \delta_\nu{}^\alpha \qquad \Rightarrow \Lambda^\mu{}_\nu\ g_{\mu\beta}\ g^{\alpha\tau}\ \Lambda^\beta{}_\tau = \delta_\alpha{}^\nu = \Lambda^{T\nu}{}_\mu\ g_{\mu\beta}\ \Lambda^\beta{}_\tau\ g^{\tau\alpha} = (\Lambda^T g \Lambda g)^\nu{}_\alpha$$

$$\Rightarrow\ \Lambda^T g\Lambda g = 1\ \Rightarrow\ \det(1) = 1 = \det(\Lambda^T g\Lambda g) = \det(\Lambda)^2 \text{ so } \det(\Lambda) = +1 \text{ for our LG piece.}$$

Now we want to talk about **covariant spinors**, so far we have only seen contravariant ones above:

spinor of the first kind (undotted indices)	$\xi'^a = \Lambda^a{}_b\xi^b$	$\Lambda^a{}_b = D^{(1/2,0)}(g)_{ab} = g$
spinor of the 2nd kind (dotted indices)	$\xi'^{\dot a} = \Lambda^{\dot a}{}_{\dot b}\xi^{\dot b}$	$\Lambda^{\dot a}{}_{\dot b} = D^{(0,1/2)}(g)_{ab} = \overline{g}$

The covariant stuff starts with the fact that there is an automorphism of SL(2,C)

$$g \rightarrow (g^T)^{-1} = \varepsilon\, g\, \varepsilon^{-1} \qquad \varepsilon = i\sigma_2 \qquad \varepsilon 2 = -1$$

$$\varepsilon^{-1} = \varepsilon^T = \varepsilon^\dagger = -\varepsilon \qquad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where ε is the matrix we talked about above in the Moore stuff. So we have

$$(g^T)^{-1} = \varepsilon\, g\, \varepsilon^{-1} \quad \Rightarrow \quad g^T = \varepsilon\, g^{-1}\, \varepsilon^{-1} \quad \Rightarrow \quad g^T\, \varepsilon = \varepsilon\, g^{-1} \quad \Rightarrow g^T\, \varepsilon\, g = \varepsilon$$

This matrix ε is the "spinor metric tensor" because of this fact:

$$a'^T\ \varepsilon\ b' = (ga)^T\varepsilon\ (gb) = a^T g^T \varepsilon g\ b = a^T\ \varepsilon\ b$$

To see why this is, let's define

$$a_\alpha \equiv \varepsilon_{\alpha\beta}\ a^\beta \qquad = \text{covariant first kind spinor}$$

Then the above equation becomes

$$a'_\alpha b'^\alpha = a_\alpha b^\alpha$$

so you get the same idea in as in the vector case that summing on tilted indices is a *scalar*. Somehow in the Dirac representation, you get this same idea but we write it this way

$$\psi'^\dagger \gamma^0 \varphi' = (S\psi)^\dagger \gamma^0 S\varphi = \psi^\dagger S^\dagger \gamma^0 S\varphi = \psi^\dagger \gamma^0 (\gamma^0 S^\dagger \gamma^0) S\varphi = \psi^\dagger \gamma^0 S^{-1} S\varphi = \psi^\dagger \gamma^0 \varphi$$

Unlike the 4-vector 4x4 metric tensor which is diagonal and symmetric, the 2x2 tensor ε is off-diagonal and anti-symmetric! This antisymmetry of ε is the reason for the following rule: If you want to lower a spinor index, apply $\varepsilon_{\alpha\beta}$ on the left, but if you want to raise an index, apply from the right, with adjacent indices matching.

$$a_\alpha \equiv \varepsilon_{\alpha\beta} a^\beta \qquad a^\beta = a_\tau \varepsilon^{\tau\beta} \qquad \text{but } \varepsilon_{\alpha\beta} = \varepsilon^{\alpha\beta}$$

Then we have

$$a_\alpha = \varepsilon_{\alpha\beta} a^\beta = \varepsilon_{\alpha\beta} a_\tau \varepsilon_{\tau\beta} = \varepsilon_{\alpha\beta} a_\tau \varepsilon^{T}_{\beta\tau} = - \varepsilon_{\alpha\beta} a_\tau \varepsilon_{\beta\tau} = - \varepsilon_{\alpha\beta}\varepsilon_{\beta\tau} a_\tau = - (\varepsilon^2)_{\alpha\tau} a_\tau = + \alpha_\alpha$$

Notice what the invariant is when you contract a spinor with itself:

$$a_\alpha a^\alpha = \varepsilon_{\alpha\beta} a^\beta a^\alpha = 0 \qquad \text{due to symmetry}$$

This subject is broached in the **Bade Jehle paper** I have, and I notice that they use the word "spinor" to mean both the basic first and second kind spinors, but also the "spinor tensors" you form out of these. That is, they don't say "spinor tensor" they just say "spinor". So our ε thing which I call the "spinor metric tensor" is to them a certain kind of "spinor" . This paper has everything you ever wanted to know about spinors, I think, including "spinor analysis" with the usual covariant derivative and all that stuff which comes up in general relativity and probably string theory. If I need to learn more about spinors, this would be a good paper to start with.

18. Computation of the finite rotation and boost matrices in the Dirac representation

We learned in Section XX that the Dirac generators are these:

$$J_i' = J_i = \tfrac{1}{2} \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} = \tfrac{1}{2} \Sigma_i = \tfrac{1}{2} \sigma_{jk} = \tfrac{1}{2} \sigma^{jk} \qquad \text{where ijk = cyclic}$$

$$K_i' = - \gamma_6 K_i = \tfrac{1}{2} i \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} = \tfrac{1}{2} i \alpha_i = \tfrac{1}{2} i \gamma_0 \gamma^i = \tfrac{1}{2} \sigma^{0i} = \tfrac{1}{2} \sigma_{i0}$$

and of course our finite transformations of interest are

$$R(\mathbf{r}) = \exp(-i\mathbf{r}\bullet\mathbf{J}) = \cos(\mathbf{r}\bullet\mathbf{J}) - i \sin(\mathbf{r}\bullet\mathbf{J}) \qquad B(\mathbf{b}) = \exp(-i\mathbf{b}\bullet\mathbf{K}')$$

Let's start with the rotation. We can write

$$\mathbf{r}\bullet\mathbf{J} = (r/2)\begin{pmatrix} \hat{\mathbf{r}}\bullet\boldsymbol{\sigma} & 0 \\ 0 & \hat{\mathbf{r}}\bullet\boldsymbol{\sigma} \end{pmatrix} \qquad (\mathbf{r}\bullet\mathbf{J})^2 = (r/2)^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (\mathbf{r}\bullet\mathbf{J})^3 = (r/2)^3\begin{pmatrix} \hat{\mathbf{r}}\bullet\boldsymbol{\sigma} & 0 \\ 0 & \hat{\mathbf{r}}\bullet\boldsymbol{\sigma} \end{pmatrix}$$

Then

$$\cos(\mathbf{r}\bullet\mathbf{J}) = \mathbf{1} - (\mathbf{r}\bullet\mathbf{J})^2/2! + (\mathbf{r}\bullet\mathbf{J})^4/4! - \ldots = \mathbf{1} - (r/2)^2\,\mathbf{1}/2! + (r/2)^4\,\mathbf{1}/4! - \ldots = \cos(r/2)\,\mathbf{1}$$

$$\sin(\mathbf{r}\bullet\mathbf{J}) = (\mathbf{r}\bullet\mathbf{J}) - (\mathbf{r}\bullet\mathbf{J})^3/3! + \ldots$$

$$= \begin{pmatrix} \hat{\mathbf{r}}\bullet\boldsymbol{\sigma} & 0 \\ 0 & \hat{\mathbf{r}}\bullet\boldsymbol{\sigma} \end{pmatrix}((r/2) - (r/2)^3/3! + \ldots.) = \begin{pmatrix} \hat{\mathbf{r}}\bullet\boldsymbol{\sigma} & 0 \\ 0 & \hat{\mathbf{r}}\bullet\boldsymbol{\sigma} \end{pmatrix}\sin(r/2)$$

So our result is then

$$R(\mathbf{r}) = \exp(-i\mathbf{r}\bullet\mathbf{J}) = \cos(r/2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\sin(r/2)\begin{pmatrix} \hat{\mathbf{r}}\bullet\boldsymbol{\sigma} & 0 \\ 0 & \hat{\mathbf{r}}\bullet\boldsymbol{\sigma} \end{pmatrix}$$

$$= \begin{pmatrix} \cos(r/2) - i\,\hat{\mathbf{r}}\bullet\boldsymbol{\sigma}\sin(r/2) & 0 \\ 0 & \cos(r/2) - i\,\hat{\mathbf{r}}\bullet\boldsymbol{\sigma}\sin(r/2) \end{pmatrix} = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$$

where $R = \cos(r/2) - i\,\hat{\mathbf{r}}\bullet\boldsymbol{\sigma}\sin(r/2)$. which is of course just the direct sum of the well-known 2x2 result which I should have in TK but don't! This is a rotation of r degrees about the $\hat{\mathbf{r}}$ axis.

Now what about the boosts?

$$B(\mathbf{b}) = \exp(-i\mathbf{b}\bullet\mathbf{K}') = \cos(\mathbf{b}\bullet\mathbf{K}') - i\sin(\mathbf{b}\bullet\mathbf{K}')$$

$$\mathbf{b}\bullet\mathbf{K}' = i\,(b/2)\begin{pmatrix} 0 & \hat{\mathbf{b}}\bullet\boldsymbol{\sigma} \\ \hat{\mathbf{b}}\bullet\boldsymbol{\sigma} & 0 \end{pmatrix} \quad (\mathbf{b}\bullet\mathbf{K}')^2 = -(b/2)^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (\mathbf{r}\bullet\mathbf{J})^3 = -i\,(b/2)^3\begin{pmatrix} 0 & \hat{\mathbf{b}}\bullet\boldsymbol{\sigma} \\ \hat{\mathbf{b}}\bullet\boldsymbol{\sigma} & 0 \end{pmatrix}$$

$$\cos(\mathbf{b}\bullet\mathbf{K}') = \mathbf{1} - (\mathbf{b}\bullet\mathbf{K}')^2/2! + (\mathbf{b}\bullet\mathbf{K}')^4/4! - \ldots = \mathbf{1} + (b/2)^2\,\mathbf{1}/2! + (b/2)^4\,\mathbf{1}/4! - \ldots = \mathrm{ch}(b/2)\,\mathbf{1}$$

$$\sin(\mathbf{b}\bullet\mathbf{K}') = (\mathbf{b}\bullet\mathbf{K}') - (\mathbf{b}\bullet\mathbf{K}')^3/3! + \ldots.$$

$$= i\begin{pmatrix} 0 & \hat{\mathbf{b}}\bullet\boldsymbol{\sigma} \\ \hat{\mathbf{b}}\bullet\boldsymbol{\sigma} & 0 \end{pmatrix}((b/2) + (r/2)^3/3! + \ldots.) = i\begin{pmatrix} 0 & \hat{\mathbf{b}}\bullet\boldsymbol{\sigma} \\ \hat{\mathbf{b}}\bullet\boldsymbol{\sigma} & 0 \end{pmatrix}\mathrm{sh}(b/2)$$

So our result is then

$$B(\mathbf{b}) = \exp(-i\,\mathbf{b}\bullet\mathbf{K}') = \mathrm{ch}(b/2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathrm{sh}(b/2)\begin{pmatrix} 0 & \hat{\mathbf{b}}\bullet\boldsymbol{\sigma} \\ \hat{\mathbf{b}}\bullet\boldsymbol{\sigma} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \text{ch}(b/2) & \hat{\mathbf{b}} \bullet \boldsymbol{\sigma}\, \text{sh}(b/2) \\ \hat{\mathbf{b}} \bullet \boldsymbol{\sigma}\, \text{sh}(b/2) & \text{ch}(b/2) \end{pmatrix} = \text{ch}(b/2) \begin{pmatrix} 1 & \hat{\mathbf{b}} \bullet \boldsymbol{\sigma}\, \text{th}(b/2) \\ \hat{\mathbf{b}} \bullet \boldsymbol{\sigma}\, \text{th}(b/2) & 1 \end{pmatrix}$$

=>

$$B(\mathbf{b}) = \text{ch}(b/2) \begin{pmatrix} 1 & D \\ D & 1 \end{pmatrix} \qquad \text{where } D = \text{th}(b/2)\ \hat{\mathbf{b}} \bullet \boldsymbol{\sigma}$$

Now let's write out the specific cases for rotations

$$R(\mathbf{r}) = \exp(-i\mathbf{r}\bullet\mathbf{J}) = C_{r/2} - i\, \hat{\mathbf{r}} \bullet \boldsymbol{\sigma}\, S_{r/2} \oplus C_{r/2} - i\, \hat{\mathbf{r}} \bullet \boldsymbol{\sigma}\, S_{r/2} = R \oplus R$$

So we can just worry about the 2x2 subspace R for now. We have

$$R_1(r) = C_{r/2} - i\,\sigma_1\, S_{r/2} = C_{r/2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\, S_{r/2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} C_{r/2} & -i\, S_{r/2} \\ -i\, S_{r/2} & C_{r/2} \end{pmatrix}$$

$$R_2(r) = C_{r/2} - i\,\sigma_2\, S_{r/2} = C_{r/2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\, S_{r/2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} C_{r/2} & -S_{r/2} \\ S_{r/2} & C_{r/2} \end{pmatrix}$$

$$R_3(r) = C_{r/2} - i\,\sigma_3\, S_{r/2} = C_{r/2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\, S_{r/2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \exp(-ir/2) & 0 \\ 0 & \exp(+ir/2) \end{pmatrix}$$

and these agree exactly with my "Rotation Matrices" page.

Now consider for example

$$R(\theta,\varphi) = R_z(\varphi)\, R_y(\theta) = \begin{pmatrix} \exp(-i\varphi/2) & 0 \\ 0 & \exp(+i\varphi/2) \end{pmatrix} \begin{pmatrix} C_{\theta/2} & -S_{\theta/2} \\ S_{\theta/2} & C_{\theta/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\varphi/2} C_{\theta/2} & -e^{-i\varphi/2} S_{\theta/2} \\ e^{i\varphi/2} S_{\theta/2} & e^{i\varphi/2} C_{\theta/2} \end{pmatrix}$$

If you try to express this matrix in terms of n_x, n_y, and n_z, it is not very nice. Scratch shows that

$$C^2_{\theta/2} = (n_z+1)/2 \qquad C^2_{\varphi/2} = (n_x+n_T)/2n_T \qquad \text{where } n_T = \sqrt{n_x^2 + n_y^2}$$
$$S^2_{\theta/2} = (n_z-1)/2 \qquad S^2_{\varphi/2} = (n_x-n_T)/2n_T$$

so probably simplest to leave it in terms of θ and φ which are the polar coordinates of $\hat{\mathbf{s}}$.

As for the specific boosts, we can write

$$B(\mathbf{b}) = \text{ch}(b/2) \begin{pmatrix} 1 & D \\ D & 1 \end{pmatrix} \qquad \text{where } D = \text{th}(b/2)\ \hat{\mathbf{b}} \bullet \boldsymbol{\sigma}$$

so

$$D_1 = th(b/2)\ \hat{\mathbf{b}} \bullet \boldsymbol{\sigma} = th(b/2)\ \sigma_1 = th(b/2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & th(b/2) \\ th(b/2) & 0 \end{pmatrix}$$

$$D_2 = th(b/2)\ \hat{\mathbf{b}} \bullet \boldsymbol{\sigma} = th(b/2)\ \sigma_2 = th(b/2) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i\ th(b/2) \\ i\ th(b/2) & 0 \end{pmatrix}$$

$$D_3 = th(b/2)\ \hat{\mathbf{b}} \bullet \boldsymbol{\sigma} = th(b/2)\ \sigma_3 = th(b/2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} th(b/2) & 0 \\ 0 & -th(b/2) \end{pmatrix}$$

So we can write

$$B_1(b) = ch(b/2) \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & th(b/2) \\ th(b/2) & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & th(b/2) \\ th(b/2) & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

$$B_2(b) = ch(b/2) \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & -i\ th(b/2) \\ i\ th(b/2) & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -i\ th(b/2) \\ i\ th(b/2) & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

$$B_3(b) = ch(b/2) \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} th(b/2) & 0 \\ 0 & -th(b/2) \end{pmatrix} \\ \begin{pmatrix} th(b/2) & 0 \\ 0 & -th(b/2) \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

where the first one agrees with BD1 page 29 but they have -b since a passive boost, mine are active. Notice that the BD exponential involves σ_{01} , but I use $\sigma_{10} = -\sigma_{01}$. So their result is correct for a passive boost both in the exponential and in the shown matrix.

Summary of the results of this section: These are the 4x4 Dirac rep active rotation and boost matrices:

DIRAC ROTATIONS:

$$R(\mathbf{r}) = R_{\hat{\mathbf{r}}}(r) = \exp(-i\mathbf{r}\bullet\mathbf{J}) = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} \qquad \mathbf{J} = \tfrac{1}{2}\,\boldsymbol{\Sigma} = \tfrac{1}{2} \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \qquad R = \cos(r/2) - i\ \hat{\mathbf{r}} \bullet \boldsymbol{\sigma} \sin(r/2)$$

$$R_1(r) = \begin{pmatrix} \begin{pmatrix} C_{r/2} & -i\,S_{r/2} \\ -i\,S_{r/2} & C_{r/2} \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} C_{r/2} & -i\,S_{r/2} \\ -i\,S_{r/2} & C_{r/2} \end{pmatrix} \end{pmatrix}$$

$$R_2(r) = \begin{pmatrix} \begin{pmatrix} C_{r/2} & -S_{r/2} \\ S_{r/2} & C_{r/2} \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} C_{r/2} & -S_{r/2} \\ S_{r/2} & C_{r/2} \end{pmatrix} \end{pmatrix}$$

$$R_3(r) = \begin{pmatrix} \begin{pmatrix} \exp(-ir/2) & 0 \\ 0 & \exp(+ir/2) \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} \exp(-ir/2) & 0 \\ 0 & \exp(+ir/2) \end{pmatrix} \end{pmatrix}$$

DIRAC BOOSTS:

$$B(\mathbf{b}) = R_{\hat{\mathbf{b}}}(b) = \exp(-i\,\mathbf{b}\bullet\mathbf{K}') = ch(b/2)\begin{pmatrix} 1 & D \\ D & 1 \end{pmatrix} \qquad \mathbf{K}' = \tfrac{1}{2}\, i\, \boldsymbol{\alpha} = \tfrac{1}{2}\, i \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix} \qquad D = th(b/2)\ \hat{\mathbf{b}} \bullet \boldsymbol{\sigma}$$

$$B_1(b) = ch(b/2)\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & th(b/2) \\ th(b/2) & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & th(b/2) \\ th(b/2) & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

$$B_2(b) = ch(b/2)\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & -i\,th(b/2) \\ i\,th(b/2) & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -i\,th(b/2) \\ i\,th(b/2) & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

$$B_3(b) = ch(b/2)\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} th(b/2) & 0 \\ 0 & -th(b/2) \end{pmatrix} \\ \begin{pmatrix} th(b/2) & 0 \\ 0 & -th(b/2) \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

19. Computation of the spinors like u(p,s) used in Bjorken and Drell

BD use spinors like u(p,s) where p and s are 4-vectors. I think this is how one of these would be defined

$$u(p,s) = B(\mathbf{p})\, R(\theta,\varphi)\, w^1(0)$$

where $R(\theta,\varphi)$ rotates the rest frame spin $\hat{\mathbf{s}}_0 = \hat{\mathbf{z}}$ into some desired rest frame spin $\hat{\mathbf{s}}$, and then we do an active boost to get to u(p,s). I think this equation above is what led me to write this whole document, so I really ought to do this calculation! [This caused me to write Section 18 above.]

Consider first the rotation R. We know that $R(\theta,\varphi)$ moves $\hat{\mathbf{z}}$ to $\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}_z = \cos\theta$, etc, and so we identify $\hat{\mathbf{n}}$ with our desired $\hat{\mathbf{s}}$. Notice that when we write $R = \cos(r/2) - i\ \hat{\mathbf{r}} \bullet \boldsymbol{\sigma} \sin(r/2)$, we are doing a rotation *about* the $\hat{\mathbf{r}}$ vector, we are not rotating $\hat{\mathbf{z}}$ *to* $\hat{\mathbf{r}}$. The point is that $\hat{\mathbf{n}} \neq \hat{\mathbf{r}}$ **.**

<u>Side Note:</u> If we think spherical coordinates, then the above rotation can be written

$$R_{\hat{\boldsymbol{\varphi}}}(\theta) = C_{\theta/2} - i\, S_{\theta/2}\, \hat{\boldsymbol{\varphi}} \bullet \boldsymbol{\sigma} \qquad \text{and spin } \hat{\mathbf{s}} = \hat{\mathbf{r}} \qquad \text{meaning spherical } \hat{r}$$

but I don't think this fact is too helpful because $\hat{\boldsymbol{\varphi}}$ itself is a function of φ.

From Section 18 above we learned that

$$R(\theta,\varphi) = R \oplus R = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} \qquad \text{where} \quad R = \begin{pmatrix} e^{-i\varphi/2}C_{\theta/2} & -e^{-i\varphi/2}S_{\theta/2} \\ e^{i\varphi/2}S_{\theta/2} & e^{i\varphi/2}C_{\theta/2} \end{pmatrix}$$

$$B(\mathbf{b}) = R_{\hat{\mathbf{b}}}(b) = \exp(-i\,\mathbf{b}\bullet\mathbf{K}') = ch(b/2)\begin{pmatrix} 1 & D \\ D & 1 \end{pmatrix} \qquad \text{where } D = th(b/2)\ \hat{\mathbf{b}}\bullet\boldsymbol{\sigma}$$

so we then have

$$u(p,s) = B(\mathbf{p})\,R(\theta,\varphi)\,w^1(0) = ch(b/2)\begin{pmatrix} 1 & D \\ D & 1 \end{pmatrix}\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} w^1(0)$$

If we just have spin up, then R = 1 and we get

$$u(p,\hat{z}) = ch(b/2)\begin{pmatrix} 1 & D \\ D & 1 \end{pmatrix} w^1(0) = ch(b/2)\begin{pmatrix} 1 & D \\ D & 1 \end{pmatrix}\begin{pmatrix} u \\ 0 \end{pmatrix} = ch(b/2)\begin{pmatrix} u \\ Du \end{pmatrix} \qquad \text{where } u = (1,0)$$

so the part of most interest is this:

$$u(p,\hat{z})_{\mathtt{lower}} = ch(b/2)\,th(b/2)\ \hat{\mathbf{b}}\bullet\boldsymbol{\sigma}\ u$$

We know these hyperbolic facts from our particle momentum page with c = 1

$$sh(b/2) = \sqrt{\frac{E-m}{2m}} \qquad ch(b/2) = \sqrt{\frac{E+m}{2m}} \qquad th(b/2) = \sqrt{\frac{E-m}{E+m}}$$

$$sh(b) = p/m = \beta\gamma \qquad ch(b) = E/m = \gamma \qquad th(b) = v = \beta = p/E$$

We can see then that

$$th^2(b/2) = (E-m)(E+m) / \{(E+m)(E+m)\} = p^2/(E+m)^2 \quad \Rightarrow \quad th(b/2) = p/(E+m)$$

Then since $\hat{\mathbf{b}} = \hat{\mathbf{p}}$, we have $th(b/2)\,\hat{\mathbf{b}} = \mathbf{p}/(E+m)$ which brings us to

$$u(p,\hat{z})_{\mathtt{lower}} = ch(b/2)\{\ \mathbf{p}\bullet\boldsymbol{\sigma}\ /(E+m)\ \}\ u \qquad \text{but we know: } \mathbf{p}\bullet\boldsymbol{\sigma} = \begin{pmatrix} p^z & p^- \\ p^+ & -p^z \end{pmatrix}$$

so

$$u(p,\hat{z})_{\mathtt{lower}} = \sqrt{\frac{E+m}{2m}}\ /(E+m)\begin{pmatrix} p^z & p^- \\ p^+ & -p^z \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \sqrt{\frac{E+m}{2m}}\ (1/(E+m))\begin{pmatrix} p^z \\ p^+ \end{pmatrix}$$

$$u(p,\hat{z})_{\mathtt{upper}} = ch(b/2)\ u = \sqrt{\frac{E+m}{2m}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and this agrees with the first column of BD1 page 30 (3.7).

So, our 4x4 matrix with spin up is this

$$B(\mathbf{p})\,R(0,0) = B(\mathbf{p}) = \mathrm{ch}(b/2)\begin{pmatrix} 1 & D \\ D & 1 \end{pmatrix}$$

where

$$D = \mathrm{th}(b/2)\ \hat{\mathbf{b}}\bullet\boldsymbol{\sigma} = \mathbf{p}\bullet\boldsymbol{\sigma}\ /(E+m)$$

so we then get

$$B(\mathbf{p})\,R(0,0) = B(\mathbf{p}) = \sqrt{\frac{E+m}{2m}}\begin{pmatrix} 1 & \mathbf{p}\bullet\boldsymbol{\sigma}/(E+m) \\ \mathbf{p}\bullet\boldsymbol{\sigma}/(E+m) & 1 \end{pmatrix} \quad \mathbf{p}\bullet\boldsymbol{\sigma} = \begin{pmatrix} p^z & p^- \\ p^+ & -p^z \end{pmatrix} \quad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and this is exactly the matrix in 3.7 BD1 page 30 just mentioned. Then the 4 columns are the four spinors. So we pass this "check" along the way.

With general spin in the θ,φ direction, our result is more complicated. One way to write it is this:

$$B(\mathbf{p})\,R(\theta,\varphi) = \sqrt{\frac{E+m}{2m}}\begin{pmatrix} 1 & D \\ D & 1 \end{pmatrix}\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} = \sqrt{\frac{E+m}{2m}}\begin{pmatrix} R & DR \\ DR & R \end{pmatrix}$$

where

$$D = \mathbf{p}\bullet\boldsymbol{\sigma}\ /(E+m) = \begin{pmatrix} p^z & p^- \\ p^+ & -p^z \end{pmatrix}/(E+m) \qquad p^{\pm} = p^x \pm i\,p^y$$

$$R = \begin{pmatrix} e^{-i\varphi/2}C_{\theta/2} & -e^{-i\varphi/2}S_{\theta/2} \\ e^{i\varphi/2}S_{\theta/2} & e^{i\varphi/2}C_{\theta/2} \end{pmatrix} \qquad \hat{\mathbf{s}} = R(\theta,\varphi)\,\hat{\mathbf{z}}$$

I had Maple compute DR ignoring the /(E+m) with the result

$$DR := \begin{bmatrix} pz\,\mathbf{e}^{\left(-\frac{1}{2}I\phi\right)}\cos\left(\frac{1}{2}\theta\right) + pm\,\mathbf{e}^{\left(\frac{1}{2}I\phi\right)}\sin\left(\frac{1}{2}\theta\right) & -pz\,\mathbf{e}^{\left(-\frac{1}{2}I\phi\right)}\sin\left(\frac{1}{2}\theta\right) + pm\,\mathbf{e}^{\left(\frac{1}{2}I\phi\right)}\cos\left(\frac{1}{2}\theta\right) \\ pp\,\mathbf{e}^{\left(-\frac{1}{2}I\phi\right)}\cos\left(\frac{1}{2}\theta\right) - pz\,\mathbf{e}^{\left(\frac{1}{2}I\phi\right)}\sin\left(\frac{1}{2}\theta\right) & -pp\,\mathbf{e}^{\left(-\frac{1}{2}I\phi\right)}\sin\left(\frac{1}{2}\theta\right) - pz\,\mathbf{e}^{\left(\frac{1}{2}I\phi\right)}\cos\left(\frac{1}{2}\theta\right) \end{bmatrix}$$

and we can compact this down to say

$$DR = \begin{pmatrix} p^z e^- C + p^- e^+ S & -p^z e^- S + p^- e^+ C \\ p^+ e^- C - p^z e^+ S & -p^+ {}^z e^- S - p^z e^+ C \end{pmatrix}/(E+m) \qquad e^{\pm} = \exp(\pm i\varphi/2) \qquad C = \cos(\theta/2) \qquad S = \sin(\theta/2)$$

$$R = \begin{pmatrix} e^- C & -e^- S \\ e^+ S & e^+ C \end{pmatrix}$$

So here is our grand result

$$B(\mathbf{p})\,R(\theta,\varphi) = \sqrt{\frac{E+m}{2m}}\ x$$

$$\begin{pmatrix} \begin{pmatrix} e^-C & -e^-S \\ e^+S & e^+C \end{pmatrix} & \begin{pmatrix} p^z e^-C + p^- e^+ S & -p^z e^- S + p^- e^+ C \\ p^+ e^- C - p^z e^+ S & -p^+ e^- S - p^z e^+ C \end{pmatrix} /(E+m) \\ \begin{pmatrix} p^z e^-C + p^- e^+ S & -p^z e^- S + p^- e^+ C \\ p^+ e^- C - p^z e^+ S & -p^+ e^- S - p^z e^+ C \end{pmatrix} /(E+m) & \begin{pmatrix} e^-C & -e^-S \\ e^+S & e^+C \end{pmatrix} \end{pmatrix}$$

and we can then read the columns as being our four spinors:

$$(\qquad u(p,s) \qquad u(p,-s) \qquad v(p,-s) \qquad v(p,s) \qquad)$$

I can see why BD don't ever write this thing out because it is a bit mess, but it was a good Lorentz exercize I think to derive it. If you were not doing spin averages, you might need the above result!