

1. The Lie Algebra for the abstract Lorentz Group generators $J^{\mu\nu}$ is derivable from the known coordinate transformations of special relativity and is this:

$$[J^{\mu\nu}, J^{\alpha\beta}] = -i (g^{\nu\beta} J^{\mu\alpha} + g^{\mu\alpha} J^{\nu\beta} - g^{\mu\beta} J^{\nu\alpha} - g^{\nu\alpha} J^{\mu\beta}) \quad \text{index} = 0,1,2,3 \quad g = \text{diag}(1,-1,-1,-1)$$

from which it can be shown that each $J^{\mu\nu}$ is antisymmetric when its labels are switched, $J^{\mu\nu} = -J^{\nu\mu}$. An $N \times N$ matrix has $(N^2 - N)/2$ elements in its upper or lower triangle, which here is $(4^2 - 4)/2 = 6$. Thus, there are only 6 distinct $J^{\mu\nu}$ generators, as befits the 6 parameter Lorentz Group.

2. The traditional rotation and boost generators are these.

$$\begin{array}{ll} \text{rotation} & J_i = J_{jk} \\ \text{boost} & K_i = J_{i0} \end{array} \quad (\text{where } jk \text{ completes the cyclic order begun by } i)$$

and the Lie Algebra shown above becomes

$$[J_i, J_j] = +i \epsilon_{ijk} J_k \quad [J_i, K_j] = +i \epsilon_{ijk} K_k \quad [K_i, K_j] = -i \epsilon_{ijk} J_k$$

The first two commutators just say that J and K are rotational 3-vectors, while the last says that boosts don't commute and results in effects like the Thomas Precession.

3. The Lorentz Group Lie Algebra can be "decoupled" simply by defining

$$L_i \equiv \frac{1}{2} (J_i + iK_i) \quad R_i \equiv \frac{1}{2} (J_i - iK_i)$$

with inverses

$$J_i = (L_i + R_i) \quad K_i = i(R_i - L_i)$$

and we then have

$$[L_i, L_j] = i \epsilon_{ijk} L_k \quad [R_i, R_j] = i \epsilon_{ijk} R_k \quad [L_i, R_j] = 0$$

which says that L_i and R_i (left and right) are normal rotation group generators and are completely independent. It follows that the finite dimensional irreducible representations of the Lorentz Group can be written as direct products of rotation group representations. This fact is summarized by the notation

$$j_L \otimes j_R \quad \text{where} \quad \begin{array}{ll} L^2 = j_L(j_L + 1) \cdot 1 & R^2 = j_R(j_R + 1) \cdot 1 \\ \dim = (2j_L + 1) & \dim = (2j_R + 1) \end{array}$$

where 1 is the identity matrix in whatever space we are in.

4. Apart from the trivial representation $0 \otimes 0$, the lowest two representations are $\frac{1}{2} \otimes 0$ and $0 \otimes \frac{1}{2}$ which are sometimes called the left and right Weyl spinor representations. In the $\frac{1}{2} \otimes 0$ we set $R_i = 0$ to get the trivial group representation $D^R = \exp(0) = 1$, and we then have

$$\begin{aligned} J_i &= (L_i + R_i) = L_i = \sigma_i/2 \\ K_i &= i(R_i - L_i) = -iL_i = -i\sigma_i/2 \end{aligned}$$

In the $0 \otimes \frac{1}{2}$ representation we have $L_i = 0$ and the results are the same as above except the boost generators have the opposite sign

$$\begin{aligned} J_i &= (L_i + R_i) = R_i = \sigma_i/2 \\ K_i &= i(R_i - L_i) = +iR_i = +i\sigma_i/2 \end{aligned}$$

5. A Majorana 4-spinor has the form $\psi = (f_L, g_R)$ where f and g are 2-spinors which transform according to our left and right Weyl representations. For such a spinor, we would have the following 4x4 rotation and boost generator matrices, based on what has been said above:

$$\begin{aligned} J_i &= \sigma_i/2 \oplus \sigma_i/2 = \frac{1}{2} \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} = \frac{1}{2} \Sigma_i = \frac{1}{2} \varepsilon_{ijk} J_{jk} = J_{jk} \text{ (cyclic)} \\ K_i &= -i\sigma_i/2 \oplus i\sigma_i/2 = \frac{1}{2} \begin{pmatrix} -i\sigma_i & 0 \\ 0 & i\sigma_i \end{pmatrix} = J_{i0} \end{aligned}$$

and such a spinor then belongs to the representation $(\frac{1}{2} \otimes 0) \oplus (0 \otimes \frac{1}{2})$ of the Lorentz Group where $\oplus$ indicates a direct sum, as shown in the matrices above.

6. This is not, however, the official Dirac representation. We have to do a "shuffle" on the boost generators (only) as follows

$$K'_i = -\gamma_6 K_i \quad \text{where} \quad \gamma_6 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Thus "shuffle" does not affect the underlying Lie Algebra, and it is easy to show that

$$[J_i, J_j] = +i \varepsilon_{ijk} J_k \quad [J_i, K'_j] = +i \varepsilon_{ijk} K'_k \quad [K'_i, K'_j] = -i \varepsilon_{ijk} J_k$$

and we end up with

$$\begin{aligned} J_i &= \frac{1}{2} \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} = \frac{1}{2} \sigma^{jk} & K'_i &= i \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} = \frac{1}{2} \sigma^{i0} \\ \Rightarrow J^{\mu\nu} &= \frac{1}{2} \sigma^{\mu\nu} & \text{where} & \sigma^{\mu\nu} \equiv i/2 [\gamma^\mu, \gamma^\nu] \end{aligned}$$

and this is the official Dirac representation of the generators, which is seen to be "in covariant form" with respect to the gamma matrices. The corresponding Lorentz Transformation in this 4x4 Dirac space is then given by

$$U(\omega) = \exp(-i/2 \omega_{\mu\nu} J^{\mu\nu}) = \exp[-i(\mathbf{r} \cdot \mathbf{J} + \mathbf{b} \cdot \mathbf{K}')] = \exp(-i/4 \omega_{\mu\nu} \sigma^{\mu\nu})$$

where we have the usual antisymmetric (when both indices are lower) parameter matrix

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & b1 & b2 & b3 \\ -b1 & 0 & r3 & -r2 \\ -b2 & -r3 & 0 & r1 \\ -b3 & r2 & -r1 & 0 \end{pmatrix} \quad \text{but} \quad \omega^{\mu}_{\nu} = g^{\mu\alpha} \omega_{\alpha\nu} = \begin{pmatrix} 0 & b1 & b2 & b3 \\ b1 & 0 & -r3 & +r2 \\ b2 & +r3 & 0 & -r1 \\ b3 & -r2 & +r1 & 0 \end{pmatrix}$$

7. For contrast, the normal 4-vector representation of the Lorentz group has generators given by

$$(J^{\mu\nu})^{\alpha\beta} = i(g^{\mu\alpha} g^{\nu\beta} - g^{\nu\alpha} g^{\mu\beta}) \quad // \text{ note that } J^{\mu\nu} = -J^{\nu\mu}$$

and the resultant 4x4 Lorentz Transformation is then

$$\Lambda(\omega) = \exp(-i/2 \omega_{\mu\nu} J^{\mu\nu}) = \exp[-i(\mathbf{r} \cdot \mathbf{J} + \mathbf{b} \cdot \mathbf{K})] = \exp(\omega)$$

where the final form is so compact that the generators appear to have vanished altogether, but this is just due to their extreme "thinness" in this representation, their being made of nothing but metric tensors. The J and K generator matrices have the following typical forms

$$\begin{array}{ll} \text{rotation} & J_i = J_{jk} \\ \text{boost} & K_i = J_{i0} \end{array}$$

$$(J_1)^{\mu}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \quad (K_1)^{\mu}_{\nu} = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which we know exponentiate into finite rotations and boosts

$$\exp(-irJ_1)^{\mu}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(r) & -\sin(r) \\ 0 & 0 & \sin(r) & \cos(r) \end{pmatrix} \quad \exp(-ibK_1)^{\mu}_{\nu} = \begin{pmatrix} \cosh(b) & \sinh(b) & 0 & 0 \\ \sinh(b) & \cosh(b) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The 4-vector representation just stated is in fact a "shuffle" of the $\frac{1}{2} \otimes \frac{1}{2}$ irreducible representation of the Lorentz group which of course has dimensionality $(2j_L + 1)(2j_R + 1) = 2 \cdot 2 = 4$. In this direct product one can represent the left $\frac{1}{2}$ by a 2x2 matrix g , and the right $\frac{1}{2}$ by g^* . These 2x2 g matrices are in fact group elements of $SL(2, \mathbb{C})$ [S = special $\Rightarrow \det(g) = 1$] which we know is $\approx SO(3, 1)$ = the Lorentz Group. So our representation of $\frac{1}{2} \otimes \frac{1}{2}$ is then $D^L \otimes D^R = g \otimes g^*$. The "shuffle" is then given by the following equation in which one can see the "two g 's" appearing on the right hand side.

$$\Lambda^{\mu}_{\nu} = \frac{1}{2} \text{tr}(\sigma_{\mu} g \sigma_{\nu} g^{\dagger}) \quad \text{where } g = g(\omega) \text{ and } \Lambda = \Lambda(\omega).$$

This "shuffle" arises from the 2x2 transformation

$$X' = gXg^\dagger \quad \text{where} \quad X = x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \quad \det X = x^\mu x_\mu$$

in which one sees that $\det X' = \det X$ so the transformation results in $x'^\mu x'_\mu = x^\mu x_\mu$. The Λ^μ_ν equation above is obtained by right multiplying $X' = gXg^\dagger$ by σ_ν and taking the trace of both sides.

8. Consider the following commutator relation which we assume exists in some unspecified space,

$$[J^{\mu\nu}, p^\alpha] = -i (g^{\alpha\mu} p^\nu - g^{\alpha\nu} p^\mu)$$

where $J^{\mu\nu}$ are the Lorentz generators, and p^α is some object with $\alpha = 0,1,2,3$. By doing the exponentiation, the above implies that

$$\exp(+i/2 \omega_{\mu\nu} J^{\mu\nu}) p^\alpha \exp(-i/2 \omega_{\mu\nu} J^{\mu\nu}) = U^{-1} p^\alpha U = \exp(\omega)^\alpha_\nu p^\nu = \Lambda^\alpha_\nu p^\nu$$

or just

$$U^{-1} p^\alpha U = \Lambda^\alpha_\nu p^\nu$$

which says that p^μ transforms as a 4-vector object in whatever space p^μ happens to live. If p^μ lives in the vector representation space, then it is a regular 4-vector such as momentum. But the equation is valid in any space in which the starting commutator relation is valid.

In the Dirac space, we have

$$J^{\mu\nu} = \frac{1}{2} \sigma^{\mu\nu} = i/4 [\gamma^\mu, \gamma^\nu]$$

and we have the commutator relation

$$[J^{\mu\nu}, \gamma^\alpha] = -i (g^{\alpha\mu} \gamma^\nu - g^{\alpha\nu} \gamma^\mu)$$

so in this example we have $p^\alpha = \gamma^\alpha$ = a 4-index object living in the Dirac space. It follows then that

$$U^{-1} \gamma^\alpha U = \Lambda^\alpha_\nu \gamma^\nu \quad \text{where} \quad U(\omega) = \exp(-i/2 \omega_{\mu\nu} J^{\mu\nu}) = \exp(-i/4 \omega_{\mu\nu} \sigma^{\mu\nu})$$

One implication of this fact is another fact, namely

$$U^{-1} a^\mu U = a^\mu \quad \text{where} \quad a^\mu \equiv a_\mu \gamma^\mu$$

which we prove as follows:

$$U^{-1} a^\mu U = U^{-1} a'_\mu \gamma^\mu U = (a'_\mu) (U^{-1} \gamma^\mu U) = (\Lambda_\mu^\beta a_\beta) (\Lambda^\mu_\nu \gamma^\nu)$$

$$= (\Lambda_{\mu}^{\beta} \Lambda^{\mu}_{\nu}) a_{\beta} \gamma^{\nu} = \delta^{\beta}_{\nu} a_{\beta} \gamma^{\nu} = a_{\beta} \gamma^{\beta} = \not{a}$$

where we use our general orthogonality rule for *any* curvilinear coordinate transformation $\delta_{\mathbf{b}}^{\mathbf{c}} = T_{\mathbf{a}}^{\mathbf{c}} T^{\mathbf{a}}_{\mathbf{b}}$, of which the Lorentz transformation is just a linear example.

9. Armed with the above, it is trivial to show the "covariance" of the Dirac equation, which is this in "the primed frame" (where the 4x4 matrix aspect of the equation is implied),

$$(\hat{\mathbf{p}}' - m)\psi'(x') = 0$$

where $\hat{\mathbf{p}}^{\mu} \equiv i\partial^{\mu}$, the differential operator. Since ψ belongs to our Dirac representation, we know that

$$\psi'(x') = U\psi(x) \quad \text{where } U(\omega) = \exp(-i/2 \omega_{\mu\nu} J^{\mu\nu}) = \exp(-i/4 \omega_{\mu\nu} \sigma^{\mu\nu}) = 4x4 \text{ Dirac LT}$$

then we can apply U^{-1} from the left to get

$$U^{-1}(\hat{\mathbf{p}}' - m) U\psi(x) = 0 \Rightarrow (U^{-1} \hat{\mathbf{p}} U - m) \psi(x) = 0 \Rightarrow (\hat{\mathbf{p}} - m) \psi(x) = 0 \quad \text{QED}$$

Thus, the Dirac equation has the same "form" in both the primed and unprimed frames which are related by a Lorentz Transformation.

Bjorken and Drell volume 1 expends an entire chapter (Chapter 2) deriving this fact.

10. Finally, as a coda, we comment more on the Left and Right spinor representations of the Lorentz Group, where we now use the symbol Λ to represent 2x2 matrices: ($\bar{g}$ means g^*)

$$\begin{aligned} \text{spinor of the first kind (undotted indices)} \quad \xi^{\mathbf{a}} &= \Lambda^{\mathbf{a}}_{\mathbf{b}} \xi^{\mathbf{b}} & \Lambda^{\mathbf{a}}_{\mathbf{b}} &= D^{(1/2, 0)}(g)_{\mathbf{ab}} = g \\ \text{spinor of the 2nd kind (dotted indices)} \quad \xi^{\dot{\mathbf{a}}} &= \Lambda^{\dot{\mathbf{a}}}_{\dot{\mathbf{b}}} \xi^{\dot{\mathbf{b}}} & \Lambda^{\dot{\mathbf{a}}}_{\dot{\mathbf{b}}} &= D^{(0, 1/2)}(g)_{\mathbf{ab}} = \bar{g} \end{aligned}$$

The dots are necessary so one can tell how a given index transforms when one constructs "spinor tensors" and watches them transform. For example, here is a mixed rank-2 "spinor tensor" with one index of each kind

$$X^{\mathbf{a}\dot{\mathbf{b}}} = \Lambda^{\mathbf{a}}_{\mathbf{a}'} \Lambda^{\dot{\mathbf{b}}}_{\dot{\mathbf{b}}'} X^{\mathbf{a}'\dot{\mathbf{b}}'}$$

where we can regard

$$\Lambda^{\mathbf{a}}_{\mathbf{a}'} \Lambda^{\dot{\mathbf{b}}}_{\dot{\mathbf{b}}'} = [D^{(1/2, 1/2)}(g)]^{\mathbf{a}}_{\mathbf{a}'} \Lambda^{\dot{\mathbf{b}}}_{\dot{\mathbf{b}}'} = [D^{(1/2, 0)}(g)]^{\mathbf{a}}_{\mathbf{a}'} [D^{(0, 1/2)}(g)]^{\dot{\mathbf{b}}}_{\dot{\mathbf{b}}'} = g_{\mathbf{aa}'} \bar{g}_{\mathbf{bb}'}$$

as described earlier. By putting indices in proper matrix order, we can write the above "spinor tensor" transformation in the following more compact notation.

$$X' = g X g^{\dagger}$$

Clearly $X^{\mathbf{a}\dot{\mathbf{b}}}$ transforms according to the $\frac{1}{2} \otimes \frac{1}{2}$ representation of the Lorentz Group. The Pauli matrices can be written $(\sigma_{\mu})^{\mathbf{a}\dot{\mathbf{b}}}$ and transform exactly as such rank 2 "spinor tensors" and the four σ^{μ} thus form a

"basis" for this $\frac{1}{2} \otimes \frac{1}{2}$ representation. One can then construct a linear combination $X = x_{\mu} \sigma^{\mu}$ and this leads to the homomorphism result quoted above that $\Lambda^{\mu}_{\nu} = \frac{1}{2} \text{tr}(\sigma_{\mu} g \sigma_{\nu} g^{\dagger})$ which relates the 4-vector representation Λ^{μ}_{ν} to the canonical $\frac{1}{2} \otimes \frac{1}{2}$ representation.

As presented above, the 2-spinors of either type are written with contravariant upper indices. For either type of spinor, one can define a covariant lower index spinor as follows,

$$a_{\alpha} \equiv \varepsilon_{\alpha\beta} a^{\beta} \quad = \text{covariant first kind spinor}$$

where

$$\varepsilon \equiv i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad = \text{"the spinor metric tensor"}$$

Whereas the 4-vector metric tensor $g_{\mu\nu}$ is diagonal and symmetric, the " spinor metric tensor " is off diagonal and antisymmetric. The compatible rule for raising a spinor index is this

$$a^{\beta} = a_{\tau} \varepsilon^{\tau\beta} \quad \varepsilon^{\mu\nu} \equiv \varepsilon_{\mu\nu}$$

where summed indices are always put adjacent. It is easy to show that the matrix ε acts as a similarity transformation changing g to $(g^T)^{-1}$, which we could call an "automorphism of $SL(2, \mathbb{C})$ ". So we have

$$(g^T)^{-1} = \varepsilon g \varepsilon^{-1} \quad \text{note: } \varepsilon^{-1} = \varepsilon^T = \varepsilon^{\dagger} = -\varepsilon \quad \text{and} \quad \varepsilon^2 = -1$$

$$\Rightarrow g^T \varepsilon g = \varepsilon$$

Suppose ψ and ϕ are each 2-spinors of the first kind, which means $\psi' = g\psi$ for transformation. Then from the above

$$\psi'^T \varepsilon \phi' = (g\psi)^T \varepsilon (g\phi) = \psi^T g^T \varepsilon g \phi = \psi^T \varepsilon \phi$$

or

$$\psi'^{\alpha} \phi'_{\alpha} = \psi^{\alpha} \phi_{\alpha}$$

which shows that the contraction of contravariant with covariant spinors produces a world scalar, just as happens with 4-vectors in the vector representation of the Lorentz Group. If one contracts a spinor with itself, the result is 0 due to the antisymmetry of the spinor metric tensor

$$\psi^{\alpha} \psi_{\alpha} = \psi^{\alpha} \varepsilon_{\alpha\beta} \psi^{\beta} = \varepsilon_{\alpha\beta} \psi^{\alpha} \psi^{\beta} = 0$$

so all spinors have norm 0 if we define $\|\psi\|^2 = \psi^{\alpha} \psi_{\alpha}$.

Just as with regular tensors, one can combine "spinor tensors" in various complicated ways, and the only complication is that there is only one basic vector, but there are two basic spinors, so one must use the dots on indices as needed. Just as in "tensor analysis" one has "spinor analysis" in which usual zoo animals appear, such as the covariant derivative involving Christoffel symbols $\Gamma^{\rho}_{\alpha s}$,

$$\psi_{\alpha; s} \equiv \partial_s \psi_{\alpha} - \Gamma^{\rho}_{\alpha s} \psi_{\rho}$$