Lai Continuum Mechanics Notes PhL 1.16.12

Motivation: This is the right place to start, THEN you can do quantum field theory and general relativity.

In the four Parts A,B,C,D below I have pasted in at the start the apropos section of Lai's official TOC.

The first 70 pages of the 500-page Lai book are "math preliminaries" which the author needs to get written down.

The subheadings below are just those where I took some notes, where something was new to me.

Comment: The following tensor analysis notions do NOT appear in this 70 page section:
metric tensor, contravariant/covariant vectors, up and down indices (it is mostly all Cartesian)
only n = 2 and n=3 are considered. All indices are "down".
There is no notion of a general transformation, but Q is used as my R for the linearized xform.
No tangent or reciprocal vectors. No notion of equation covariance.
Only spherical and cylindrical curvilinears are considered, no general formulas.
No comments on the ε tensor, no tensor densities.

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From my review of books:

Lai 1993 has 3 authors. It opens with tensors and curvilinear coordinates, which I think I like. Then comes strain and then stress. Then "the elastic solid" and yes, here are wave discussions. Eventually we get off into viscous fluids, wow, a freebie. This one is the best book so far in my collection.

The book is **Lai, Ruben and Krempl** which I shall call **Lai**. I see there is a **4th edition** (my copy is 3rd). I just brought in the 4th Ed from here:

http://search.4shared.com/postDownload/2KIBRu6e/introduction_to_continuum_mech.html

It has the same structure, so I guess I would go with this later edition if I get launched. Date is **2009**. It does not have a TOC so I had to get that from elsewhere. Amazon has the book for $60 free shipping and there are reviews. But not many reviews, and one is a major pan, lots of typos, etc, in 3rd Ed. But this is 4th Ed some 18 years later, so I would assume things are cleaned up.

I will give Lai 4th Edition a go. I could find no online errata at the Lai site. I think this book can be bought for **$12** including shipping from a used place, but in "new" condition. I might do that at some point, but right now let's use the PDF and see what we see.

This book has no TOC! Maybe this is some strange draft? I will scan for other downloads. It is here with TOC http://www.scribd.com/doc/61554693/-3PfiF1Gdb but it blocks me for money. I go to home page, and am already logged in to **scribd**. This thing is not in the search list for scribd somehow (notice the Swiss bank account identifier above.) I was able to use the above to create my own detailed TOC in a separate doc. Fine.

Note Added 1.18.12. I just found a new, 4th Ed, copy of this at phatcampus for $7 + $4 ship = $11 and I just ordered it via PayPal with instant $ transfer.

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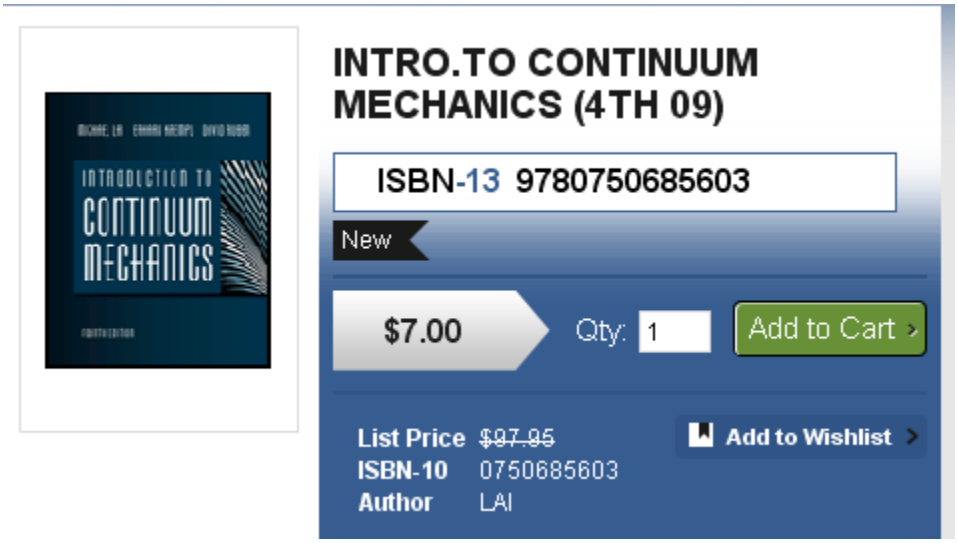
I included a "message" saying: I assume Lai et al book is 4th Ed, new and clean. If that is not the case, please let me know. It claims to be new on their website: (notice the word "new")



This book must have been a poor seller? Why is it $7 pray tell? I got the lead on Amazon where it was $8. Other places have it as well for $10-$20. The official Amazon price is $104 marked to $60.05. Be careful when buying a book on Amazon! I wonder why only 5 comments in Amazon, one of which is horrible, another says it is great. // On Jan 27 I checked on my order since I heard nothing, and it Phatcampus did not recognize me at all, so I had to sent them a customer service email. We shall see. // On Jan 28 it arrived and is in perfect condition. I sent Phat a second email belaying the first, and then registered there as xmission email and ginger. It came from Sterling book wholesaler in Florida with a cellophane wrap

Chapter 1: Introduction

The idea is to neglect microscopic details. CM is the study of how a "continuous" system **responds to loading**. There are two parts to the theory and method. First, **general facts** like energy conservation that apply to all systems. Second, specific "**constitutive equations**" for a specific system. This latter is what distinguishes an elastic solid from a viscous liquid, for example. Lai claims that there are **two types** of relations of interest: integral form ones which involve a volume, and differential ones at a "point". The book will look at **four different models**: linear and non-linear elastic solid (Ch 5), Newtonian viscous fluid (Ch 6), non-Newtonian fluid (Ch 8).

Chapter 2: Tensors

Part A: index notation

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Comment: This stuff is very well done for the undergraduate, I like it, it would make teaching easier. There are lots of examples AND problems every step along the way. You know that typical undergrads need to be hand-held along the way and this book does just that. This book is for all students, not just for the smart students.

I already know 95% of all this stuff very well, BUT there is 5% that is new to me and I will comment on below.

Repeated indices called Einstein notation I am happy to see. Index on both sides of an equation is called "a free index", very good. Kronecker delta. The ε tensor is called "the permutation tensor".

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Use of bold font for matrices.

Here is the first sign of strangeness:

$$\begin{aligned}\mathbf{Te}_1 &= T_{11}\mathbf{e}_1 + T_{21}\mathbf{e}_2 + T_{31}\mathbf{e}_3,\\ \mathbf{Te}_2 &= T_{12}\mathbf{e}_1 + T_{22}\mathbf{e}_2 + T_{32}\mathbf{e}_3,\\ \mathbf{Te}_3 &= T_{13}\mathbf{e}_1 + T_{23}\mathbf{e}_2 + T_{33}\mathbf{e}_3,\end{aligned} \qquad (2.7.1)$$

$$\mathbf{Te}_i = T_{ji}\mathbf{e}_j. \qquad (2.7.2)$$

The three RHS's of 2.7.1 are the three *column vectors* of matrix T. For example

$$\mathbf{T}\,\mathbf{e_1} = \mathbf{T}\,[\,\mathbf{e_1}\,] = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} T_{11} \\ T_{21} \\ T_{31} \end{pmatrix} = T_{11}\mathbf{e_1} + T_{21}\mathbf{e_2} + T_{31}\mathbf{e_3} = \Sigma_k T_{k1}\mathbf{e_k}$$

OK, my confusion is that this set of authors wants to represent a matrix with a bold letter **T** perhaps to distinguish that from a scalar. This is something I would NEVER do, but they do it. Fine. Perhaps think of their T as $\mathbf{T}$. I wonder if this bold matrix notation is "standard" in this field? What is especially confusing is that this $\mathbf{Te_1}$ notation *looks like* the later **ab** dyadic notation in which the two bolded objects are vectors, whereas in $\mathbf{Te_1}$ the left object is a tensor and the right a vector.

Now consider

$$[\mathbf{Te_1}]_j = [\,\Sigma_k T_{k1}\mathbf{e_k}\,]_j = \Sigma_k T_{k1}\,\delta_{kj} = T_{j1}$$

It is true that the object $[\mathbf{Te_1}]$ is itself a vector with these components (ie, it is the first column of T), and you could argue that since it is a vector, you should bold the whole thing.

$$\mathbf{C_1} = [\mathbf{Te_1}]$$

so I guess that is their justification. But I really think they just mean T and they want to bold it, as in

Let **T** be a transformation that transforms any vector into another vector. If **T** transforms **a** into **c** and **b** into **d**, we write $\mathbf{Ta} = \mathbf{c}$ and $\mathbf{Tb} = \mathbf{d}$.

If **T** has the following linear properties:

$$\mathbf{T}(\mathbf{a}+\mathbf{b}) = \mathbf{Ta} + \mathbf{Tb}, \qquad (2.6.1)$$

$$\mathbf{T}(\alpha\mathbf{a}) = \alpha\mathbf{Ta}, \qquad (2.6.2)$$

Now I like to write

$$T_{ij} = \langle i \,|\, \mathbf{T} \,|\, j \rangle = \mathbf{e}_i \bullet \mathbf{T}\,\mathbf{e}_j \qquad \mathbf{T} = \text{operator} \qquad T = \text{matrix}$$

On and on we to, transpose is $T^{\mathbf{T}}$. Then we have

$T^{T}_{ij} = \langle i \mid \mathbf{T}^{T} \mid j \rangle = Te_i \bullet \mathbf{e}_j$ **T** = operator T = matrix

$T_{ji} = \mathbf{e}_j \bullet Te_i$

More generally

$\mathbf{a} \bullet T\mathbf{b} = \Sigma_{ij}\, a_i T_{ij} b_j = \Sigma_{ij}\, a_i T^{T}_{ji} b_j = \Sigma_{ij}\, T^{T}_{ji} a_i b_j = T^{T}\mathbf{a} \bullet \mathbf{b}$

as they say (with role of a and b reversed) ,

$$\mathbf{a} \cdot \mathbf{T}^{\mathrm{T}} \mathbf{b} = \mathbf{b} \cdot (\mathbf{T}^{\mathrm{T}})^{\mathrm{T}} \mathbf{a}. \qquad (2.11.5)$$

Dyadic Product of two vectors

Section 2.12 is on the **dyadic stuff**, pay special attention please! For me, a dyadic is a rank-2 tensor you obtain by taking the outer product of two rank-1 tensors, so for example

$W_{ij} = a_i b_j$

The dyadic people *want to write this* in the following notation

$W = \mathbf{ab}$

and since Lai bolds matrices, he writes this as

$\mathbf{W} = \mathbf{ab}$

So in Lai's book, you cannot tell by looking at an equation which bolded things are vectors and which are matrices. For example, he would also write $\mathbf{W} = \mathbf{AB}$ as the product of two matrices. For example

The two products are, in general, different. Thus, it is clear that in general $\mathbf{TS} \neq \mathbf{ST}$. That is, in general, the

He has never defined a vector. They seem to use lower case for vectors when both are present, but not always. He does note that people write $\mathbf{a} \otimes \mathbf{b}$ and this makes more sense to me, this is really what the outer or direct product is. So yes, we now have

$$[\mathbf{e}_1\mathbf{e}_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\mathbf{e}_1\mathbf{e}_2] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ldots . \qquad (2.12.6)$$

where sometimes people write $\mathbf{e}_1\mathbf{e}_1 = \hat{\mathbf{1}}\ \hat{\mathbf{1}}$.

I suppose the benefit of writing the dyadic **a**b is that you don't have to come up with another name for this matrix , like $\mathbf{W} = \mathbf{ab}$. Later we will see examples where $\mathbf{a} = \nabla$ and then $\nabla\mathbf{a}$ is a dyadic.

I would write

$$W = ab^T = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} (b_1\ b_2\ b_3\ ) \quad = \ldots$$

and they write this out for me:

$$[\mathbf{W}] = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} [b_1 \quad b_2 \quad b_3\,]. \qquad (2.12.5)$$

I must admit that my notation is then inconsistent. I really should write

$W = \mathbf{ab}^T$

Since my vectors are bolded, I know they are not matrices. But then I do have two adjacent bolded vectors with no operator between them. OK OK.

Notice this fact:

$\det(\mathbf{ab}) = \varepsilon_{ij}(\mathbf{ab})_{1i}(\mathbf{ab})_{2j} = \varepsilon_{ij}a_1b_ia_2b_j = a_1\,a_2\varepsilon_{ij}b_ib_j = 0$ by symmetry

$\mathrm{tr}(\mathbf{ab}) = a_ib_i = \mathbf{a} \bullet \mathbf{b}$

In their "inverse of a matrix" section, they fail to state the formula for A^{-1}! They are just saying that A^{-1} might exist, and identity I always exists.

Notice this dyadic rule:

$$(\mathbf{ab})\mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c}). \qquad (2.12.1)$$

I will prove it here: [I now write this as $(\mathbf{ab}^T)\mathbf{c} = \mathbf{a}\,(\mathbf{b}^T\mathbf{c}) = \mathbf{a}(\mathbf{b}\bullet\mathbf{c})$ so all obvious.]

$[(ab)\mathbf{c}]_i = (ab)_{ij}c_j$ = matrix acting on a vector $= a_ib_jc_j = a_i(\mathbf{b}\bullet\mathbf{c})$

Therefore $[(ab)\mathbf{c}]_i = a_i(\mathbf{b}\bullet\mathbf{c}) \Rightarrow (ab)\mathbf{c} = \mathbf{a}\,(\mathbf{b}\bullet\mathbf{c})$ or $W\mathbf{c} = \mathbf{a}\,(\mathbf{b}\bullet\mathbf{c})$ = a vector

Orthogonal matrices and similarity/congruence transformations

Then we have orthogonal tensors $Q^{-1} = Q^T$, example of a z rotation. Eventually we have

$$[\mathbf{T}]' = [\mathbf{Q}]^{\mathrm{T}}\,[\mathbf{T}]\,[\mathbf{Q}]. \qquad (2.18.5)$$

which I would write as

$T' = Q^T T Q$

The matrix elements of T' can then be written

$T'_{ij} = \langle e'_i | \mathbf{T} | e'_j \rangle$ where

$$\mathbf{e}'_i = Q_{mi}\mathbf{e}_m,$$

M&M call this $T' = Q^T T Q$ a "congruence transformation" so that T and T' are congruent. (M&M use an over-twiddle for transpose.). A similarity is $T' = Q^{-1} T Q$ and if Q is orthogonal, then congruence and similarity are the same. Lai uses none of these names.

Transformation of general tensors

Eventually we get to tensors of arbitrary rank:

$$\begin{aligned}
&\alpha' = \alpha && \text{zeroth-order tensor (or scalar),}\\
&a'_i = Q_{mi}a_m && \text{first-order tensor (or vector),}\\
&T'_{ij} = Q_{mi}Q_{nj}T_{mn} && \text{second-order tensor (or tensor),}\\
&S'_{ijk} = Q_{mi}Q_{nj}Q_{rk}S_{mnr} && \text{third-order tensor,}\\
&C'_{ijkl} = Q_{mi}Q_{nj}Q_{rk}Q_{sl}C_{mnrs} && \text{fourth-order tensor,}
\end{aligned} \qquad (2.19.1)$$

Q = my R

so things are tensors with respect to a linear transformation Q (which is the R of my tensor doc). Lai then shows how tensors "transform".

Well be careful. This book does not address the subject of contra and covariant vectors! If I assume that these down indices are really contravariant, as in my "developmental notation", then one would have

$a'_i = R_{ij}a_j$ or $\mathbf{a}' = R\mathbf{a}$ => $Q_{ij} = (R^T)_{ij}$ or $Q = R^T$

This seems a slightly strange way to do things.

The Dual Vector associated with an Asymmetric Matrix

Next new item is the dual vector in their Section 2.21. Consider

$T\,\mathbf{a} = \mathbf{t}^{dual} \times \mathbf{a}$

which is applied ONLY to an antisymmetric matrix T. *First of all*, I claim that you can represent any antisymmetric 3x3 matrix in this manner (such a matrix has only 3 distinct elements)

$T_{ij} = \varepsilon_{ijk}b_k$

To prove, this we need to find vector b. So

$T_{12} = b_3$
$T_{23} = b_1$
$T_{31} = b_2$ all done!

Then

$$b_i = (1/2)\,\varepsilon_{ijk}T_{jk}$$

$$b_1 = (1/2)\,\varepsilon_{123}T_{23} + (1/2)\,\varepsilon_{132}T_{32} = \varepsilon_{123}T_{23} = T_{23} \quad \text{etc.}$$

So now consider the claimed dual equation

$$T\,\mathbf{a} = \mathbf{t}^{dual} \times \mathbf{a}$$

This says

$$T_{ij}a_j = [\varepsilon_{ijk}b_k]\,a_j = [\mathbf{t}^{dual} \times \mathbf{a}]_i = \varepsilon_{ijk}\,\mathbf{t}^{dual}{}_j\,a_k$$

or

$$\varepsilon_{ijk}b_k\,a_j = \varepsilon_{ijk}\,t^d{}_j\,a_k = \varepsilon_{ikj}\,t^d{}_k\,a_j = -\,\varepsilon_{ijk}\,t^d{}_k\,a_j$$

so then

$$\varepsilon_{ijk}\,a_j\,b_k = -\,\varepsilon_{ijk}\,a_j\,t^d{}_k$$

and I conclude that

$$b_k = -\,t^d{}_k$$

and then

$$T_{12} = b_3 = -t^d{}_3 = -\,T_{21} \quad \Rightarrow \quad t^d{}_3 = T_{21}$$

which agrees with Lai. In general

$$t^d{}_i = -\,b_i = -\,(1/2)\,\varepsilon_{ijk}T_{jk}$$

$$2\,t^d{}_i = -\,\varepsilon_{ijk}T_{jk}$$

The vector version of this would be

$$2\,\mathbf{t}^d = \Sigma_i 2\,t^d{}_i\,\hat{\mathbf{i}} = \Sigma_i[-\,\varepsilon_{ijk}T_{jk}]\hat{\mathbf{i}} = -\,\varepsilon_{ijk}T_{jk}\hat{\mathbf{i}}$$

with implied summation on index i. Thus they say

$$2\mathbf{t}^{A} = -\varepsilon_{ijk} T_{jk} \mathbf{e}_i . \qquad (2.21.4)$$

Comments: The vector I called $\mathbf{t}^{dual}$ is associated with the antisymmetric matrix T and has nothing to do with the scratch vector **a** used in the work above. It seems very strange that the write t^{A} for this dual vector, since the superscript A seems to mean nothing at all. Maybe it means Antisymmetric!

I am not sure where this dual vector idea will be used.

Eigenvectors and Eigenvalues of a Real Symmetric Matrix

We move then to E^3 eigenvalue stuff. Lai writes

$$\mathbf{Tn} = \lambda \mathbf{n} = \lambda \mathbf{In}, \qquad (2.22.3)$$

$$(\mathbf{T} - \lambda \mathbf{I})\mathbf{n} = \mathbf{0} \quad \text{with} \quad \mathbf{n} \cdot \mathbf{n} = 1. \qquad (2.22.4)$$

where **n** are supposed to be orthonormal eigenvectors. In order for the second equation NOT to be invertible for **n** (which would give **n** = 0), we must have det(T-λ1) = 0 which is a cubic for λ and that is how you find the EV's. They call this the **characteristic equation**. They write α_i as the components of n instead of n_i, very strange.

For a **real symmetric matrix** (a Hermitian matrix), Lai quotes the famous results. EV's are real, EF's are in general orthogonal. The EF's are called "**principle directions**" . They are of course unit vectors each pointing in some direction. The EV's are $\mathbf{n}_1$ $\mathbf{n}_2$ and $\mathbf{n}_3$ and if you use those for new basis unit vectors, the new T will be diagonal, yes yes. [I think in the Lai application of the the EF/EV world, the eigenvectors will in fact be *directions in 3-space*, perhaps why he uses the letter **n.**]

Diagonal elements of T matrix bounded by eigenvalues

They then prove something I did not know: If you write down T_{ij} in every possible basis and examine the diagonal elements, you will find that $\max(T_{ii}) = \lambda_{max}$ and $\min(T_{ii}) = \lambda_{min}$. How would I prove this myself? I think any two bases must be related by a rotation since we are talking orthonormal basis, so we really have

$T = R^{-1} \Lambda R$ where Λ is $\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)$

Then look at a diagonal element

$$T_{ii} = [R^{-1} \Lambda R]_{ii} = \Sigma_{jk} \, R^{-1}{}_{ij} \, \Lambda_{jk} R_{ki} = \Sigma_{jk} \, R^{-1}{}_{ij} \, \delta_{jk} \, \lambda_k R_{ki} = R^{-1}{}_{ik} \, \lambda_k R_{ki}$$

Therefore

$$\Sigma_i T_{ii} = \Sigma_i \Sigma_k R_{ki} R^{-1}_{ik} \lambda_k = \Sigma_k \delta_{kk} \lambda_k = \Sigma_k \lambda_k$$

I have just shown that **trace is invariant under a rotatio**n which I guess I knew. so

$$T_{11} + T_{22} + T_{33} = \lambda_1 + \lambda_2 + \lambda_3$$

OK, I will ignore this for now and do it later if need be, they prove it on page 40. I showed above that

$$T_{ii} = \Sigma_k R^{-1}_{ik} \lambda_k R_{ki} = \Sigma_k \lambda_k (R_{ki})^2$$

If we write $\lambda_1 \geq \lambda_2 \geq \lambda_3$ then

$$T_{ii} = \Sigma_k \lambda_k (R_{ki})^2 \leq \lambda_1 \Sigma_k (R_{ki})^2$$

But now

$$\Sigma_k (R_{ki})^2 = \Sigma_k R_{ki} R_{ki} = \Sigma_k R^{-1}_{ik} R_{ki} = \delta_{ii} = 1$$

Therefore I get that, if λ_1 is the largest EV,

$$T_{ii} \leq \lambda_1 = \text{max of the } \lambda_i$$

Similarly,

$$T_{ii} = \Sigma_k \lambda_k (R_{ki})^2 \geq \lambda_3 \Sigma_k (R_{ki})^2$$

$$T_{ii} \geq \lambda_3 = \text{min of the } \lambda_i$$

<u>So I have proven that</u>

$$\min(\lambda_k) \leq T_{ii} \leq \max(\lambda_k) \qquad i = 1,2,3$$

which says that if we consider ALL matrices T obtained by rotating the diagonal one, the diagonal elements are all bounded as shown from above and below.

Three invariants which are functions of the three eigenvalues

Invariants: I have just shown above that the tr(T) is an invariant under rotations of T. I know that the determinant is also, since $\det(T) = \det(R^{-1}\Lambda R) = \det(\Lambda) = \lambda_1\lambda_2\lambda_3$. And $\text{tr}(T) = \lambda_1 + \lambda_2 + \lambda_3$. They quote a third invariant that is another linear combination. First we need

$$\text{tr}(T^2) = \text{Tr}(TT) = \text{Tr}(R^{-1}\Lambda R\, R^{-1}\Lambda R) = \text{Tr}(R^{-1}\Lambda^2 R) = \text{Tr}(R\ R^{-1}\Lambda^2) = \text{Tr}(\Lambda^2) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

$$(1/2)\,[\,\{\text{tr}(T)\}^2 - \text{tr}(T^2)\,\} = (1/2)\,[\,\{\lambda_1 + \lambda_2 + \lambda_3\}^2 - \{\lambda_1^2 + \lambda_2^2 + \lambda_3^2)\}\,\}$$

$= (1/2)\,[\,2\{\,\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1\}\,] = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$

and so now I have all three of these invariants computed:

$I_1 = \mathrm{tr}(T) = \lambda_1 + \lambda_2 + \lambda_3$

$I_2 = (1/2)\,[\,\{\mathrm{tr}(T)\}^2 - \mathrm{tr}(T^2)\,\} = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$

$I_3 = \det(T) = \lambda_1\lambda_2\lambda_3$

$$\begin{aligned} I_1 &= \lambda_1 + \lambda_2 + \lambda_3, \\ I_2 &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \\ I_3 &= \lambda_1\lambda_2\lambda_3. \end{aligned} \qquad (2.25.5)$$

I have no idea what these three combinations of EV's might be used for.

(My notes got a little out order here, but I will just plough ahead for now.)

Part C: Tensor Calculus

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This section on "tensor calculus" deals with various kinds of *derivatives* acting on scalars, vectors and matrices. No *integrals* appear.

There is stuff here I am not used to, so more notes! It involves time derivatives.

Facts involving time derivatives of an orthogonal matrix.

P 46: consider for the case that Q is an orthogonal matrix,

$0 = \partial_t 1 = \partial_t(QQ^T) = (\partial_t Q)Q^T + Q(\partial_t Q^T)$

$\Rightarrow \quad [(\partial_t Q)Q^T] = -Q(\partial_t Q^T) = -Q\,(\partial_t Q)^T = -[(\partial_t Q)Q^T]^T$

The first thing to note is that ∂_t and ^{T} commute:

$$[\partial_t Q^T]_{ij} \equiv \partial_t Q^T_{ij} \equiv [\partial_t Q_{ij}]^T$$

That is, $\partial_t A$ is must a matrix of derivatives of the A_{ij}. Above I have just proven this theorem:

Theorem 1: if $Q^T = Q^{-1}$ then $(\partial_t Q)Q^T$ is an antisymmetric matrix.

The Stakgold rigid body fact that dr/dt = ω x r

Now consider $\mathbf{r}(t) = R(t)\mathbf{r}_0$ where R is a rotation matrix which depends on time. Then I would say

$$\partial_t \mathbf{r} = (\partial_t R)\ \mathbf{r}_0 = (\partial_t R)R^{-1}\mathbf{r} = [(\partial_t R)R^T]\ \mathbf{r} \equiv T\ \mathbf{r}$$

Note added: Suppose $R = \exp(-i\theta\hat{\mathbf{n}}\bullet\mathbf{J}) = \exp(-i\boldsymbol{\Omega}\bullet\mathbf{J})$ where $\boldsymbol{\Omega} = \theta\hat{\mathbf{n}}$. Then

$$T = (\partial_t R)R^T = \exp(-i\boldsymbol{\Omega}\bullet\mathbf{J})\ (-i\partial_t\ \boldsymbol{\Omega}\bullet\mathbf{J})\ \exp(+i\boldsymbol{\Omega}\bullet\mathbf{J}) = -i(\partial_t\boldsymbol{\Omega})\bullet\mathbf{J} = -\,i(\partial_t\boldsymbol{\Omega})_a\, J_a$$

$$T_{ij} = -\,i(\partial_t\boldsymbol{\Omega})_a\,(J_a)_{ij} = -\,i(\partial_t\boldsymbol{\Omega})_a\ [-i\,\varepsilon_{aij}] = -\,(\partial_t\boldsymbol{\Omega})_a\,\varepsilon_{aij} = -\,\varepsilon_{ija}\,(\partial_t\boldsymbol{\Omega})_a = \text{antisym!}$$

Then given the equation below we get

$$t^d_i = -\,(1/2)\,\varepsilon_{ijk}T_{jk} = -\,(1/2)\,\varepsilon_{ijk}\{-\varepsilon_{jka}\,(\partial_t\boldsymbol{\Omega})_a\} = (1/2)\,\varepsilon_{ijk}\,\varepsilon_{jka}\,(\partial_t\boldsymbol{\Omega})_a$$

$$(1/2)\,\varepsilon_{ijk}\,\varepsilon_{ajk}\,(\partial_t\boldsymbol{\Omega})_a = (1/2)\,2\,\delta_{i,a}(\partial_t\boldsymbol{\Omega})_a = (\partial_t\boldsymbol{\Omega})_i = \boldsymbol{\omega}_i \quad \Rightarrow \quad \mathbf{t}^d = \boldsymbol{\omega}$$

So we can write the above as

$$d\mathbf{r}/dt = T\ \mathbf{r} = \mathbf{t}^d \times \mathbf{r} = \boldsymbol{\omega} \times \mathbf{r} = (d\boldsymbol{\Omega}/dt) \times \mathbf{r}$$

Thus $\partial_t\mathbf{r}$ is an antisymmetric matrix acting on **r**! But above I showed that when T is antisymmetric

$$T\ \mathbf{a} = \mathbf{t}^d \times \mathbf{a} \qquad \text{where} \qquad t^d_i = -\,(1/2)\,\varepsilon_{ijk}T_{jk}$$

Therefore we know that

$$\partial_t\mathbf{r} = \mathbf{t}^d \times \mathbf{r} \qquad \text{where} \qquad t^d_i = -\,(1/2)\,\varepsilon_{ijk}T_{jk} = -\,(1/2)\,\varepsilon_{ijk}[(\partial_t R)R^T]_{jk}$$

One says that t^d is the dual vector of the matrix $(\partial_t R)R^T$.

Now how do I make the connection that $\mathbf{t}^d = \boldsymbol{\omega}$? Here is my derivation from Goldstein notes, where I am using $d\boldsymbol{\Omega} \equiv \theta\hat{\mathbf{n}}$, and I use the usual $R = \exp(-i\theta\hat{\mathbf{n}}\bullet\mathbf{J})$ [correcting my Goldstein notes here only]

$$d\mathbf{r} = [-i\theta\hat{\mathbf{n}}\bullet\mathbf{J}]\ \mathbf{r} \quad \Rightarrow \quad \text{// } \theta \text{ is a small angle so doing } e^x = 1 + x$$

$$dr_j = [-i\theta \hat{\mathbf{n}}\bullet\mathbf{J}]_{jk}\, r_k = [-d\boldsymbol{\Omega}\bullet i\mathbf{J}]_{jk}\, r_k = [\, -d\Omega_m\, iJ_m]_{jk}\, r_k = -d\Omega_m\, (iJ_m)_{jk}\, r_k = -d\Omega_m\, \varepsilon_{mjk} r_k$$

$$= -\varepsilon_{jkm}\, r_k\, d\boldsymbol{\Omega}_m = -[\, \mathbf{r} \times \mathbf{d\Omega}]_j \;\Rightarrow\; d\mathbf{r} = -\mathbf{r} \times \mathbf{d\Omega} = \mathbf{d\Omega} \times \mathbf{r}$$

Therefore, dividing both sides by dt

$$d\mathbf{r}/dt = \mathbf{d\Omega}/dt \times \mathbf{r} = \boldsymbol{\omega} \times \mathbf{r} \qquad \boldsymbol{\omega} = \mathbf{d\Omega}/dt = (d\theta(t)/dt)\, \hat{\mathbf{n}}(\mathbf{t})$$

In this equation, then, **ω** is the instantaneous rotational status of a rigid body. It says what axis it is rotating around at that instant, and at what angular frequency ω. The Goldstein idea is that the most general motion of a rigid object with a point fixed is a rotation. You could take that fixed point as the center of mass, for example. So the notion of **ω** is specific to some selected fixed point.

Gradient of a scalar

Comment on gradient of a scalar:

$$d\varphi = \nabla\varphi \bullet d\mathbf{r} = |\nabla\varphi|\cos\theta\; dr$$

$$d\varphi_{max} = |\nabla\varphi|\, dr$$

So if you select d**r** normal to ∇φ , you get the max change in φ for a given scalar dr motion. I am quite familiar with this fact. For this direction, note that quantity $|\nabla\varphi|$ is what appears:

$$|\mathbf{v}|^2 = \mathbf{v}\bullet\mathbf{v} = \Sigma_i\, v_i^2$$
$$|\nabla\varphi|^2 = \nabla\varphi\bullet\nabla\varphi = \Sigma_i\, (\partial_i\varphi)^2 = (\partial_x\varphi)^2 + (\partial_y\varphi)^2 + (\partial_z\varphi)^2$$

$$|\nabla\varphi| = [\Sigma_i\, (\partial_i\varphi)^2]^{1/2}$$

A nice example is given for linear heat flow rule with temperature field specified where k = scalar thermal conductivity. Lai calls this the **Fourier Heat Conduction law**, $\mathbf{q} = -k\,\nabla\Theta$. This tells you that the heat flow is normal to the isotherms of temperature field Θ.

He then considers a non-isotropic medium where k becomes matrix K and $\mathbf{q} = -K\,\nabla\,\Theta$, and of course here in an example Lai shows that q is NOT normal to the isotherms.

Gradient of a vector, object ∇v, dyad but indices reversed.

Next comes the idea I noted above that we have a dyad derivative (but not quite!!)

$$\text{matrix} = \nabla\mathbf{v} \qquad \text{matrix}_{ij} = (\nabla\mathbf{v})_{ij} = \partial_j v_i = \partial v_i/\partial x_j$$

Notice the order of his indices which matches the longer partial derivative notation (and which is in my book the reverse of the usual dyadic ordering)

Later Lai claims that

$d\mathbf{v} = (\nabla\mathbf{v})\, d\mathbf{r}$ // proof on next line...; note also that $d\mathbf{v} = d(d\mathbf{x}/dt) = (\nabla\mathbf{v})\, d\mathbf{x} = D_t(d\mathbf{x})$

which says

$dv_i = (\nabla\mathbf{v})_{ij}\, dr_j = \partial_j v_i\, dx_j$ = the partial chain rule QED

Now consider this interesting claim:

$\nabla\bullet\, \mathbf{v} = \text{tr}(\nabla\mathbf{v})$

LHS $= \partial_x v_x + ... = \partial_j v_j$

RHS $= \partial_x v_x + ...$

I have never seen this $\nabla\mathbf{v}$ notation before, it looks just fine though.

Note Added: Here is why Lai treats $\nabla\mathbf{v}$ in this manner, first consider this "normal" statement:

$du_i = u_i(\mathbf{r} + d\mathbf{r}) - u_i(\mathbf{r}) = \partial_j u_i(r)\, dx_j = \nabla u_i \bullet d\mathbf{r}$ i = 1,2,3...n

We can get rid of the hanging index and make it all vector notation like so

$d\mathbf{u} = \mathbf{u}(\mathbf{r} + d\mathbf{r}) - \mathbf{u}(\mathbf{r}) = \partial_j u_i(r)\, dx_j = (\nabla\mathbf{u})_{ij}\, dx_j = (\nabla\mathbf{u})\, d\mathbf{r}$ = matrix acting on a vector

Notice that the "matrix index ordering" in $(\nabla\mathbf{u})_{ij}\, dx_j = T_{ij}b_j$ only works if we define matrix $(\nabla\mathbf{u})$the way Lai did. This structure du will appear many times in CM.

Note: I went off and spend about 4 days "learning about" this object $\nabla\mathbf{v}$ and how to express it in curvilinear coordinates. This resulted in a large new section 7 (w) in my tensor doc.

Divergence of a Matrix -- a vector formed by taking the divergence of the row vectors.

Next Lai wants to think about the divergence of a matrix! Again, something I have never considered. He wants it to be this:

$$\text{div}\, \mathbf{T} = \left(\partial T_{ij}/\partial x_j\right)\mathbf{e}_i.$$

This says that

$\nabla\bullet\, T = \Sigma_{ij}\, \partial_j T_{ij}\, \hat{\mathbf{i}} = \partial_j T_{1j}\, \hat{\mathbf{1}} + \partial_j T_{2j}\, \hat{\mathbf{2}} + \partial_j T_{3j}\, \hat{\mathbf{3}}$

Here, $\partial_j T_{1j}$ is the divergence of the vector which is the first row of matrix T. So you take these three divergences and create a vector from them. I have no idea where such a concept would be useful. Let's now **verify** his theorem that

$$(\text{div } \mathbf{T}) \cdot \mathbf{a} \equiv \text{div}\left(\mathbf{T}^{\mathrm{T}}\mathbf{a}\right) - \text{tr}\left(\mathbf{T}^{\mathrm{T}}\nabla\mathbf{a}\right).$$

LHS $= (\Sigma_{ij}\ \partial_j T_{ij}\ \hat{\mathbf{i}})\ \bullet\ \mathbf{a}\ =\ (\Sigma_{ij}\ \partial_j T_{ij})\ (\ \hat{\mathbf{i}} \bullet\ \mathbf{a})\ =\ (\Sigma_{ij}\ a_i\ \partial_j T_{ij})$

$\text{RHS}_1 = \text{div}(T^{\mathrm{T}}\mathbf{a})\ = \Sigma_i\ \partial_i\ (T^{\mathrm{T}}\mathbf{a})_i = \Sigma_{ij}\ \partial_i\ (T^{\mathrm{T}}{}_{ij} a_j)\ = \Sigma_{ij}\ \partial_i\ (T_{ji}\ a_j)\ = \Sigma_{ij}\ \partial_j\ (T_{ij}\ a_i)$

$=\ \Sigma_{ij}(\ \partial_j\ T_{ij})\ a_i\ + \Sigma_{ij}(\ \partial_j\ a_i)\ T_{ij}$

$\text{RHS}_2 = \text{tr}(T^{\mathrm{T}}\nabla a)\ = \Sigma_i\ (T^{\mathrm{T}}\nabla a)_{ii} = \Sigma_{ij}\ (T^{\mathrm{T}}{}_{ij}[\nabla a]_{ji})\ = \Sigma_{ij}\ T_{ji}(\partial_i a_j)$

$= \Sigma_{ij}\ T_{ij}(\partial_j a_i) =$ second term of RHS_1

Therefore

$\text{RHS}_1 - \text{RHS}_2\ = \Sigma_{ij}(\ \partial_j\ T_{ij})\ a_i = \text{LHS}$ QED

So I imagine that the more general definition can be used in curvilinear coordinates. We shall see.

Consider this little exercise:

$\text{div}\ (\alpha\ T)\ = \Sigma_{ij}\ \partial_j(\alpha T_{ij})\ \hat{\mathbf{i}}\ = \Sigma_{ij}\ (\partial_j\alpha) T_{ij}\ \hat{\mathbf{i}}\ + \Sigma_{ij}\ \alpha(\partial_j T_{ij})\ \hat{\mathbf{i}}\ = T1 + T2$

$T1 = \Sigma_{ij}\ (\partial_j\alpha) T_{ij}\ \hat{\mathbf{i}}\ = \Sigma_{ij}\ T_{ij}\ (\nabla\alpha)_j\ \hat{\mathbf{i}}\ = \Sigma_i\ [\ T\ \nabla\alpha\ ]_i\ \hat{\mathbf{i}}\ =\ T\ \nabla\alpha\ =$ matrix act on vector = vector

$T2 = \alpha\ \Sigma_{ij}\ (\partial_j T_{ij})\ \hat{\mathbf{i}}\ = \alpha\ \text{div}\ T$

so we have shown that

$\text{div}\ (\alpha\ T) = T\ \nabla\alpha + \alpha\ \text{div}\ T$

$$\text{div}\ (\alpha\mathbf{T}) = \mathbf{T}(\nabla\alpha) + \alpha\ \text{div}\ \mathbf{T}. \qquad (2.29.7)$$

Curl associated with a dual vector

Next: the curl is associated with that dual vector:

$[\text{curl v}\ ]_i\ = \varepsilon_{ijk}\ \partial_j v_k$ curl $\mathbf{v}\ = \nabla$ x $\mathbf{v}$

Recall that for antisymmetric T,

$\mathbf{T\,a} = \mathbf{t}^{\mathbf{d}} \times \mathbf{a}$ $\qquad 2\,t^{d}_{i} = -\varepsilon_{ijk}T_{jk}$

The claim is that if we decompose

$\nabla\mathbf{v} = (\nabla\mathbf{v})^{\mathbf{S}} + (\nabla\mathbf{v})^{\mathbf{A}}$

then curl $\mathbf{v} = 2\,\mathbf{t}^{\mathbf{d}}$ where $\mathbf{t}^{\mathbf{d}}$ goes with $(\nabla\mathbf{v})^{\mathbf{A}}$.

Proof: Start with (I happened to select the wrong sign here)

$T_{jk} \equiv (\nabla\mathbf{v})^{\mathbf{A}}_{jk} = (1/2)\,[\,\partial_j v_k - \partial_k v_j]$

Then

$2\,t^{d}_{i} = -\varepsilon_{ijk}T_{jk} = -\varepsilon_{ijk}\,(1/2)\,[\,\partial_j v_k - \partial_k v_j] = -\varepsilon_{ijk}\partial_j v_k = -\,[\text{curl } v\,]_i$

so I have shown that

$-\,[\text{curl } v\,]_i = 2\,t^{d}_{i}$

so we disagree by a sign. I think Lai has defined $(\nabla\mathbf{v})^{\mathbf{A}}_{jk}$ with the opposite sign, as in

$T_{jk} \equiv (\nabla\mathbf{v})^{\mathbf{A}}_{jk} = -\,(1/2)\,[\,\partial_j v_k - \partial_k v_j] = (1/2)[\,\partial v_j/\partial x_k - \partial v_k/\partial x_j\,]$

which is natural if you write out the partial derivative in full. Lai has failed to write his $(\nabla\mathbf{v})^{\mathbf{A}}_{jk}$ down except in a detailed matrix which I see does agree with what I just said.

Summary of the above: The curl of a vector **v** can be related to the object **∇v.** The relation is this:

curl $\mathbf{v} = 2\,\mathbf{t}^{\mathbf{d}}$ where $\mathbf{t}^{\mathbf{d}}$ is the dual vector associated
with the antisymmetric matrix $(\nabla\mathbf{v})^{\mathbf{A}}_{jk} \equiv (1/2)\,[\,\partial_k v_j - \partial_j v_k]$,
which is just the antisymmetric part of matrix **∇v .**
One then has $2\,t^{d}_{i} = -\varepsilon_{ijk}(\nabla\mathbf{v})^{\mathbf{A}}_{jk}$.

M&F use this concept as well. It is not clear to me what the payoff is for this unusual connection. I think later Lai will use this fact to compute curl **v** in curvilinear coordinates given **∇v** !

Comment: Here is an inconsistency in Lai notation. First, for dyad he says

$\mathbf{W} = \mathbf{ab},$ $\qquad \Rightarrow \qquad W_{ij} = a_i b_j, \qquad \Rightarrow \qquad (\mathbf{ab})_{ij} = a_i b_j$

From this you would conclude, setting **a** = **∇**,

$$\mathbf{W} = \nabla\mathbf{b} \quad \Rightarrow \quad W_{ij} = \nabla_i b_j = \partial_i b_j$$

But instead Lai has

$$(\nabla\mathbf{v})_{ij} = \mathbf{e}_i \cdot (\nabla\mathbf{v})\mathbf{e}_j = \mathbf{e}_i \cdot \frac{\partial \mathbf{v}}{\partial x_j} = \frac{\partial(\mathbf{v}\cdot\mathbf{e}_i)}{\partial x_j} = \frac{\partial v_i}{\partial x_j},$$

Admittedly by doing it this non-dyad way he gets the desired result

$$d\mathbf{v} = \mathbf{v}(\mathbf{r} + d\mathbf{r}) - \mathbf{v}(\mathbf{r}) = (\nabla\mathbf{v})d\mathbf{r}.$$

Part D: Curvilinear Coordinates

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Polar Coordinates: Lai obtains these results:

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta.$$

$$[\nabla\mathbf{v}] = \begin{bmatrix} \dfrac{\partial v_r}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_r}{\partial \theta} - v_\theta\right) \\ \dfrac{\partial v_\theta}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_\theta}{\partial \theta} + v_r\right) \end{bmatrix}.$$

$$\text{div } \mathbf{v} = \text{tr}(\nabla\mathbf{v}) = \frac{\partial v_r}{\partial r} + \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right).$$

$$\text{curl } \mathbf{v} = \left(\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)\mathbf{e}_3.$$

$$(\text{div}\,\mathbf{T})_r = \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{T_{rr} - T_{\theta\theta}}{r}.$$

n derive

$$(\text{div}\,\mathbf{T})_\theta = \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{T_{r\theta} + T_{\theta r}}{r}.$$

$$\nabla^2 f = \text{div}\,\nabla f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r}\frac{\partial f}{\partial r}.$$

$$(\nabla^2 \mathbf{v})_r = \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{v_r}{r^2},$$

$$(\nabla^2 \mathbf{v})_\theta = \frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2}.$$

I have not verified all these results, but could easily do so.

Comments: In my tensor doc, I never consider objects like (**∇v**), so let's consider that here.

$(\mathbf{\nabla v})_{ij} = \partial_j v_i = [\text{grad } v_i]_j$

I will try to copy paste and edit my grad section from tensor doc and apply it here:

STOP. OK, this is where I spent my 4 days. Since $\partial_j v_i$ is not a rank 2 tensor, you cannot just use the new general formulas I recently developed for expansion. The upshot of all this time is the new Section 7 (w) in my tensor doc where everything is explained including **∇v** as a non-tensor example. I got a nice expression for **∇v** in any curvilinear system, and got Maple going on it. All this work is now in a little folder called Lai grad-v work.

Cylindrical Coordinates: A similar set of results

Spherical Coordinates: A similar set of results.

Problems for Part D. 2.70 to 2.80