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3.1 Descriptions of Motions of a Continuum (69)

[70] In writing $\mathbf{x} = \mathbf{x}(\mathbf{X},t)$, one interpretation would be to think of $\mathbf{x}$ as the displacement of the "continuum particle" which at rest is located at position $\mathbf{X}$. I have seen such displacement called $\boldsymbol{\xi}(\mathbf{x},t)$ as for example in discussion of sound waves. This is probably a viable view for an "elastic solid" where you imagine

there is some "rest position". But maybe this view does not work for a lump of silly putty or a viscous fluid where the "rest position" is not so well defined. In this case, I think Lai wants to say **X** is the position the particle *had* at time t_0 , and at a later time t this particle has "flowed" to a new position **x**(**X**,t). Particles which start at other positions **X** in the fluid end up at their new positions too. This then is why we see

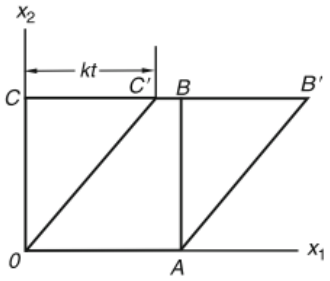
$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \quad \text{with} \quad \mathbf{X} = \mathbf{x}(\mathbf{X}, t_0), \tag{3.1.4}$$

and again, **X** is the position of the particle at earlier time t_0. The coordinates X are called **material coordinates**, whereas **x** seems to be the "position" and I would expect $\partial_t \mathbf{x} = \mathbf{v}$ to be the velocity of a chunk of material of some tiny dV volume.

So I will define a "**particle of a continuum**" as a tiny chunk of matter at some current position **x**.

Comment Added: (1) Displacement as **u** = **x**-**X** will be considered below; (2) the equation **x** = **x**(**X**,t) describes the **trajectory** of a continuum particle in time, it is just that simple. Such a particle is a tiny blob of material which, during its trajectory flight, may change density and perhaps may change other parameters. As a macro blob of material moves from time t_0 to t, the particle at **X**(t_0) moves to **x**(t) along some path. The microblobs are *labeled* by their starting position **X**, so we have **x**(t; **X**) .

Example 3.1.1 It took me a while to understand the picture. (axes are lower case x_1 and x_2)



At first I thought it was a 3D picture of a cube, but now I realize that the entire picture can be interpreted as the front face of the cube (in the x_1x_2 plane) which face starts as a square and "flows" into a parallelogram. So the square OABC flows into the 2-piped 0AB'C'. The example's "flow equation" or equation of "motion" is this [this equation gives the trajectory of every microblob!!]

x(**X**,t) = **X** +??? or $x_1 = X_1 + ktX_2, \quad x_2 = X_2, \quad x_3 = X_3.$

All microblobs flow only in the $\hat{\mathbf{e}}_1$ direction, and the amount of "displacement" is proportional to the X_2 coordinate. This is why the top edge of the square face (the "material line" BC) moves the most to the right, because it has the max X_2 coordinate. [I don't know how to write the above as a vector equation.]

Comment: The above example is an example of a possible equation form for **x**(**X**,t). As each particle moves according to this function, the macro object makes some corresponding motion, and in this case the macro square becomes a skewed macro 2-piped. Each micro particle simply moves to the right, but since they move at different speeds, you get the macro shape change.

3.2 Material Description and Spatial Description (72)

Now we get the very famous "two points of view". Here is one of those two:

I. Lagrangian = material = reference point of view

1. Following the particles, i.e., we express Θ, $\mathbf{v}$, $\mathbf{T}$ as functions of the particles [identified by the material coordinates (X_1, X_2, X_3)] and time t. In other words, we express

$$\begin{aligned} \Theta &= \hat{\Theta}(X_1, X_2, X_3, t), \\ \mathbf{v} &= \hat{\mathbf{v}}(X_1, X_2, X_3, t), \\ \mathbf{T} &= \hat{\mathbf{T}}(X_1, X_2, X_3, t). \end{aligned} \tag{3.2.1}$$

Such a description is known as the *material description*. Other names for it are the *Lagrangean description* and the *reference description*.

For example, the temperature here is that of the particle which started at position **X** at t_0 . Lai says "following the particles" because here our property like temperature is that of a specific volume of material as it moves along. This is the **Lagrangian view** (notice their odd spelling -ean). You could tag a chunk of fluid with some red dye containing temperature transmitters and this is what you would get as data.

Comment on Lagrangian: If we tag a microblob with red dye, we can have a thermometer embedded in the moving microblob to read the red microblob temperature. But *instead of this*, imagine we have a Tom temperature gun and we just point it at this red microblob as it moves and we read back the temperature of the red microblob. This is Lagrangian because we treat the microblob just the way we treat ordinary particles in Lagrangian dynamics as presented in Goldstein. In similar fashion, imagine a special velocity-reading radar gun we could point at the red microblob to read out its velocity and acceleration and any other property. All these readings of all such properties Q are of the nature $Q_i = \hat{Q}_i(t; \mathbf{X})$ where Q could be a scalar, vector, tensor, what have you. The HAT ^ indicates the functional form, hard to see above. Again, the **X** argument serves to label the microblob! It is the one that started at position **X** at time t_0.

II. Eulerian = spatial point of view.

Here is the other view where we have a temperature transmitter locked to a point in space and it reports out the temperature of whatever fluid particle is there at the moment. The **Eulerian** or spatial view.

2. Observing the changes at fixed locations, i.e., we express Θ, $\mathbf{v}$, $\mathbf{T}$ as functions of fixed position and time. Thus,

$$\begin{aligned} \Theta &= \tilde{\Theta}(x_1, x_2, x_3, t), \\ \mathbf{v} &= \tilde{\mathbf{v}}(x_1, x_2, x_3, t), \\ \mathbf{T} &= \tilde{\mathbf{T}}(x_1, x_2, x_3, t). \end{aligned} \tag{3.2.2}$$

Such a description is known as a *spatial description* or *Eulerian description*. The triple (x_1, x_2, x_3) locates the fixed position of points in the physical space and is known as the *spatial coordinates*. The spatial coordinates x_i of a particle at any time t are related to the material coordinates X_i of the particle by Eq.

I can see that these are very different functions say for temperature, totally different meanings, and good to get this clarified at Square Zero of the game.

Comments on Eulerian: Now we use our same Tom temperature gun and radar gun, BUT now we keep our gun pointed at a fixed point in space **x**. As we look at point **x**, we might see our red microblob flow by at some time, but lots of other microblobs come before and after the red one. So we are no longer measuring the temperature or velocity or acceleration of the same microblob (and thus there will be issues with time derivatives of these quantities!) This is totally different. This is why we write $Q_i = \tilde{Q}_i(t; \mathbf{x})$ where **x** is the point we are looking at, and $\tilde{Q}$ is the new functional form.

So what do we mean by saying $Q_i = \hat{Q}_i(t; \mathbf{X}) = \tilde{Q}_i(t; \mathbf{x})$? Well, *if* the red microblob which starts at location **X** at t_0 is at **x** at time t, then the two functions are equal. So what you really mean is

$$\hat{Q}_i(t; \mathbf{X}) = \tilde{Q}_i(t; \mathbf{x}(t; \mathbf{X})) \qquad \text{// with same } \mathbf{X} \text{ in both places.}$$

and of course you could do it the other way

$$\tilde{Q}_i(t; \mathbf{x}) = \hat{Q}_i(t; \mathbf{X}(t; \mathbf{x}))$$

where X(t;x) is the inverse trajectory of **x(t; X)**. But Lai does not use this.

In some machine or application, I could imagine that what you really want to know is the temperature or pressure at some point in space **x** and you don't care about where the microblob which is at **x** came from at some earlier time. So I suspect that the Eulerian view is more useful in general! Especially if the material is "uniform" in its composition, such as air or steel or water.

An example then shows how you can compute one form from the other. I think you have to make sure you always compute the velocity first as $v_i = \partial_t x_i$ and then transform.

Comment: Notice how the authors have *carefully* distinguished the two functional forms such that the Lagrangian one has a hat ^, while the Eulerian one has a ~ . I heartily approve.

High Level Comment: I did not see this in Lai, so I will say it here. You have a blob of material and it consists of 10^{24} particles. One physical description would be to give each particle a mass and position and velocity, but this gives way too many variables to deal with to characterize "a blob". So we look for some much smaller set of variables to describe the essential features of the blob. We promise not to worry about the nanoscale, only the microscale. Therefore even our microblob will have a large number of nanoscale particles in it so the microblob is then a statistical average system of some sort. Since the microblob is statistical, it is allowed to have thermodynamic properties like temperature. Our first simplification is to think of the blob as some large number of microblobs instead of some much larger number of nanoblobs. Each microblob has a position **x**(t). As the macroblob moves during our little video, every single microblob follows some trajectory, so we write **x**(t; **X**) where $\mathbf{X} = \mathbf{x}(t_0)$. We are thus assuming that our microblob does not "split up" say into 4 pieces as it might in an explosion, so we are assuming something generally smooth about the motion in our video. Similarly our microblob has other properties like temperature and pressure and density and velocity and acceleration. For all these properties, we have to

specify how we with to "parameterize" the property: Lagrangian or Eulerian. Our blob could be a solid, liquid or a gas or even a plasma I suppose. Certain properties not listed above are rank-2 tensors like various strain tensors and they too can be part of our description of the blob over time. It seems that for such a tensor, you still have to say whether you are thinking Lagrangian or Eulerian. The set of parameters we choose and the way we think of their measurement comprise the "kinematics" of blob motion which is the apt title of this section. In Lagrangian mechanics of regular particles, such kinematics might include Euler angles of a rigid body, for example. What variables to you want to use?

3.3 The Material Derivative. This is going to take some pondering! First they say

$$D\Theta/Dt = [\partial_t \hat{\Theta}(\mathbf{X}, t)]_{\mathtt{X=fixed}}$$

Note: D/Dt is nothing more than the usual total derivative d/dt, but Lai failed to say that.

Well, as you track a particle of fluid or solid as it moves, **X** is a constant (in time) vector which just says where this particle started from. **X** is just the microblob label. Thus, the only thing that DΘ/Dt picks up is the explicit time dependence from the second argument of $\hat{\Theta}(\mathbf{X}, t)$. So imagine we have a temperature-reporting dye dot that is moving with a blob of fluid which started at position **X**. During time dt, the particle moves some small distance d**x** *and* at the same time changes temperature. Perhaps we are in a fridge expansion chamber and we follow a little piece of Freon and as that piece moves to a lower pressure area, it also cools off. I think you could just write this as $d\hat{\Theta}/dt$. [yes]

Now it seems pretty clear that in the Eulerian case,

$$d_t\tilde{\Theta}(\mathbf{x(t)}, t) = \partial_t\tilde{\Theta}(\mathbf{x(t)}, t) + \partial\tilde{\Theta}/\partial x_i \, \partial x_i/\partial t$$
$$= \partial_t\tilde{\Theta}(\mathbf{x(t)}, t) + (\partial_i\tilde{\Theta})\, v_i = \partial_t\tilde{\Theta} + (\nabla\tilde{\Theta}) \bullet \mathbf{v}$$

where ∂_t only picks up explicit dependence from the time argument t, and ∇ relates to the spatial coordinates. Here d_t is the usual d/dt "total derivative". This last form works for any coordinate system if you install the correct gradient. [Recall that [grad f](**x**) • **B(x)** is explicitly treated in tensor doc!]

Now why do they want to call this D/Dt instead of d/dt? I have shown that

$$d\Theta/dt = \partial_t\hat{\Theta}(\mathbf{X}, t) = \partial_t\tilde{\Theta}(\mathbf{x},t) + \mathbf{v} \bullet \nabla\tilde{\Theta}(\mathbf{x},t) \qquad \mathbf{v} = \partial_t\mathbf{x}(t)$$

The main idea is that you are working with a tiny piece of volume which is flowing! Notice that the two objects here like $\partial_t\hat{\Theta}(\mathbf{X}, t)$ and $[\partial_t\tilde{\Theta}(\mathbf{x},t) + \mathbf{v} \bullet \nabla\tilde{\Theta}(\mathbf{x},t)]$ are always exactly equal so in that sense there is only "one object". But they are computed differently due to the different functional forms.

Question: Is there any distinction between the notation dΘ/dt and DΘ/Dt ? If not, then what is the purpose of introducing the capital letter D? Here is a quote from a Google book

Inviscid incompressible flow
By Jeffrey S. Marshall

It is common practice to drop the "hats" over the functions in expressions such as (3.1.6) and (3.1.7). It is clear that the material derivative is simply the usual total derivative of calculus, and in fact many fluid mechanics texts write equations such as (3.1.6) and (3.1.7) with the standard lowercase *d* notation often used for the total derivative in mathematics texts. The capital *D* notation is used for the material derivative in the current text (following Batchelor, 1967) to remind us that the function being differentiated depends on the current position **x** rather than the reference position $\boldsymbol{\xi}$ of the fluid particles.

The above book exactly reverses Lai's meaning of ~ and ^, and uses ξ in place of X. I think the author makes these two points: (1) Yes, D/Dt *really is the same* as the normal total derivative d/dt, as I have suspected. (2) The reason for using the D/Dt notation to indicate that the functional form of the function acted on is the Eulerian form and no ^ or ~ appears on the function. So really when you see Dg/Dt, you think of function g as being g = g(**x**,t) and not g(**X**,t). In this case. D/Dt always has "those two terms".

Comments:

(1) In the Eulerian view, think of our moving red microblob. If it is moving extremely slowly, then our Tom temperature gun is seeing mostly the $\partial_t\widetilde{\Theta}$ term which indicates temp change for the red microblob itself. In the limit of no motion, this is all there is. But consider another situation where every microblob retains its initial temperature magically, but there is lots of motion and different microblobs have different temperatures. Then $\partial_t\widetilde{\Theta} = 0$, but because of the different microblobs passing by, we do see the temperature varying at a fixed observation point **x**. This is the $v \bullet \nabla\widetilde{\Theta}$ term! For one dimension this term would be $v_x(\partial_x\widetilde{\Theta})$. Thus, if there is some spatial gradient in the $\widetilde{\Theta}(\mathbf{x})$ field at time instant t, then the flow causes the observing Tom temperature gun to get a non-zero reading even when each microblob retains its own initial starting temperature. In general, both terms contribute to $d_t\Theta$ in the Eulerian view. In the Lagrangian view, only the first term $\partial_t\hat{\Theta}$ ever contributes, there IS not other term.

(2) Suppose the property were a vector $\widetilde{\mathbf{p}}$ instead of a scalar $\widetilde{\Theta}$. We would say

$$d_t\widetilde{p}_j(\mathbf{x(t)}, t) = \partial_t\widetilde{p}_j(\mathbf{x(t)}, t) + \partial\widetilde{p}_j/\partial x_i \, \partial x_i/\partial t$$
$$= \partial_t\widetilde{p}_j(\mathbf{x(t)}, t) + (\partial_i\widetilde{p}_j)\, v_i = \partial_t\widetilde{p}_j + (\nabla\widetilde{\mathbf{p}})_{ji}\, v_i = \partial_t\widetilde{p}_j + [(\nabla\widetilde{\mathbf{p}})\mathbf{v}]_j$$

$$\Rightarrow d_t\widetilde{p}_j(\mathbf{x(t)}, t) = \partial_t\widetilde{p}_j(\mathbf{x(t)}, t) + (\nabla\widetilde{p}_j) \bullet \mathbf{v} \qquad \text{// one way to write it (as if } \widetilde{p}_j \text{ were a scalar)}$$

$$\Rightarrow d_t\widetilde{\mathbf{p}}(\mathbf{x(t)}, t) == \partial_t\widetilde{\mathbf{p}} + (\nabla\widetilde{\mathbf{p}})\mathbf{v} \qquad \text{// vector notation available, and no } \bullet \text{ !}$$

This then shows the *utility* of Lai's convention that $(\nabla\mathbf{a})_{ij} = \partial_i a_j$. It turns out this object has a complicated transformation rule going to x'-space (ie, getting itself expressed in curvilinear coordinates) because it is a non tensor! [I ended up studying the $(\nabla\mathbf{a})$ object for many days !!]

Comment: Here is yet another notation:

$$d_t\tilde{p}_j(\mathbf{x(t)}, t) = \partial_t\tilde{p}_j(\mathbf{x(t)}, t) + (\partial_i\tilde{p}_j)\, v_i = \partial_t\tilde{p}_j(\mathbf{x(t)}, t) + v_i\,(\partial_i\tilde{p}_j)$$

$$= \partial_t\tilde{p}_j(\mathbf{x(t)}, t) + (\mathbf{v} \bullet \nabla)\,\tilde{p}_j$$

so that

$$d_t\tilde{\mathbf{p}}(\mathbf{x(t)}, t) = \partial_t\tilde{\mathbf{p}}(\mathbf{x(t)}, t) + (\mathbf{v} \bullet \nabla)\,\tilde{\mathbf{p}}$$

This is the notation used on page 38 of my fluid dynamics book. We have

$$(\mathbf{v} \bullet \nabla) \equiv \Sigma_i v_i \partial_i$$

so that last term is a scalar differential operator acting on the vector $\tilde{\mathbf{p}}$. In contrast, in the Lai form where we have $d_t\tilde{\mathbf{p}} = \partial_t\tilde{\mathbf{p}} + (\nabla\tilde{\mathbf{p}})\mathbf{v}$ we have a regular matrix $\nabla\tilde{\mathbf{p}}$ acting on the vector $\mathbf{v}$.

Special case: suppose $\tilde{\mathbf{p}} = \mathbf{x} = \tilde{\mathbf{x}}(\mathbf{x},t)$. In this case, the notation is nonsense because a function cannot be a function of itself. For example, we might say z = z(x,y) as a function of two variables, but if we write then that x = x(x,y) we really mean x = x(y). Thus, in this case we have $\tilde{\mathbf{p}} = \mathbf{x} = \tilde{\mathbf{x}}(t)$ so that

$$d_t\tilde{\mathbf{x}}(t) = \partial_t\tilde{\mathbf{x}}(t)$$

There is no "convective contribution" in this case. To try to "get" a convective term, you would think of an example where each microblob has a fixed but different **x** (as we said with temperature), and then we get a reading as the blobs flow by. That works for temperature. A microblob can have a fixed temperature as it moves, but it cannot have a fixed **x** as it moves! Imagine two microblobs flow by and each has a different temperature, then you get a convective Δ temp reading. But each of these two microblobs has exactly the same **x** when it flows by so you can never get a Δ**x** reading! This is because you are observing at point **x**.

So the upshot is that the position vector is different from every other possible vector property in that there will be no convective term in $d_t\mathbf{x}$ in the Eulerian picture.

(3) Suppose the property were a rank-2 tensor $\tilde{p}_{jk}$? Then the above becomes

$$d_t\tilde{p}_{jk}(\mathbf{x(t)}, t) = \partial_t\tilde{p}_{jk}(\mathbf{x(t)}, t) + \partial\tilde{p}_{jk}/\partial x_i\, \partial x_i/\partial t$$

$$= \partial_t\tilde{p}_{jk}(\mathbf{x(t)}, t) + (\partial_i\tilde{p}_{jk})\, v_i = \partial_t\tilde{p}_j + \nabla\tilde{p}_{jk} \bullet \mathbf{v}$$

Lai has not given us any "fancy notation" so I guess we just leave it at that, and the same for any higher rank tensor property. [But I now know in principle HOW to express something like $\nabla\tilde{p}_{jk}$ in any curvilinear coordinates!]

3.4 Acceleration of a Particle (76)

Velocity. Lai has sort of skipped over this and did temperature instead, then suddenly they go to acceleration. So I have to ask: what is the meaning of "velocity" ? If you want to know the velocity of the "particle" of fluid, it must be $\mathbf{v} = \partial_t \mathbf{x}(t)$ where $\mathbf{x}(t) = \mathbf{x}(\mathbf{X},t)$. Since $\mathbf{X}$ is a constant, there is no issue of any other term in the velocity. This would also be the average velocity of "atoms" in the fluid.

Now the position vector does not have "two views". That is, we don't have

$\mathbf{x} = \hat{\mathbf{x}}(\mathbf{X},t)$ // position of a particle that starts at $\mathbf{X}$

$\mathbf{x} = \tilde{\mathbf{x}}(\mathbf{x},t)$ // position of the fluid at position $\mathbf{x}$?

This second line seems to say nothing other than the identity $\mathbf{x} = \mathbf{x}$. The position of the fluid if you look at position $\mathbf{x}$ is going to be $\mathbf{x}$ no matter what. So I think for this particular vector, there is no twiddle functional form. Therefore, we can regard these two things as the same thing: $\hat{\mathbf{x}}(\mathbf{X},t)$ and $\mathbf{x}(\mathbf{X},t)$.

Now what about velocity? Since we have only the first form, we write

$$\mathbf{v} = d\mathbf{x}/dt = d\hat{\mathbf{x}}(\mathbf{X},t)/dt = \partial_t \hat{\mathbf{x}} = \partial_t \mathbf{x}(t)$$

and we have two other ways to write this same object

$$d\mathbf{x}/dt = d_t \mathbf{x} = D/Dt\,(\mathbf{x}).$$

The velocity of the fluid viewed at fixed point $\mathbf{x}$ is the same as the velocity of the red-dyed blob which is at position $\mathbf{x}$ at that time. So we can write it we want,

$$\mathbf{v} = d\mathbf{x}/dt = D\mathbf{x}/Dt = \hat{\mathbf{v}}(\mathbf{X},t) = \tilde{\mathbf{v}}(\mathbf{x},t)$$

That is to say, there *are* two different functional forms for the velocity (just not for x!)

Acceleration. [This is just an application of our general vector formula (2) above.]

Now apply d/dt here and what happens?

$$d\mathbf{v}/dt = d\hat{\mathbf{v}}(\mathbf{X},t)/dt = \partial_t \hat{\mathbf{v}}(\mathbf{X},t)$$

$$d\mathbf{v}_i/dt = d\tilde{\mathbf{v}}_i(\mathbf{x},t)/dt = \partial_t \tilde{\mathbf{v}}_i(\mathbf{x},t) + \partial \tilde{\mathbf{v}}_i(\mathbf{x},t)/\partial x_j \; \partial x_j/\partial t$$

$$d\mathbf{v}_i/dt = d\tilde{\mathbf{v}}_i(\mathbf{x},t)/dt = \partial_t \tilde{\mathbf{v}}_i(\mathbf{x},t) + \partial_j \tilde{\mathbf{v}}_i(\mathbf{x},t)\; v_j$$

$$d\mathbf{v}_i/dt = d\tilde{\mathbf{v}}_i(\mathbf{x},t)/dt = \partial_t \tilde{\mathbf{v}}_i(\mathbf{x},t) + (\nabla\tilde{\mathbf{v}})_{ij}\, v_j$$

$$d\mathbf{v}/dt = d\tilde{\mathbf{v}}(\mathbf{x},t)/dt = \partial_t \tilde{\mathbf{v}}(\mathbf{x},t) + (\nabla\tilde{\mathbf{v}})\mathbf{v}$$

So we get two forms here

$$d\mathbf{v}/dt = \partial_t \hat{\mathbf{v}}(\mathbf{X},t)$$

$$d\mathbf{v}/dt = \partial_t \tilde{\mathbf{v}}(\mathbf{x},t) + (\nabla\tilde{\mathbf{v}}(\mathbf{x},t))\; \tilde{\mathbf{v}}(\mathbf{x},t)$$

Lai reports this out as

$$\mathbf{a} = \left(\frac{\partial \mathbf{v}}{\partial t}\right)_{X_i-\text{fixed}} \equiv \frac{D\mathbf{v}}{Dt}. \qquad (3.4.3)$$

$$\mathbf{a} = \frac{\partial \boldsymbol{\nu}}{\partial t} + (\nabla \boldsymbol{\nu})v. \qquad (3.4.8)^*$$

For some strange reason they use ν for v, and left off the bold on the last v. Maybe the ν thing is a new way to distinguish the two functional forms. $\mathbf{v} = \hat{\mathbf{v}}(\mathbf{X},t)$ and $\mathbf{v} = \tilde{\mathbf{v}}(\mathbf{x},t)$. This is consistent with the two lines shown above. [Maple also displays v's as nu's. TNR says vee = v and nu = ν , almost identical. This is a subject with web discussion, some fonts make them different.]

Comment: $\mathbf{a} = d\mathbf{v}/dt$ is later going to appear in some form of $\mathbf{F} = m\mathbf{a}$, Newton's Law for a microblob, so you know that this $\mathbf{a} = \partial_t\mathbf{v} + (\nabla v)\mathbf{v}$ is going to be important, and so (∇v) will play an important role in what lies ahead. So I am glad I spent time understanding it and getting it into other coordinates.

Now a very interesting **footnote** regarding the last equation above,

*In dyadic notation, the preceding equation is written as $\mathbf{a} = \frac{\partial \boldsymbol{\nu}}{\partial \mathbf{t}} + \boldsymbol{\nu} \cdot (\bar{\nabla}\boldsymbol{\nu})$, where $\bar{\nabla} = (\mathbf{e}_m \partial/\partial x_m)$.

I would interpret this second term as:

$[\mathbf{w} \bullet (\bar{\nabla}\mathbf{v})]_j = [\mathbf{w}^T (\bar{\nabla}\mathbf{v})]_j = \Sigma_i w_i (\bar{\nabla}\mathbf{v})_{ij} = \Sigma_i w_i \partial_i v_j$ // regular dyad order!!!!

whereas

$[(\nabla\mathbf{v})\mathbf{w}]_j = (\nabla\mathbf{v})_{ji} w_i = \Sigma_i \partial_i v_j w_i$ = the same as above. // Lai reverse dyad order!!!

So here Lai is using $\bar{\nabla}\mathbf{v}$ to indicate the "normal dyad order object" and $\nabla\mathbf{v}$ to be the "reverse order dyad object". So here are four different ways to write the same equation:

$d_t\tilde{\mathbf{p}} = \partial_t\tilde{\mathbf{p}} + \mathbf{v} \bullet (\bar{\nabla}\tilde{\mathbf{p}})$	$(\bar{\nabla}\tilde{\mathbf{p}})_{ij} = \partial_i p_j$	// dyad notation
$d_t\tilde{\mathbf{p}} = \partial_t\tilde{\mathbf{p}} + \mathbf{v}^T(\bar{\nabla}\tilde{\mathbf{p}})$	$(\bar{\nabla}\tilde{\mathbf{p}})_{ij} = \partial_i p_j$	// transpose notation
$d_t\tilde{\mathbf{p}} = \partial_t\tilde{\mathbf{p}} + (\nabla\tilde{\mathbf{p}})\mathbf{v}$	$(\nabla\tilde{\mathbf{p}})_{ij} = \partial_j p_i$	// Lai notation
$d_t\tilde{\mathbf{p}} = \partial_t\tilde{\mathbf{p}} + (\mathbf{v}\bullet\nabla)\tilde{\mathbf{p}}$		// Schaum notation

You would think that the Schaum form was the best since it shows action on the vector $\tilde{\mathbf{p}}$ as in the other two terms of the equation, so there must be some good reason for the Lai form. I have been through the curvilinear issues with his form. The last form would be

$v_i \partial_i \tilde{\mathbf{p}}_j$

which is different from my tensor doc $v_i(\partial_i \varphi)$ which is a true scalar object. So maybe it is just easier to do the curvilinear business in the Lai form.

What about the little rule that $d\mathbf{v} = (\nabla\mathbf{v})d\mathbf{r}$ which I know is valid and reasonable. How would this appear in the other notations?

$d\mathbf{v} = (\nabla\mathbf{v}(\mathbf{r}))d\mathbf{r}$ => $dv_i = (\nabla\mathbf{v})_{ij}\, dr_j = (\partial_j v_i)\, dr_j = dr_j(\partial_j v_i) = (d\mathbf{r} \bullet \nabla)\, v_i$

=> $d\mathbf{v} = (d\mathbf{r} \bullet \nabla)\, \mathbf{v}$ // Schaum

$d\mathbf{v} = (\nabla\mathbf{v}) \bullet d\mathbf{r}$ // dyadic dot here

Here the Schaum notation does not put $d\mathbf{r}$ on the right where I think we might want it to be.

Now here is another *powerful* use of the Lai notation:

$$d\mathbf{f} = \mathbf{f}(\mathbf{x} + d\mathbf{x}) - \mathbf{f}(\mathbf{x}) = (\nabla\mathbf{f})d\mathbf{x}$$

This is just a generalization of the previous $d\mathbf{v} = (\nabla\mathbf{v})d\mathbf{r}$ to an arbitrary vector field f. Proof:

$df_i = f_i(\mathbf{x} + d\mathbf{x}) - f_i(\mathbf{x}) = [(\nabla\mathbf{f})d\mathbf{x}]_i = (\nabla\mathbf{f})_{ij}\, dx_j = (\partial_j f_i)\, dx_j = (\partial f_i/\partial_{x_j})\, dx_j$ = chain rule

I think this is the real motivation for using this object $\nabla\mathbf{f}$ with the Lai definition.

Notice in the above 6" that we are talking about functions like $\mathbf{v}(\mathbf{r})$ that are not functions of time.

We then have a Cartesian, cylindrical, then spherical coordinates example of acceleration. Various other good examples I may peruse later.

Note added 6.25.12. Lai omits to state the general notion of the material derivative, so I will state it right here. Let p be some vector quantity and I leave off the ~ topping since we assume everywhere that objects are functions of current position, not starting position. So we have these two ways to write things.

$D_t\mathbf{p} = \partial_t\mathbf{p} + (\nabla\mathbf{p})\mathbf{v}$ $\quad (\nabla\mathbf{p})_{ij} = \partial_j p_i$ // Lai notation

$D_t\mathbf{p} = \partial_t\mathbf{p} + (\mathbf{v}\bullet\nabla)\mathbf{p}$ // Schaum notation

The operator form is this

$D_t = \partial_t + (\mathbf{v}\bullet\nabla)$ // convective derivative

I cannot find any source which writes it using $(\nabla\mathbf{p})$ notation other than Lai (in this convective derivative context). But I think I would find $(\nabla\mathbf{u})$ and $(\nabla\mathbf{v})$. I just added a comment on this to tensor doc, and released both PDF's just now with today;s date and emailed the one guy who might read them.

3.5 Displacement Field -- totally clear

$$\mathbf{u}(\mathbf{X},t) = \mathbf{x}(\mathbf{X},t) - \mathbf{X}. \qquad (3.5.1)$$

3.6 Rigid body motion. Again, totally clear.

3.7 Infinitesimal Deformation (p 84)

There is some trickiness going on here. We have a clean displacement definition,

$\mathbf{u}(\mathbf{X},t) \equiv \mathbf{x}(\mathbf{X},t) - \mathbf{X}$ (3.7.1)

$\mathbf{x}(\mathbf{X},t) = \mathbf{u}(\mathbf{X},t) + \mathbf{X}$

Now we think of d**x** and d**X** as <u>small isochronous flow-probe-vectors</u> so that

$\mathbf{x} + d\mathbf{x} = \mathbf{u}(\mathbf{X}+d\mathbf{X},t) + \mathbf{X} + d\mathbf{X}$

$\mathbf{u}(\mathbf{X},t) + \mathbf{X} + d\mathbf{x} = \mathbf{u}(\mathbf{X}+d\mathbf{X},t) + \mathbf{X} + d\mathbf{X}$

$d\mathbf{x} - d\mathbf{X} = \mathbf{u}(\mathbf{X}+d\mathbf{X},t) - \mathbf{u}(\mathbf{X},t) = (\nabla\mathbf{u})\, d\mathbf{X}$ $\quad \nabla$ means $\nabla^{(\mathbf{X})}$

$d\mathbf{x} = d\mathbf{X} + (\nabla\mathbf{u})\, d\mathbf{X}$ (3.7.4)

We are not talking total differentials here, we are talking short vector mapping. If we have some dX resulting in some dx, then I would claim that

$d\mathbf{u} = d\mathbf{x} - d\mathbf{X}$ (PL1)

would be the way the little short vector changed. If this is correct, then I claim we have

$d\mathbf{u} = (\nabla\mathbf{u})\, d\mathbf{X} \qquad \mathbf{u} = \mathbf{u}(\mathbf{X},t)$ (PL2)

as our short vector probe transformation rule. We also can write this to describe how vector dX maps into d**x**

$d\mathbf{x} = (\nabla\mathbf{x})\, d\mathbf{X} \qquad \mathbf{x} = \mathbf{x}(\mathbf{X},t)$ (PL3)

None of these last three equations is written in the text on page 84 but I think they are all correct.

Now insert (PL3) into (3.7.4) to get

$(\nabla \mathbf{x})\, d\mathbf{X} = d\mathbf{X} + (\nabla \mathbf{u})\, d\mathbf{X}$

Since this is valid for any dX, it must be that

$(\nabla \mathbf{x}) = 1 + (\nabla \mathbf{u})$ (PL4)

This last result also follows directly from (remember that ∂_i is $\partial/\partial X_i$)

$\mathbf{x}(\mathbf{X},t) = \mathbf{u}(\mathbf{X},t) + \mathbf{X}$

$(\nabla \mathbf{x}) = (\nabla \mathbf{u}) + 1$ (PL4)

Now suppose we make this definition

$F \equiv (\nabla \mathbf{x})$ (PL5)

Then we get

$F = (\nabla \mathbf{u}) + 1$ (3.7.7)

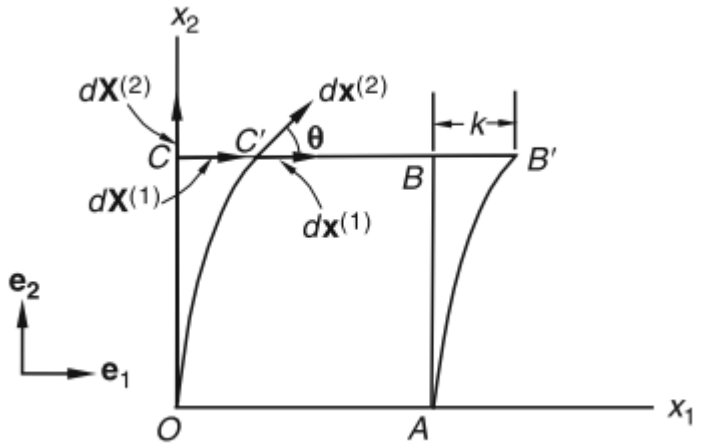
$d\mathbf{x} = (\nabla \mathbf{x})\, d\mathbf{X} = F\, d\mathbf{X}$ (3.7.6)

$d\mathbf{u} = (\nabla \mathbf{u})\, d\mathbf{X}$ (PL2)

OK, the page 86 equation text is now also happy.

Example 3.7.1 which assumes $u_1 = k X_2^2$ and $u_2 = u_3 = 0$. In this example, Lai suddenly starts using this kind of new notation: [I used $d\mathbf{x}^{(n)} = \mathbf{e}_n\, dx'^n$ in tensor doc, a similar idea.]

$d\mathbf{X}^{(1)} \equiv dX_1\, \hat{\mathbf{e}}_1$ = " a probe value of d**X** lined up with the x_1 axis" = "a short vector"



Material line CB at t_0 is at $X_2 = 1$ in this picture, so $\mathbf{u} = (k,0,0)$ for every point on this line, so the entire line is displaced without stretching from CB to C'B'. The differential vector $d\mathbf{X}^{(1)}$ points from point C to a point slightly to its right (much closer to C than to C'), but the arrow is exaggerated in length so you can see it. This arrow marks two closely spaced points on the CB material line. We then use the formula

$$d\mathbf{x} = d\mathbf{X} + (\nabla \mathbf{u}) d\mathbf{X}, \qquad (3.7.4)$$

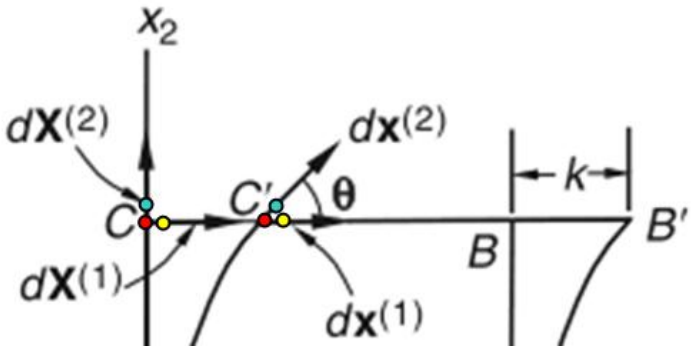
to compute $d\mathbf{x}$ and the computation gives

$d\mathbf{x}^{(1)} = dX_2\, \mathbf{e}_1$ for $d\mathbf{X}^{(1)}$

$d\mathbf{x}^{(2)} = dX_2(\, 2k\mathbf{e}_1 + \mathbf{e}_2)$ for $d\mathbf{X}^{(2)}$

The first line tells us that if we start with two points at t_0, one to the right of the other, we end up with one to the right of the other at time t, and in fact they remain the same distance apart. This is consistent with the idea that the top material line simply translates.

The second line tells us that if we start with one point *above* the other, they do NOT end up one point above the other, but rather as shown in the picture. Here are some marker points to show the above:



All this is staying is that our square is being bent as shown and this is what our function for u results in. I went through this example on second reading and did all the details, and right off the bat we see one of those $\nabla\mathbf{u}$ matrices as shown in the equation above. In the picture above, the material at point C has been "bent" by an angle $\gamma = \pi/2 - \theta$, and we are going to learn more about such angles later. Again, a "particle" that starts out square at time t_0 has been "deformed" into a parallelogram shape at time t.

OK, p 86 and now lots of action: { recall $\mathbf{u} \equiv \mathbf{x} - \mathbf{X}$, $d\mathbf{u} = (\nabla\mathbf{u})\, d\mathbf{X}$, $d\mathbf{x} = (\nabla\mathbf{x})\, d\mathbf{X}$, $F = (\nabla\mathbf{x})$ etc }

$d\mathbf{x} = F\, d\mathbf{X}$ $\qquad F = 1 + (\nabla\mathbf{u})$ = the deformation gradient tensor

$ds^2 = dx \bullet dx = (F\, d\mathbf{X}) \bullet (F\, d\mathbf{X}) = d\mathbf{X} \bullet F^{\mathbf{T}} F\, d\mathbf{X} = d\mathbf{X} \bullet C d\mathbf{X} = C_{ij} dX_i dX_j$

$C = F^{\mathbf{T}}F$ = right Cauchy-Green deformation tensor $= (1 + (\nabla\mathbf{u}))^{\mathbf{T}}(\, 1 + (\nabla\mathbf{u}))$ = 4 terms

$C = 1 + (\nabla\mathbf{u}) + (\nabla\mathbf{u})^{\mathbf{T}} + (\nabla\mathbf{u})^{\mathbf{T}} (\nabla\mathbf{u}) = 1 + 2\, E^*$

E* = Lagrange strain tensor

If changes in u are very small, then approximate as

$C \approx 1 + (\nabla \mathbf{u}) + (\nabla \mathbf{u})^{\mathbf{T}} = 1 + 2\,E$

E = infinitesimal strain tensor

Back around pp 57 we wrote out ($\nabla \mathbf{u}$) in various coordinate systems, so it is a mechanical thing then to write out E in these same systems using the above. For example, in cylindricals we found earlier that

$$[\nabla \mathbf{v}] = \begin{bmatrix} \dfrac{\partial v_r}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_r}{\partial \theta} - v_\theta\right) & \dfrac{\partial v_r}{\partial z} \\ \dfrac{\partial v_\theta}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_\theta}{\partial \theta} + v_r\right) & \dfrac{\partial v_\theta}{\partial z} \\ \dfrac{\partial v_z}{\partial r} & \dfrac{1}{r}\dfrac{\partial v_z}{\partial \theta} & \dfrac{\partial v_z}{\partial z} \end{bmatrix}.$$

and then $E = [(\nabla \mathbf{u}) + (\nabla \mathbf{u})^{\mathbf{T}}]/2$ so

$$[\mathbf{E}] = \begin{bmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{1}{2}\left(\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} - \dfrac{u_\theta}{r} + \dfrac{\partial u_\theta}{\partial r}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_r}{\partial z} + \dfrac{\partial u_z}{\partial r}\right) \\ \dfrac{1}{2}\left(\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} - \dfrac{u_\theta}{r} + \dfrac{\partial u_\theta}{\partial r}\right) & \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta} + \dfrac{u_r}{r} & \dfrac{1}{2}\left(\dfrac{\partial u_\theta}{\partial z} + \dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_r}{\partial z} + \dfrac{\partial u_z}{\partial r}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_\theta}{\partial z} + \dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta}\right) & \dfrac{\partial u_z}{\partial z} \end{bmatrix}. \qquad (3.7.20)$$

Connection #1 to tensor doc. Consider **X = X(x,t)** as the inverse of our usual transformation **x** = **x**(**X**,t). This inverse transformation maps into our tensor doc language if we just take **X**→ **x'** and **x** → **x**. The coordinates x_i can be thought of as our normal Cartesian space coordinate x_i with metric tensor g = 1. Let's refer to this inverse transformation as the forward transformation of tensor doc which we will call $\mathcal{F}$ and reserve letter F for the way Lai defines F. So consider:

$d\mathbf{x} = F\, d\mathbf{X}$ // Lai
$d\mathbf{x} = S\, d\mathbf{x}'$ // tensor doc

Therefore, we identify Lai's matrix F with my matrix S. Next, above we see

$dS^2 = C_{ij} dX_i dX_j$ // lai
$dS^2 = g'_{ij} dx'_i dx'_j$ // tensor doc

Therefore, I now identify this right Cauchy-Green with my metric tensor g'_{ij}. Moreover

$C = F^T F$ // Lai
$g' = S^T S$ // tensor doc

Connection #2 to tensor doc. Now consider $\mathbf{x} = \mathbf{x}(\mathbf{X},t)$ as the tensor doc transformation, so this time we have some transformation I will call F and the mapping is now $\mathbf{X} \rightarrow \mathbf{x}$ and $\mathbf{x} \rightarrow \mathbf{x}'$. Going in this direction we imagine that X-space has g = 1. Then

$d\mathbf{x} = F\, d\mathbf{X}$ // Lai
$d\mathbf{x}' = R\, d\mathbf{x}$ // tensor doc

So in this direction, we associate the same Lai F with my matrix R. He then defines the left Cauchy as

$B = FF^T$ // Lai
$g' = RR^T$ // tensor doc

So although he never computed ds^2, I know that B is the metric tensor for this direction transformation.

$ds^2 = B_{ij} dx_i dx_j$
$ds^2 = g'_{ij} dx'_i dx'_j$ // tensor doc

So I can interpret Lai's flows as transformations of the tensor doc form, and I can then identify the pieces as just outlined. I did not grok this until today 2.16.12. Lai says nothing about this stuff.

I just added the above stuff as Section 5 (o) in tensor doc, I like it.

3.8 (p 88) Meaning of E.

Consider for any two items [E is the infinitesimal Lagrange strain tensor]

$$d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} = d\mathbf{X}^{(1)} \cdot d\mathbf{X}^{(2)} + 2d\mathbf{X}^{(1)} \cdot \mathbf{E} d\mathbf{X}^{(2)}. \qquad \text{ok} \qquad (3.7.18)$$

If they are the same we get

$d\mathbf{x} \bullet d\mathbf{x} = d\mathbf{X} \bullet d\mathbf{X} + 2\, d\mathbf{X} \bullet E\, d\mathbf{X}$

Now let d**X** = dS **n** so we get

d**x** • d**x** = d**X** • d**X** + 2 (dS **n)** • E (dS **n)**

$ds^2 = dS^2 + 2\, dS^2\, \mathbf{n} \bullet E\mathbf{n}$ ds^2 means $(ds)^2$

I then am fine to this point

$$\frac{ds - dS}{dS} \text{ ok} = \mathbf{n} \cdot \mathbf{En} = E_{nn} \text{ (no sum on } n\text{).} \qquad (3.8.2)$$

where E_{nn} I think is appropriate if we thing of **n** = **1**,**2** or **3** and not general **n .**

[I just ordered this book from Phatcampus, was $7 new + $4 shipping.]

I grok now the "meaning of E_{ij}", see the text pp 88-89.

Comment: These tensors like E are **specific to a particular "flow"** as described by **x** = **x**(**X**,t). The flow can be finite, for then a differential d**X** will still map into a differential d**x** (flow must be bounded). In particular, the tensors (matrices) like E are specific to the start and end of the finite flow. After some specific finite flow, E_{11} is the unit elongation in the x direction of the flowing matter, and $2E_{12}$ is the angle by which the right angle between x and y of the material has deformed. The particular matrix E applies to flows which can be finite, but in which things only change by a small amount. The deformations of the material are called "strains". Nothing is said about what might cause such a flow or strain. This is still just "kinematics".

Comment: Lai intends this notation to be valid for any unit vector n :

n • E**n** = E_{nn} which we could write as $\mathbf{n}^T E\mathbf{n}$ or $\langle \mathbf{n} | E | \mathbf{n} \rangle$

How do we distinguish E_{nn} as shown above from matrix element E_{nn} ??? This gets to the issue of E being an operator in Hilbert Space versus a matrix, and this is why I think Lai bolds the operator E. But he really has no notation to show coordinate system. For example, he writes [**∇u**] to indicate this matrix in any coordinate system, but I would like to say $(\nabla \mathbf{u})^{r\theta\varphi}$ to indicate the matrix in sphericals, say.

So my tensor doc is weak on this point. // I have added a section on this subject.

Comment: In tensor doc, I introduce the 2 kinds of vectors with dx_i being any coordinate system. I put subscripts on my rank-2 tensors in section 5 (e), but these are not specifically Cartesian as Picture A shows. When I write the matrix form like M' = R M R^T , this is just matrix multiplication and objects R_{ab} appear, but still not associated with Cartesian. I don't think I ever discuss "basis" for a matrix anywhere. I just represent a rank-2 tensor as some M_{ij} or g_{ij} . What then happens in section 7 (w) ? There is nothing Cartesian about my direct product. // I think this is all now discussed in Appendix E.

So one might disambiguate Lai by saying: (superscript indicates the basis)

$\mathbf{n}_i \bullet E\mathbf{n}_j = E^{(n)}{}_{ij}$

and then the Cartesian choice would have

$\mathbf{u}_i \bullet E\mathbf{u}_j = E_{ij}$

3.9 Principle Strain (93)

When you go to a basis which diagonalizes E (infinitesimal strain tensor), the diagonal elements are called the principle strains, giving unit stretch in each basis direction.

Now we have an expansion of the characteristic equation in powers of λ, and the coefficients are given special names I_n . I will try now to verify these. I try this method

$$0 = \det(E-\lambda) = \varepsilon_{ijk}(E_{1i}-\lambda\delta_{1i})\,(E_{2j}-\lambda\delta_{2j})\,(E_{3k}-\lambda\delta_{3k})$$

$$= \varepsilon_{ijk}(E_{1i}\,E_{1j}\,E_{1k}) - \lambda\,\varepsilon_{ijk}[E_{1i}E_{2j}\delta_{3k} + E_{1i}\delta_{2j}E_{3k} + \delta_{1i}\,E_{2j}\,E_{3k}]$$

$$+ \varepsilon_{ijk}\,\lambda^2\,[\delta_{1i}\delta_{2j}\,E_{3k} + \delta_{1i}\,E_{2j}\delta_{3k} + E_{1i}\delta_{2j}\delta_{3k}] - \varepsilon_{ijk}\,\lambda^3\,\delta_{1i}\delta_{2j}\delta_{3k}$$

Write this as

$$= A + \lambda B + \lambda^2 C + \lambda^3 D$$

Then we have one at a time

$$A = \varepsilon_{ijk}(E_{1i}\,E_{2j}\,E_{3k}) = \det(E)$$

$$B = -\,\varepsilon_{ijk}[E_{1i}E_{2j}\delta_{3k} + E_{1i}\delta_{2j}E_{3k} + \delta_{1i}\,E_{2j}\,E_{3k}]$$

$$= -\,\varepsilon_{ijk}E_{1i}E_{2j}\delta_{3k} - \varepsilon_{ijk}\,E_{1i}\delta_{2j}E_{3k} - \varepsilon_{ijk}\,\delta_{1i}\,E_{2j}\,E_{3k}$$

$$= -\,\varepsilon_{ij3}E_{1i}E_{2j} - \varepsilon_{i2k}\,E_{1i}E_{3k} - \varepsilon_{1jk}E_{2j}\,E_{3k}$$

Now $\varepsilon_{1jk}E_{2j}\,E_{3k}$ is the cofactor and minor of the upper left element of $E-\lambda = \det\begin{pmatrix} E_{22} & E_{23} \\ E_{32} & E_{33} \end{pmatrix}$ so it seems to me that B = sum of the three dets with - signs.

$$-B = \det\begin{pmatrix} E_{22} & E_{23} \\ E_{32} & E_{33} \end{pmatrix} + \text{the other two}$$

Next,

$$C = \varepsilon_{ijk}\,[\delta_{1i}\delta_{2j}\,E_{3k} + \delta_{1i}\,E_{2j}\delta_{3k} + E_{1i}\delta_{2j}\delta_{3k}]$$

$$= \varepsilon_{ijk}\,\delta_{1i}\delta_{2j}\,E_{3k} + \varepsilon_{ijk}\,\delta_{1i}\,E_{2j}\delta_{3k} + \varepsilon_{ijk}\,E_{1i}\delta_{2j}\delta_{3k}$$

$$= \varepsilon_{12k}\,E_{3k} + \varepsilon_{1j3}\,E_{2j} + \varepsilon_{i23}\,E_{1i} =$$

$$= \varepsilon_{123}\,E_{33} + \varepsilon_{123}\,E_{22} + \varepsilon_{123}\,E_{11} \quad = E_{11} + E_{22} + E_{33}$$

and finally

$$D = -\,\varepsilon_{ijk}\,\delta_{1i}\delta_{2j}\delta_{3k} = -\varepsilon_{123} = -1.$$

So I think find that

$$\det(E-\lambda) = A + \lambda B + \lambda^2 C + \lambda^3 D$$

$$A = \det(E)$$

$$-B = \det\begin{pmatrix} E_{22} & E_{23} \\ E_{32} & E_{33} \end{pmatrix} + \text{the other two}$$

$$C = E_{11} + E_{22} + E_{33}$$

$$D = -1$$

Now if $\det(E-\lambda) = 0$, that means that

$$-A - \lambda B - \lambda^2 C - \lambda^3 D = 0$$

$$-\det(E) + \{\det\begin{pmatrix} E_{22} & E_{23} \\ E_{32} & E_{33} \end{pmatrix} + \text{the other two}\}\lambda - (E_{11} + E_{22} + E_{33})\lambda^2 + \lambda^3 = 0$$

or

$$-I_3 \quad + I_2\,\lambda \qquad\qquad -I_1\lambda^2 \qquad + \lambda^3 = 0$$

so that

$$I_3 = \det(E)$$

$$I_2 = \det\begin{pmatrix} E_{22} & E_{23} \\ E_{32} & E_{33} \end{pmatrix} + \text{the other two}$$

$$I_1 = E_{11} + E_{22} + E_{33}$$

Comment: Looking at d**x** = **F** d**X** (F is the deformation gradient), you see that the matrix F and all the others derived from it are functions of a point in space and of time. We start with some small separation of points d**X** and see where the two points end up after some flow. The two points at flow start define a little vector, and the most that can happen to this thing is it gets rotated and perhaps also stretched. Where does the flow take two adjacent points, that is the issue here. This seems to be a Lagrangian picture notion more than Eulerian. I have to follow the pair of points to see what happens. For a given material blob and flow situation, I can imagine there is some nonlinear **x** = $\boldsymbol{\mathcal{F}}$(**X**) which describes how every particle in the blob flows. But this is just **x** = **x**(**X**,t). Then we linearize at each point in space to get d**x** = **F** d**X** and then I

suppose this matrix is then my R matrix of tensor doc if, in tensor doc, I take x→X ,x'→x, **R** → **F**. So here is another application tensor doc's general theory of transformations. But I don't think Lai is going to be thinking about tensors with respect to this transformation! [I formalized this above]

3.10 Dilatation (p 94). The main result here is this: (recall **u** is the displacement = **x**-**X**)

$$e \equiv \Delta(dV)/dV = \Sigma_i E_{ii} = \text{div } \mathbf{u}$$

and this is then written out in a few coordinate systems. Lai derives this in a simple manner, and it is assumed that the directions i = 1,2,3 are the principle directions. But I think that if you were to take some other set of directions, you get there by a rotation, and the trace of a matrix is invariant to rotations, so this result is then true for any matrix E.

$$\text{tr}(R^{-1}ER) = \text{tr}(ERR^{-1}) = \text{tr}(E) = \Sigma_i E_{ii} = e = \text{div } \mathbf{u}$$

3.11 Infinitesimal Rotation Vector (p 94) Earlier we used $\boldsymbol{\Omega} = \theta\hat{\mathbf{n}}$ but here Ω is a matrix which is antisymmetric. Yes, it must therefore have a $\mathbf{t}^{\mathbf{d}}$ vector as shown. I know that if Ω were a rotation matrix, then $\mathbf{t}^{\mathbf{d}}$ would be **ω**, and its direction would be the axis about which something is instantaneously rotating. But here Ω is just the AS part of Δu so I don't quite follow the interpretation given last sentence of p 94. I agree with all the equations.

3.12 D(dx)/Dt. The big equation here says this: [and the main object of interest is (**∇v**)]

$$D(d\mathbf{x})/Dt = (\nabla\mathbf{v})\, d\mathbf{x} \qquad \text{ie} \qquad d_t(d\mathbf{x}) = (\nabla\mathbf{v})\, d\mathbf{x}$$

which seems very strange to me. If d**x** is the distance between two adjacent points in the fluid, then this tells the rate at which d**x** is changing in time. And this is in the spatial or Eulerian view. So this is the "rate of change of a differential vector between two points". I am not used to such strange notation!

Derivation. There is something very slippery and unclear here, so do each step "with words" :

1. What is the meaning of d**x** ? (only the one functional form exists for **x**)

$$d\mathbf{x} \equiv \mathbf{x}(\mathbf{X}+d\mathbf{X},t) - \mathbf{x}(\mathbf{X}, t)$$

You start with two dots separated by d**X** at time t_0 and they map into this d**x** at time t.

2. Apply the total time derivative to the above equation

$$d_t d\mathbf{x} \equiv d_t\mathbf{x}(\mathbf{X}+d\mathbf{X},t) - d_t\mathbf{x}(\mathbf{X}, t)$$

3. As explained on page 7 above, we know that

$$\mathbf{v} = d_t\mathbf{x} = \hat{\mathbf{v}}(\mathbf{X},t) = \tilde{\mathbf{v}}(\mathbf{x},t)$$

Thus from 2 we have that

$$d_t d\mathbf{x} = \tilde{\mathbf{v}}(\mathbf{x}+d\mathbf{x},t) - \tilde{\mathbf{v}}(\mathbf{x},t) = \hat{\mathbf{v}}(\mathbf{X}+d\mathbf{X},t) - \hat{\mathbf{v}}(\mathbf{X}\,t) = d\mathbf{v}$$

4. Earlier we showed that $d\mathbf{v} = (\nabla\mathbf{v})d\mathbf{x}$ gives the way the velocity field at time t changes with position. But that is exactly what we have on the above line with the twiddle functions, so

$$d_t d\mathbf{x} = d\mathbf{v} = (\nabla\mathbf{v})d\mathbf{x}$$

And this is the desired result, which we can write as

$$D/Dt\,(d\mathbf{x}) = (\nabla\mathbf{v})d\mathbf{x} = D(dx)/Dt = d_t(d\mathbf{x}) = d(dx)/dt$$

3.13 The Rate of Deformation Tensor (95)

If you just write out the velocity gradient $\nabla\mathbf{v}$ = D + W = sym + antisym, then D is called the **rate of deformation tensor** and W is called the **spin tensor**. Note the slight parallel here:

$$D = [\,(\nabla\mathbf{v}) + (\nabla\mathbf{v})^T]/2$$
$$E = [\,(\nabla\mathbf{u}) + (\nabla\mathbf{u})^T]/2$$

Lai shows that

$$ds\,D(ds)/Dt = d\mathbf{x} \bullet D \bullet d\mathbf{x}$$

Suppose you put $d\mathbf{x}$ in one of the three directions which diagonalize D. Call that $d\mathbf{x} = ds\,\mathbf{n}$. Then

$$(1/ds)\,D(ds)/Dt = D_{nn}$$
$$(1/dS)\,\Delta(ds) = E_{nn} \qquad (3.8.2)$$

so the diagonal elements of D are similar to those of E, but here they give the RATE of unit length elongation in direction n. Called the **rate of extension**. The **off-diagonals** give the rate of angle change between a pair of direction base vectors, similar to E. These angle changes are **rates of shear**.

Problem 3.46. (Ref p 97 ; on p 150).

(a) My plan is to compute $D_t(d\mathbf{x}_1 \bullet d\mathbf{x}_2)$ two different ways and then set them equal. One way is

$$D_t(d\mathbf{x}_1 \bullet d\mathbf{x}_2) = D_t(ds_1\mathbf{n}) \bullet ds_2\,\mathbf{m} + ds_1\mathbf{n} \bullet D_t(ds_2\,\mathbf{m}) \qquad // \; d\mathbf{x}_1 = ds_1\mathbf{n} \quad d\mathbf{x}_2 = ds_2\,\mathbf{m}$$

$$= \{\, D_t(ds_1)\,\mathbf{n} + ds_1\,D_t(\mathbf{n})\,\} \bullet ds_2\,\mathbf{m} + ds_1\mathbf{n} \bullet \{\, D_t(ds_2)\,\mathbf{m} + ds_2\,D_t(\mathbf{m})\}$$

$$= [D_t(ds_1)\,ds_2 + ds_1\,D_t(ds_2)\,]\,(\mathbf{n}\bullet\mathbf{m}) + ds_1\,ds_2\,[D_t(\mathbf{n})\bullet\mathbf{m} + \mathbf{n} \bullet D_t(\mathbf{m})]$$

$= [(1/ds_1)D_t(ds_1) + (1/ds_2)D_t(ds_2)\ ]\ (\mathbf{n}\bullet\mathbf{m})\ ds_1 ds_2 + ds_1\ ds_2\ D_t(\mathbf{n}\bullet\ \mathbf{m})$

The second way is this

$D_t(d\mathbf{x}_1\ \bullet\ d\mathbf{x}_2) = D_t(d\mathbf{x}_1)\bullet\ d\mathbf{x}_2 + D_t(d\mathbf{x}_2)\bullet\ d\mathbf{x}_1$

$= [(\nabla\mathbf{v})d\mathbf{x}_1] \bullet d\mathbf{x}_2\ + d\mathbf{x}_1 \bullet [(\nabla\mathbf{v})d\mathbf{x}_2]$

$= d\mathbf{x}_1 \bullet (\nabla\mathbf{v})^{\mathbf{T}} d\mathbf{x}_2\ + d\mathbf{x}_1 \bullet [(\nabla\mathbf{v})d\mathbf{x}_2]$

$= d\mathbf{x}_1 \bullet [(\nabla\mathbf{v})\ +(\nabla\mathbf{v})^{\mathbf{T}}]\ d\mathbf{x}_2$

$= 2\ d\mathbf{x}_1 \bullet D\ d\mathbf{x}_2$

$= 2\ ds_1 ds_2\ \mathbf{n} \bullet D\mathbf{m}$

Se then set things equal, cancel the $ds_1 ds_2$ and we have

$[(1/ds_1)D_t(ds_1) + (1/ds_2)D_t(ds_2)\ ]\ (\mathbf{n}\bullet\mathbf{m})\ + D_t(\mathbf{n}\bullet\ \mathbf{m})\ =\ 2\ \mathbf{n} \bullet D\mathbf{m}$

We then set $\mathbf{n}\bullet\mathbf{m} = \cos\theta$ and then $D_t(\mathbf{n}\bullet\ \mathbf{m}) = D_t(\cos\theta) = -\sin\theta\ D_t\theta$ to get

$[(1/ds_1)D_t(ds_1) + (1/ds_2)D_t(ds_2)\ ]\ \cos\theta\ -\sin\theta\ D_t\theta\ =\ 2\ \mathbf{n} \bullet D\mathbf{m}\ \ = 2\ D\mathbf{n} \bullet \mathbf{m}$

and this agrees with the quoted result of part (a).

(b) If $d\mathbf{x}_1 = d\mathbf{x}_2$ this says $ds_1 = ds_2$ and $\mathbf{n} = \mathbf{m}$ and $\cos\theta = 1$ and $\sin\theta = 0$, so

$2\ (1/ds_1)D_t(ds_1)\ = 2\ \mathbf{n} \bullet D\mathbf{n}$

which just replicates our known result 3.13.12. Fine.

(c) Now go with **n** and **m** at right angles so $\cos\theta = 0$ and $\sin\theta = 1$:

$-\ \sin\theta\ D_t\theta = 2\ \mathbf{n} \bullet D\mathbf{m}$

$D_t\theta = -\ 2\ D_{\mathbf{nm}}$

=> $2D_{\mathbf{nm}}$ = rate of decrease of the angle θ

and this is the claim made on page 97 which I have now verified! // added 6.17.12

Problem 3.47. (Ref p 97; on p150) Let the $\mathbf{e}_i$ be in the principle direction unit vectors. Consider the orthogonal 3-piped spanned by $d\mathbf{x}^{(i)} = ds_i\ \mathbf{e}_i$ which has volume $dV = ds_1 ds_2 ds_3$. From 3.13.11 we know that

$ds_1\ D(ds_1)/Dt = d\mathbf{x}^{(1)} \bullet D\ d\mathbf{x}^{(1)} = (ds_1)^2 D_{11}$ and similarly for 2 and 3.

$\Rightarrow\ D(ds_1)/Dt = ds_1\ D_{11}$.

Then

$D(dV)/Dt = D(ds_1 ds_2 ds_3)/Dt = D(ds_1)/Dt * ds_2 ds_3 + \text{cyclic}$

$= (ds_1)\ D_{11}\ ds_2 ds_3 + \text{cyclic} = dV\ (\text{tr}(D))$.

Therefore

$(1/dV)\ D(dV)/Dt = \text{tr}(D)$ // which verifies 3.13.13

I penciled in a copy of p 96A which says tr(D) = div **v** , so we then end up with 3.13.14

$(1/dV)\ D(dV)/Dt = \text{tr}(D) = \text{div}\ \mathbf{v}$ (3.13.14)

$e = \Delta(dV)/dV = \text{tr}(E) = \text{div}\ \mathbf{u}$ // dilatation from (3.10.3)

So the first thing above would be the "rate of dilatation" (but Lai does not use that phrase)

3.14 The Spin Tensor (98)

Here they just say that when you do $\nabla\mathbf{v} = D + W$, the antisym matrix W has its associated dual vector and they call that vector $\boldsymbol{\omega}$ which can be computed from W, and in fact $Wd\mathbf{x} = \boldsymbol{\omega} \times d\mathbf{x}$. It was shown earlier that W does not contribute to elongation of the d**x**, so it must just rotate it, and it turns out that W rotates d**x** about instantaneous angular velocity $\boldsymbol{\omega}$. I will have to ponder this stuff on another pass.

I think this entire spin tensor action corresponds to a *rigid body rotation* of the matter particle without any kind of deformation. See notes Chapters 1,2 part C tensor calculus. Also Goldstein stuff.

3.15 Conservation of Mass (99)

From $D(\rho dV)Dt = 0$ (you track a blob and its mass cannot change), we get the rule

$D\rho/Dt = \partial\rho/\partial t + \mathbf{v} \bullet \nabla\rho = -\rho\ \text{div}\ \mathbf{v}$

which relates the time rate of change of ρ to both ρ and **v**. I think this is just my intuitive material flow picture, could write in terms of mass current density J_m etc.

If ρ = constant, then $D\rho/Dt = 0$ and thus div **v** = 0. This is the incompressible situation, and this is why we can represent $\mathbf{v} = \nabla\varphi$ in potential theory fluid work.

Question: What exactly is the meaning of the equation $D(\rho dV)/Dt = 0$?? Quantity $dm = \rho dV$ is the mass of a "particle", a tiny red blob shall we say. If we track along with our little red blob, we define dV to be the changing boundary of the blob and ρ to be the changing density inside the blob. Has Lai really stated the notion that matter stays in the blob? Well, the blob is described by $\mathbf{x}(t; \mathbf{X})$ for every point in the blob. We think of the mass of a particle as sticking with the particle! So considering all the points in a tiny particle, if they were Lagrangian particles, each would maintain its mass, and so the larger continuum particle maintains its mass. That is a very key idea that I don't think Lai stressed. Once again, even a large blob is a set of these continuum particles, each particle moves according to $\mathbf{x}(t; \mathbf{X})$ so the starting blob and the ending blob contain the same particles, it's just that they have moved and perhaps expanded or shrunk. So key idea: the boundary of a large blob in motion always contains the same particles (assuming there is no process which absorbs or creates particles somehow). Therefore even for a large blob, the mass remains constant for the blob. Therefore $D(\rho dV)/Dt = 0$ just states this fact for a tiny blob dV. The volume of the blob over time is in fact *defined* to be that boundary which encloses the same particles that you started with.

Notice that you can regard dV (volume of a tiny blob) itself as an Eulerian function $dV(\mathbf{x},t)$ and you can apply D/Dt to it just as to any Eulerian function.

I have derived all equations on page 99. Now I once wrote a doc called Mass Charge and Heat Flow stored in the Stak Ch 6 folder. It talks about the mass current density $\mathbf{P}(\mathbf{x}) = \rho(\mathbf{x})\, \mathbf{v}(\mathbf{x})$ which defines the velocity vector, and the usual divergence rule is written $\nabla \bullet \mathbf{P}(x) = -\partial_t \rho(\mathbf{x})$ which tells us that if there is a change in the mass density in a tiny volume at some point, that arises due to the mass flux leaving that volume. This then says

$$\nabla \bullet [\rho(\mathbf{x})\, \mathbf{v}(\mathbf{x})] = -\partial_t \rho(\mathbf{x}) = v \bullet \nabla\rho + \rho\, \nabla \bullet \mathbf{v}$$

$$\Rightarrow \partial_t \rho(\mathbf{x}) + \mathbf{v} \bullet \nabla\rho = -\rho\, \nabla \bullet \mathbf{v} = D_t\rho \qquad \text{// which agrees with 3.15.4}$$

Once again, there are two ways to write the same continuity equation

$$D_t\rho = -\rho\, \nabla \bullet \mathbf{v} = -\rho \operatorname{div} \mathbf{v} \qquad \text{// way \#1}$$

$$\partial_t\rho = -\nabla \bullet [\rho\, \mathbf{v}] = -\operatorname{div}(\rho \mathbf{v}) \qquad \text{// way \#2}$$

3.16 Conditions that a valid E tensor's components must satisfy (101).

We have that $E = (\nabla \mathbf{u})^{\mathbf{s}}$ so that given any field u, we can compute the matrix E, and there is never a problem if everything is reasonable. But if you exhaust all possible vector fields **u**, you do NOT exhaust all possible matrices E. For example, you can only generate symmetric E's. If you pick an E out of the blue and you want to know whether it is a "legal E", you find there are conditions this E must satisfy as shown top of page 102. These are the "equations of compatibility" or "integrability conditions" on E. All fine.

3.17 Conditions that a valid D tensor's components must satisfy (101).

This is the exact same idea, but replace **u** with **v** and $D = (\nabla \mathbf{v})^{\mathbf{S}}$

Comment: Lai is introducing something new on every single page, this is rough sailing!

3.18 The Deformation Gradient (105)

This is Lai's second pass on this subject where we have $F = \nabla \mathbf{x}$. He notes that F is invertible as one of our early flow conditions. He then claims that detF > 0. He justifies this on a notion somehow that flow cannot change handedness. I think for me an easier derivation is from the polar decomposition F = RU, now that I have invested quite heavily in its derivation. We know that detR = 1 and detU = product of eigenvalues since U is symmetric, and since it is posdef, those eigenvalues are all positive so detU > 0 and thence detF > 0. Now I guess in theory R orthogonal allows detR = -1, so we are then back to Lai's comment that we cannot smoothly move from the group parameter space of SO(3) to the parity inverted chunk of this group space. I forget the notation here. Parity transformation requires a sudden non-continuous change which we don't allow in our flow.

3.19 Local Rigid Body Motion (106)

I don't get the point of this tiny section. It seems tied to the example which precedes it. Yes, in an example it is possible that a whole plane will be doing rigid body motion. // Well, the point is that one kind of deformation is F = R, a pure rotation. In the next section we consider F = U, a pure stretch deal, and then in the section after that we look at the general combination F = RU.

3.20 Finite Deformation (106).

Soon we will look at F = RU, but here we first look at F = U, U being pos def sym. If you go to the principle directions, the eigenvalues of U are positive, but these are the amounts by which differential distances along the principle directions are stretched, such as $d\mathbf{x}^{(1)} = \lambda_1 d\mathbf{X}^{(1)}$. Thus, the matrix U is doing *no rotation at all* since vectors stay in the same direction. They do not even invert direction since the λ are positive. Of course making this statement means we have be using the appropriate principle directions at time t = t when we are looking at some point in space **x**. Note that this does not mean the stretch is > 1, it is just positive. An obvious definition of stretch is given for an arbitrary $d\mathbf{X}$.

3.21 Polar Decomposition Theorem (107).

A good example is given, picture is labeled just as I would have done it. Rotate then stretch into ellipse, or stretch first then rotate the ellipse. Notice that U and V are related by a similarity transformation by the rotation R which means that U and V have the same positive eigenvalues.

F = RU	U = right stretch tensor	$U^2 = C$ = right Cauchy-Green tensor
F = VU	V = left stretch tensor	$V^2 = B$ = left Cauchy-Green tensor

3.22 Computing R,U and V from F (111)

Part of my general proof is done here. When he gets to $U^2 = F^T F \equiv S$, he claims without proof that there is a unique pos def sym U that solves this equation. Once you get U this way, V and R are easy. Lai then

gives a simple example of computing these three objects from a specific given F. In the second "example" Lai attempts the polar decomposition uniqueness proof. Again, he has to fudge on the $U^2 = S$ uniqueness idea, and this is of course the crux of the proof. This gives $(R_1 - R_2)\, U = 0$, and since U is invertible we find that the R's must be equal, and then finally you get a specific V. So this is the second fudge on the subject, but we are told that Appendix 3.3 might lessen the fudge a bit. Lai has been very good about proving everything which I like, but I don't think he will make it on this one.

Appendix 3.3 (p 143, out of order here)

This appendix does not really show what was advertised in the text. Instead it says that if $U^2 = D$ where D is diagonal with positive diagonal elements then U can be taken to be the obvious positive diagonal matrix. The fact used in the 2D case is that $U_{nn} > 0$ for U being pos def (regardless of whether U is diagonal, since this is $\hat{n} \bullet U\hat{n}$ and all such dot products are positive. There is no mention of matrix U being symmetric, but of course in the solution it is. So for 2D and 3D this appendix demonstrates that there is only one pos def matrix which is the square root of a pos def diagonal matrix.

Let's do a little theorem here. Show that if U has pos def eigenvalues, then $\hat{\mathbf{n}} \bullet U\hat{\mathbf{n}} > 0$ for any $\hat{\mathbf{n}}$. Well, drop the hat because that is irrelevant here. Assume that $\mathbf{n} = \Sigma_i\, a_i \mathbf{e}_i$ where $\mathbf{e}_i$ is the diagonal basis. Then

$$\mathbf{n} \bullet U\mathbf{n} = \Sigma_{ij}\, a_i a_j\, e_i \bullet U e_j = \Sigma_{ij}\, a_i a_j\, \delta_{ij} U_{ij} = \Sigma_i\, a_i^2\, U_{ii} = \Sigma_i\, a_i^2\, \lambda_i > 0 \qquad \text{QED}$$

3.23 The right Green-Cauchy deformation tensor C (114)

This thing is defined as $C = U^2$ where U is our sym pos def U from the F = RU deal. We know that we can write $F^T F = U^2$ which makes if obvious that U^2 is sym and pos def (ie, easy to show that eigenvalues are positive for $F^T F$, see polar notes). Thus, C is sym and pos def. This section studies the *interpretation* of the matrix elements of C. It first derives the general fact 3.23.4 and then looks at special cases:

Special case A:

both = $d\mathbf{x} = ds_1 \mathbf{n}$ where $\mathbf{n}$ is an arbitrary unit vector

both = $d\mathbf{X} = dS_1 \mathbf{e}_1$ point separation starts as being along the 1 axis at time t = 0

We assume that C_{11} means in the $\mathbf{e}_n$ basis. Then get $C_{11} = (ds_1/dS_1)^2$ which is just the square of the stretch of the vector $d\mathbf{X}$ which was started in the $\mathbf{e}_1$ direction.

Using a different choice in the general formula, we find that

$$C_{12} = (ds_1/dS_1)\,(ds_2/dS_2) \cos\left[\, d\mathbf{x}^{(1)} \bullet d\mathbf{x}^{(2)} \right]$$

so these elements are reduced by the cosβ as shown. If angle is 90 degrees then $C_{12} = 0$.

We then have a few examples. Note that U^2 is the matrix I called S in my polar notes.

Notes and reading are now both up to page 118.

3.24 The Lagrangian (Finite) Strain Tensor E* (118)

We met this one earlier as well, it is E* = (C-I)/2. If you have no deformation at all, then the polar says that F = RU = R with U=I and of course then C = I and E* = 0, so no strain at all means E* = 0, so this is perhaps a better metric of strain. And of course the name is as noted above. Earlier, we got interpretations for the linearized version E, but here Lai gives interpretations for the E* elements. They are all directly related to those elements of C of course.

3.25 The Left Cauchy-Green Deformation Tensor (121)

This one is $B = V^2 = FF^T$ and again we get interpretations of the matrix elements of B. It seems much less "nice" because in X space instead of taking little vectors like $d\mathbf{x}^{(1)} = ds_1\mathbf{e}_1$, you have to take the uglier vectors $d\mathbf{x}^{(1)} = ds_1(R^T\mathbf{e}_1)$ where this is the R of F = RU = VR. I more or less skipped this section for now. [this is related to the forward and inverse transforms, see notes earlier]

3.26 The Eulerian Strain Tensor (125)

This one has the strange definition $e^* = (I-B^{-1})/2$. Again if F = VR = R and V=I so there is no strain, you get $B = V^2 = I$ and then of course $B^{-1} = I$ and e* = 0. For this e* matrix things are sort of simple again, but they go backwards. You take now $d\mathbf{x} = ds_1\mathbf{e}_1$ (with C we took $d\mathbf{X} = dS\mathbf{e}_1$) and then the elements of e* come out being very similar to those of C in structure. I skip this all for now, it is just another tensor representation of F.

3.27 Change of Area Due to Deformation (128)

At first I thought this section was a no-brainer because I had already dealt with it in Section 8 of tensor doc, in particular, what is now Section 8 (c) item 9. But then the formula (3.27.10) started causing trouble. I followed in full detail how Lai derived it, but I could not come up with a tensor-doc general derivation! This led to several side documents, one of which now survives called "Lai p 129 and tensor doc". This in turn led to some small additions to tensor doc, as outlined at the end of the doc just mentioned. The confusion basically arose because Lai does not put overbars on covariant vectors such as the areas dA. So now a full derviation of (3.27.10) and more generally (3.27.12) appears at the end of tensor doc Section 5 (o), so I won't repeat it here. Everything is OK now, the cofactor for the ratio is correct, the Cartesian view is correct, and so on.

Notes added 2.23.12. I have checked off results on page 128, but am having trouble on page 129. He claims that (unit vectors $\mathbf{e}_n$)

$$F\mathbf{e}_3 \bullet (F\mathbf{e}_1 \times F\mathbf{e}_2) = \det F$$

proof:

$$[F\mathbf{e}_3]_a = F_{ai}(e_3)_i = F_{ai}\delta_{3i} = F_{a3}$$

$F\mathbf{e}_3 \bullet (F\mathbf{e}_1 \times F\mathbf{e}_2) = [F\mathbf{e}_3]_a \, \varepsilon_{abc} [F\mathbf{e}_1]_b [F\mathbf{e}_2]_c$

$= F_{a3} \, \varepsilon_{abc} \, F_{b1} \, F_{c2} \;\; = \varepsilon_{abc} \, F_{b1} \, F_{c2} \, F_{a3} \;\; = \varepsilon_{abc} \, F_{a1} \, F_{b2} \, F_{c3}$ (cyclic back up) = det(F) QED

OK, I have now derived through equation 11 here.

3.28 Change of Volume Due to Deformation (128)

Tensor doc tells us that for **x** transforming to **x'**

$dV = (g')^{1/2} \, d\mathcal{V}'$ $\qquad d\mathcal{V}' \equiv \Pi_i dx'^i$ $\qquad |J| = \sigma J = g'^{1/2}$

$J(\mathbf{x}') \equiv \det(S(\mathbf{x}')) \;\; = \det(\partial x_i / \partial x'_k) \; = 1/\det(R(\mathbf{x}(\mathbf{x}')) = 1/\det(\partial x'_i / \partial x_k)$

Translated to the notion that **X** transforms into **x**, this all says

$dV = (g)^{1/2} \, d\mathcal{V}$ $\qquad d\mathcal{V} \equiv \Pi_i dx^i$ $\qquad |J| = \sigma J = g^{1/2}$

$J(\mathbf{x}) \equiv 1/\det(R(\mathbf{X})$

Now Lai thinks of dV in X space as being dV_0 and $d\mathcal{V}$ in x space is dV . So we have

$dV_0 = (g)^{1/2} \; dV \;\; = |J|_{me} \, dV = \; dV \, / |\det(R(\mathbf{X}))|$

so that

$dV = dV_0 \, |\det(R(\mathbf{X}))| \; = dV_0 \, |\det(F)| \;\; = dV_0 \, J_{Lai}$ $\qquad$ // agrees with p 130 3.28.3

Lai then refers to |det(F)| as his J, which differs from mine by J→1/J and as usual it just depends on how you define J (ie, which coordinates go on top, etc). Note also that I would write

$dV = (g)^{-1/2} dV_0$

so the usual g factor is upside down here.

One extra detail. From tensor doc I had that (translating)

$g \equiv \det(\bar{g}) \;\; = J_{me}^2$ $\qquad$ if G = 1

$= (\det(R))^{-2}$

But of course $\det(RR^T) = \det(R^T R) = [\, \det(R) \,]^2$. So in the Lai world we have R = F and

$$1/g' = J_{Lai}^2 = [\det(F)]^2 = \det(FF^T) = \det(F^TF) = \det(B) = \det(C)$$

And from essay then we can say

$$dV = (g)^{-1/2} dV_0 = J_{Lai}\, dV_0 = \sqrt{\det(B)}\; dV_0 = \sqrt{\det(C)}\; dV_0 = |\det(F)|\, dV_0$$

Since B and C are positive definite, the sign of their dets are always +.

Lai finally notes that if your material is **incompressible**, then $dV = dV_0$ and all these matrices F and B and C must have det = 1, an interesting fact. That will mean that det(U) = 1 and det(V) = 1 as well. The shape of a dV can change, but the volume of the dV cannot change.

3.29 The matrices in other coordinate systems (131)

This section is now written up in a separate doc called " Lai section 3_29 F notes.doc" .

We have so far this notion,

$$F_{ij} = (\partial x_i/\partial X_j)$$

where things are Cartesian for both **x** and **X**. Treated as a "transformation" this F_{ij} linearized version of the non-linear transform **x** = **x**(**X**,t) is NOT a coordinate transform, it is a blob motion transform. BUT we should be able to tack on coordinate transforms in this manner:

$$(\partial x'_i/\partial X'_j) = (\partial x'_i/\partial x_n)\,(\partial x_n/\partial X_m)\,(\partial X_m/\partial X'_j)$$

where **x**' are curvilinear coordinates for Cartesian **x**, and **X**' are curvilinear for Cartesian **X**, and we can choose an arbitrary curvilinear coordinate system for the two sides! We would then say

$$dx'_i = \partial'_j x'_i\; dX'_j = F'_{ij} dX'_j \qquad F'_{ij} = (\partial x'_i/\partial X'_j)$$

$$d\mathbf{x}' = F'\, d\mathbf{X}$$

So applied to the matrix F I think we can say

$$F'_{ij} = (\partial x'_i/\partial x_n)\, F_{nm}\, (\partial X_m/\partial X'_j)$$

and in this way we can define a matrix F' with respect to an arbitrary pair of coordinate systems.

Case A: both systems are cylindrical. In this case I think I know that

$$(\partial x'_i/\partial x_k) = R_{ik}$$

$$(\partial x_i/\partial x'_k) = S_{ik}$$

The equations are 123 = xyz = rθz (ρφz sometimes)

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

So it is easy then to compute S_{ik}

$S_{11} = \partial x/\partial r = \cos\theta$
$S_{12} = \partial x/\partial \theta = -r\sin\theta$
$S_{13} = \partial x/\partial z = 0$

$S_{21} = \partial y/\partial r = \sin\theta$
$S_{22} = \partial y/\partial \theta = r\cos\theta$
$S_{23} = \partial y/\partial z = 0$

$S_{31} = \partial z/\partial r = 0$
$S_{32} = \partial z/\partial \theta = 0$
$S_{33} = \partial z/\partial z = 1$

$$\Rightarrow \qquad S = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

But I seem to be marching down the wrong path here because I have trig functions appearing. If I continue, yes, I will have matrices which let me relate F to F'. But that is not really what we want to do. I don't want to see Cartesian $F_{12} = \partial x_1/\partial X_2$ appearing anywhere, for example.

So go back to

$$d\mathbf{x} = F\, d\mathbf{X}$$

and use the direct approach. How would we write d**x** in some arbitrary coordinate system? Well, think of a tensor doc transformation from x to x' just for the moment. Then recall that

$$\mathbf{V} = V'^1\mathbf{e}_1 + V'^2\mathbf{e}_2 + \ldots = \Sigma_n V'^n \mathbf{e}_n \qquad \text{where} \quad \mathbf{e}^n \bullet \mathbf{V} = V'^n \qquad \mathbf{e}^n = g'^{ni}\,\mathbf{e}_i$$

So apply this to **V** = d**x** to get

$$d\mathbf{x} = \Sigma_n\, dx'^n\, \mathbf{e}_n \qquad \text{where} \quad \mathbf{e}^n \bullet d\mathbf{x} = dx'^n \qquad \mathbf{e}^n = g'^{ni}\,\mathbf{e}_i$$

$$= \Sigma_n\, dx'^n\, h'_n\, \hat{\mathbf{e}}_n$$

In this world we will have things like

$dx'^1 = dr \qquad dx'^2 = d\theta \qquad dx'^3 = dz$

$\hat{\mathbf{e}}_1 = \hat{\mathbf{r}}$ $\hat{\mathbf{e}}_2 = \hat{\boldsymbol{\theta}}$ $\hat{\mathbf{e}}_3 = \hat{\boldsymbol{\varphi}}$ $h_1' = h_r = 1$ $h_2' = h_\theta = r$ $h_3' = h_z = 1$

Then do a similar thing on the d**X** side

$$d\mathbf{X} = \Sigma_n \, dX'^n \, h'^\circ_n \, \hat{\mathbf{e}}^\circ_n$$

In this world we will have things like

$dX^1 = dr_0$ $dX^2 = d\theta_0$ $dX^3 = dz_0$

these below should all have super 0

$\hat{\mathbf{e}}^\circ_1 = \hat{\mathbf{r}}_0$ $\hat{\mathbf{e}}^\circ_2 = \hat{\boldsymbol{\theta}}_0$ $\hat{\mathbf{e}}^\circ_3 = \hat{\boldsymbol{\varphi}}_0$ $h_1' = h_r = 1$ $h_2' = h_\theta = r_0$ $h_3' = h_z = 1$

Since we have x = r,θ,φ and we have X = r_0, θ_0, φ_0, we have to jimmy up the notation so it can handle things by putting 0 superscripts on things to say we are talking about X at time t_0.

Then we get

$$(\Sigma_n \, dx'^n \, h'_n \, \hat{\mathbf{e}}_n) = F \, (\Sigma_n \, dX'^n \, h'^\circ_n \, \hat{\mathbf{e}}^\circ_n) = \Sigma_n \, dX'^n \, h'^\circ_n \, (F \, \hat{\mathbf{e}}^\circ_n)$$

Now dot both sides with $\hat{\mathbf{e}}_m$ from the left say to get

$$dx'^m \, h'_m = \Sigma_n \, dX'^n \, h'^\circ_n \, \hat{\mathbf{e}}_m \bullet (F \, \hat{\mathbf{e}}^\circ_n)$$

Now recall that we had **x** = **x**(**X**,t) as our flow equation, and this can be interpreted in any coordinate systems you want.

STOP. I need to understand this better in a completely general way. My notation is hopelessly tangled. My tensor doc notation conflicts with the Lai notation. Maybe I should convert Lai to my notation instead of the other way around. Then I would have this flow equation.

$$\mathbf{x}' = \mathbf{x}'(\mathbf{x},t)$$

But it is not that simple since there are really three transformations still to worry about. So this will now be put off into a separate doc. I want the general solution, not just this Case A.

We then have **three appendices**. The first 3.1 concerns those compatibility conditions that I don't care about right now. The second 3.2 gives some properties of positive definite matrices. The third 3.3 gives an unsatisfactory discussion of the key step in the polar decomposition, see my notes elsewhere.

We then have no less than **86 problems!** I might come back at some point and do some of them, but right now my motivation is low concerning the seemingly obscure array of tensors like F, B, C, E, E*, e*, etc. I want to get on with the show now and see what this "kinematics" is going to buy me!