Lai Chapter 4 Notes PhL 2.17.12

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4.1 The Stress Vector (155)

A lot of new stuff here. First we have the division into **body forces** on a particle (like gravity, action at a distance) and **surface forces** (such as the stress vector we are about to deal with). We know that normal gas pressure is force/area and units nowadays are MPa where 1 atmosphere is about 1/10th of a MPa (and is also 760 torr by definition nowadays, torr used to be 1 mm Hg). The gas pressure is always normal to the surface, but now we have this new idea of a "pressure" like force which is at some angle relative to a surface. This angled pressure concept is called "**stress**".

Imagine you mentally carve out a mathematical smooth surface (perhaps an egg shape) inside a blob of continuous material. Now go to a tiny patch on that surface. As the patch gets very small, it gets flatter. Sure, you could think of a planar surface as "the first derivative fit" to the surface, and there would also be some kind of surface "curvature", and also higher derivatives. I don't know off-hand how to deal with such things in 3D, but in 2D it is pretty clear. Now the fundamental "axiom" is Cauchy's stress principle and it claims that the stress vector is independent of the higher derivatives like curvature and is a function only of the planar surface angle, which is to say $\mathbf{S} = \mathbf{f(n)}$, where for the moment I use **S** for the "stress" force per area pressure idea.

In reality, if you look at a tiny patch of area inside a blob, you can imagine that one side of this patch faces into a tiny 3D particle (my sugar cube), and the other side then faces out to the residual medium which is everything outside the particle. The outside applies **S** to the surface, and the inside applies -**S** to

the surface, so the net force on the surface is 0. Now this piece of surface could be accelerating so to make the above argument you have to say something like total stress = $\mathbf{S}_1 dA - \mathbf{S}_2 dA = m\mathbf{a}$. But the mass of the surface patch is 0 since it is infinitely thin, so the two stresses are then equal and opposite even if the patch is accelerating.

Newton's Third Law says forces are equal and opposite and I guess that is what I am saying here, when those forces can be imagined as acting on a massless piece of surface between two objects.

So in reality, there really is some stress vector on any little embedded patch and in theory you could measure it somehow, though I don't know how. You would have to somehow get inside the blob and put some probe at the patch. I will leave this as an unresolved question for now. But I do believe that such a vector exists. And I accept as reasonable Cauchy's claim that it is a function only of position in the blob and of the normal of your selected patch.

Lai uses **t** instead of my S for the stress vector, so I accept the claim that $\mathbf{t} = \mathbf{f}(\mathbf{x},\mathbf{n},t)$. By convention, Lai defines $\mathbf{t}_n$ as the force *of the exterior medium* acting on the particle patch, to $\mathbf{t}_{-n}$ is the force of the particle's patch acting outward on the residual blob.

4.2 The Stress Tensor (156)

We now have a clever argument for why the function shown above is linear in **n**. I will try to construct this argument here. I interpret the page 157 picture as being a chopped off corner of a cube whose three faces lie in the coordinate system planes and the fourth face is the diagonal one. I interpret Lai's usual $\mathbf{e}_1$ as being the unit vector $\hat{\mathbf{x}}_1$ and so on. The little distances like Δx_3 in the picture are completely clear. The triangular face in the 2-3 plane has an outfacing normal $\mathbf{n}_1 = -\mathbf{e}_1$. The force of the medium acting on this face from the outside is then $\mathbf{t}_n = \mathbf{t}_{-e1}$, so I agree with his labels of these forces for the three smaller triangles. The big face has outfacing normal **n** and the force of the blob on this face is $\mathbf{t}_n$, agreed.

The total surface force on this pyramid is then the LHS of (4.2.2) where he uses dA_1 as the area of that face with normal $-\mathbf{e}_1$ and so on. I agree we have to set this to m**a** where m is the mass of the pyramid. I agree that the mass is a constant times the product of these three areas, so if the density is finite, then as you shrink down to differentials, the mass is order ΔA^3 whereas the force is order ΔA so in this limit you get that m**a** = 0, so this gives the first very major result (4.2.3). [This does not say that a particle must have **a** = 0, however. That would be pretty drastic: particles cannot accelerate. It is just that in this equation, the m**a** term can be dropped compared to the other terms.]

Surely we can write the normal **n** as in 4.2.4.

We then come to the area projection idea. Here is a theorem I think is true but which I have never even thought of before. [I proved it in a separate doc in my geometry folder]

Theorem: Let flat area patch 1 of arbitrary shape lie in a plane with normal $\mathbf{n}_1$. Let area patch 2 be some other and larger area patch with normal $\mathbf{n}_2$ whose *projection* onto the plane of area patch 1 is precisely area patch 1. So the smaller patch 1 is the projection of the larger one 2 onto the plane of the smaller one. The claim is that the ratio of the areas is given by $A_2/A_1 = |\, \hat{\mathbf{n}}_1 \bullet \hat{\mathbf{n}}_2 .|$

So using this theorem, how can we relate ΔA_n to say ΔA_1? The latter with normal $-\mathbf{e}_1$ is the projection of the former with normal **n**, so the area ratio must be $|\, \hat{\mathbf{n}}_1 \bullet \hat{\mathbf{n}} \,| = |\, \hat{\mathbf{e}}_1 \bullet \hat{\mathbf{n}} \,| = n_1$ and so $\Delta A_1 = n_1 \Delta A_n$ and so on for the other two, and we then have 4.2.5. So insert these into 4.2.3 to get 4.2.6, I agree. Then just rewrite this as 4.2.8 and we have this result

$\mathbf{t_n} = n_1\mathbf{t_{e1}} + n_2\mathbf{t_{e2}} + n_3\mathbf{t_{e3}}$

and this then tells us that the stress force $\mathbf{t_n}$ is LINEAR in each of the n_i components. Think of maintaining the three face stress vectors $\mathbf{t_{e1,2,3}}$ (these are per unit area) and then rotating the diagonal face with normal **n** so that it perhaps becomes larger and the normal changes. The above shows the way in which the stress vector $\mathbf{t_n}$ on that diagonal face varies with the components of its normal.

This means we can write it in terms of a matrix T

$\mathbf{t_n} = T\,\hat{\mathbf{n}}$

which is then the claim of 4.2.10. Now if we set $\hat{\mathbf{n}} = -\mathbf{e_1}$ (we rotate our diagonal face so it lines up with one of the side faces) this formula must generate $\mathbf{t_{-e1}}$, so that

$\mathbf{t_{e1}} = T\,\mathbf{e_1}$ and so on

Conclusion: By doing simple geometry, we have proven that the stress vector, which previously we had written $\mathbf{t_n} = \mathbf{f}(\mathbf{x},\mathbf{n},t)$, is linear in **n**, and this means there exists a rank-2 tensor T in terms of which we can get $t_n = T(\mathbf{x},t)\,\hat{\mathbf{n}}$. So suddenly now we have a **"stress tensor"** in addition to our "stress vector". It is the **Cauchy stress tensor** (1827).

4.3 Components of the Stress Tensor. (158)

Now consider again

$\mathbf{t_{e1}} = T\,\mathbf{e_1}$

Apply $\mathbf{e_n}$ to both sides to get (note that our little pyramid's three $\mathbf{e_n}$ could have been *any* orthogonals).

$\mathbf{e_n} \bullet \mathbf{t_{e1}} = \mathbf{e_n} \bullet T\mathbf{e_1} = T_{n1}$ in this basis

The second index on T tells you which face your stress is acting upon. For example, the entire line above is talking about our little $-\mathbf{e_1}$ aligned face in Lai's picture. The other index is telling us which component of the stress force we are looking at.

Let's now nail down the sign of things. Imagine a cube made from Lai's pyramid plus the other half. When we write $\mathbf{t_{e1}} = T\,\mathbf{e_1}$ we can think of $\mathbf{t_{e1}}$ as the force of the blob acting on the right face (the one with out-facing normal $\mathbf{e_1}$). Then $T_{n1} = \mathbf{t_{e1}} \bullet \mathbf{e_n}$ is the component of that force in the n direction. So I would argue then that T_{11} is the force of the blob pulling the right face to the right. This seems to agree with the picture on page 159. If this were a gas, I would expect to have $T_{11} < 0$, but if it were steel it could have either sign. The force of the blob on the partner left face must by T_{11} to the left, also as shown. We would write $\mathbf{t_{-e1}} \bullet \mathbf{e_n} = -T_{n1}$ so that $\mathbf{t_{-e1}} \bullet \mathbf{e_1} = -T_{11}$.

Restate: $T_{n1} = \mathbf{t_{e1}} \bullet \mathbf{e_n}$ is the component of the blob force on the $\mathbf{e_1}$ face in the n direction.
More generally:

Theorem: $T_{nm} = \mathbf{e}_n \bullet \mathbf{t}_{em}$ = the force of the blob on the $\mathbf{e}_m$ face in the n direction.

second index tells you which face the blob acts on (the face with outgoing normal $\mathbf{e}_m$)
first index n tells you that T_{nm} is the nth component of that blob force on the face.

One more detail. According to $\mathbf{t}_{e1} = T\,\mathbf{e}_1$, $\mathbf{t}_{e1}$ is the outside force acting on the right face of my cube. If it is a tiny tiny cube, then the left face is at the same **x**, and then the force on the left face must be $-\mathbf{t}_{e1} = T(-\mathbf{e}_1)$. So all three components of this stress vector would then be equal and opposite to those on the positive face, if all corners of the cube were at the same **x** so we are then talking the same T. [This assumes no acceleration or that acceleration does not matter.]

4.4 Symmetry of the Stress Tensor. (159)

The picture shows our differential "cube" end on, and we are going to study the torque around the center of this cube. Since T_{ij} varies with x, the picture allows some variation as shown by the Δ adders, so the lower left corner is the reference point x. All the T objects on the right have 1 as second index as per the above discussion. The CCW torque due to the right face is $[\Delta x_2 \Delta x_3]\ (T_{21}+\Delta T_{21})\Delta x_1/2$ where [] is the area of the right face, and this is the second term on the first line.

Now what is the x_2 directed force on the left face? As noted above, all three components are the negatives of what is on the right face, except for the Δx effect. So the shear force is T_{21} down on the left, and it is $T_{21} + \Delta T_{21}$ on the right, and both these are additive in terms of torque! So this is then the first term on the first line.

Now lets do the top face which is the $\mathbf{e}_2$ face and so T_{12} is the 1 component of this force but it has the extra adder so the top arrow is drawn correctly. But, in terms of CCW torque it makes a negative contribution and this is the second term on the second line and Lai has a <u>sign error</u> there. The bottom face is then the first term on the second line, correctly shown with a - sign.

Now I would argue that the torque equation should have a RHS which would be $I\boldsymbol{\alpha}$ where I is the moment of inertia of the cube, analogous to m**a** of the previous analysis. Then the moment I would be an integral over the area of the rectangle shown. In worst case, all mass is at the edge, say, then the inertia is mR^2 as for a rim, a crude model. Think then of R = Δ, one of the distance deltas, Meanwhile, for constant ρ we would have $m = \Delta^3$ as before, so this $I\boldsymbol{\alpha}$ would be a Δ^5 term! THAT is why he quietly sets Iα = 0 and then sets torque = 0. Then we really have 4.4.1 $\approx$ 0 dropping this Δ^5 RHS relative to the Δ^3 and Δ^4 terms on the LHS. Then to balance the Δ^3 terms we get $2T_{21} - 2T_{12} = 0$ and $T_{21} = T_{12}$ and this is Lai's proof that T_{ij} must be symmetric. // I now see that bottom of page 159 he makes exactly this point and quotes the exact I for the cube in fact.

How might body forces contribute to the torque and destroy this argument? In the problem quoted, suppose you have a polarized anisotropic dielectric IN an external electric field (something the problem neglects to say). Then there is a **body torque** and that kills this little symmetry argument, fair enough.

So for general problems, T_{ij} is symmetric and there are only 6 independent components.

Comment: (added 4.7.12) This symmetry notion applies in any orthogonal coordinate system in the same way. You have your little sugar cube and you might think of directions 1,2,3 as r,θ,φ in sphericals. So when we get to sphericals, the above shows that in general $T_{r\theta} = T_{\theta r}$ etc.

Example 4.4.1. Suppose your blob is a gas. We know that there is no shear force, T is diagonal, and T_{11} would be the gas pressure of the blob "pulling out" on face e_1. So we know in this case $T_{11} < 0$ and all three diagonal elements are the same, and that is why he says T = -p 1 where p is then just the gas pressure. So this is about as simple as an example can get.

Example 4.4.2 We are given a specific T at some point. We are given in effect a certain surface patch with a certain unit normal **n** as deduced in the obvious way from the plane equation. So it is a simple matter to compute $\mathbf{t_n}$ which is the blob force on this face. The normal force is then **t•n** where **n** is a unit vector. Now I could expand T on any set of basis vectors I want, and suppose those are a set of primed unit vectors (two of which he gives). Then $T = \Sigma_{ij}\, T'^{ij}\, \mathbf{e}'_i \mathbf{e}'_j$ in dyadic notation, and of course I know from tensor doc that $T'^{ij} = \langle e'_i|T|e'_j\rangle$ etc etc.

Example 4.4.3. Imagine a cube of air of constant density ρ embedded in a larger amount of air. There is a body force ρgy pulling the block of air down, so the up pressure exerted on the bottom of the cube must exceed that applied to the top. This really means that p = p(y) = p_0 + ρgy. But this equation must apply not just at the top and bottom of the cube, but at all points on the sides of the cube (in fact, it has to apply everywhere). Since pressure is the same in all directions, that acting on the sides builds up as you move from top to bottom, just as the picture shows. If we incorporate the effect of this body force into the stress tensor, then we are saying that $-T_{nn} = p_0 + \rho gy$ on any face of the cube. Think next of the cube as being a very tiny cube at some height y. Then the situation for that cube can be described by a stress tensor T which is diagonal (since no shear forces in a gas) and has $-T_{nn} = p_0 + \rho gy$. THAT is what this example is all about. A pretty basic example indeed.

4.5 Principle Stresses (162)

It matrix T is symmetric, I agree that there are three eigen-normals that would make T diagonal. Since the diagonalizing matrix R would be real orthogonal as usual, the three eigen-normals are orthogonal. At a particular point x, T(x) would be diagonal in this basis and would then have no shear force at all. The "at least" phrase here and earlier might refer to degenerate eigenvalues. In that case, the eigen-normals are not unique (one of them would be) and then there are more than 3 candidate vectors for eigen normals, so you would say there were "more than 3 mutually perp eigen normals" which make T diagonal. From the earlier theorem, we know that the max and min of these diagonal T'_{nn} values bracket the T_{nn} values you get from any other choice of basis.

4.6 Maximum Shearing Stresses (162)

Here is the question addressed here. Suppose you start with basis so T is diagonal and shear stresses are 0. Then you rotate this T in all possible ways. What is the max off-diagonal element you might encounter? And in what situation does that max arise?

I went through Lai's derivation which is spread out over 5 pages, but really occupies about 3 pages. I understand it, I understand the use of the famous Lagrange multiplier method. The table lists off all the possible normals which cause either a max or a min in T_s^2 (shear). The mins are of course the three 0 cases. The max is whatever the max is of the three possibles shown and thus I agree with 4.6.16 which says: the max shear stress is half the difference between the max and min principle stresses.

The harder part to describe is what normals causes these maxes. Well, the normals are easy, it is the way Lai describes their planes. If you have a normal (1,1,0) how would you describe the set of planes that can have that normal? The planes are parallel to the z axis first of all. Secondly, one possible plane passes through the origin and separates quadrants I and III. Then any plane parallel to that plane is allowed. ALSO, all the planes perp to these planes (and still parallel to the z axis) are solutions. You see how the two solutions I have marked with a perp symbol are perp in this sense, (1,1,0 and (1,-1,0). So if this is the winning max value for the stress, then our prototype solution planes are those whose projections onto the x-y plane bisect the axes. The axes we are bisecting are associated with the max and min principle stresses. Wow, this is very hard theorem just to state, so my normals shown here would apply if the winner were the $|T_1-T_2|$ difference.

In order to really convince myself I would have to invoke Maple and verify the table on page 165. But really I would just need to get the first two non-trivial entries. Then I would be convinced of (4.6.16). So the normals that max out the shear stress are going to have some normal stress as well given by 4.6.3. You can see for my first solution pair that

$$n_1^2 = n_2^2 = 1/2 \qquad n_3^2 = 0 \qquad T_n^2 = [(1/2)(T_1+T_2)]^2 \text{ so } T_n = (1/2)[T_1+ T_2]$$

which is then (4.6.17) if this is the winning pair.

Full details of the calculations of this section appear later in **Appendix 4.1** p 191.

It occurs to me belatedly that one might be interested in the "maximum shear stress" in terms of some material ripping apart. Maybe this is crack formation and it would happen first at a certain surface patch that had the offending normal which maximizes shear stress.

Example 4.6.1. (166) Just a journeyman example. For a certain T matrix that is all zero except in the upper left 2x2 square, we are asked to find the eigenvalues and eigenvectors. $\lambda=0$ and $n = e_3$ is one, then you have to find the other two. Then when we do that, we use the formula to find the max stress, but in the end it is the max of the three candidates shown in 4.6.23 so result depends on detailed values of the four numbers. Notice that he does assume $T_{12} = T_{21}$. Why is this matrix called "**plane stress**"? Well, along one of the cube's axes there is no normal stress at all! You would then rotate the cube about that axis to make T be diagonal (the lower right T_{33} values stays 0 when you do this). So you will then have your little principle cube for which there is zero normal force on the top and bottom, you have normal force only on the sides of the cube. [I will try to refer to this force as stress sooner or later] Stress has the units of pressure, force/area.

4.7 Equations of Motion: Linear Momentum (168)

What this title means is that this section will derive **F** = m**a** for a particle! Momentum is never mentioned but I know what they mean. We start with our usual cube and we just add up all the forces on the 6 surfaces. The picture is tough because you can't really tell where the arrows start. A shaded picture of some sort would have been better, but I do get it. It is similar to my curvilinear divergence formula calculation and you pick up derivatives across the cube in the three directions. I have done all the steps and agree with the result (4.7.5) which can be written in the fancy divT notation where T is a rank-2 tensor and not a vector.

Was T symmetry used here? I think it was not used! T does not appear until page 169. So the derivative is on the second index of T which is the index that labels faces, just as one would expect. For symmetric T of course divergence could work on either index.

Notice that body force B is per mass, that is, the total body force on your "particle", and of course **a** is the acceleration of the particle. So here it is (an it is called **Cauchy's equations of motion**)

$\partial_j T_{ij} + \rho B_i = \rho a_i \quad \leftrightarrow \quad F_i = m a_i \quad \mathbf{divT} + \rho \mathbf{B} = \rho \mathbf{a}$

Comment: In an $\mathbf{F} = m\mathbf{a} = 0$ statics problem, you draw a picture of your ladder or whatever and make the vector forces add up to zero. In a CM statics problem, you will have $\partial_j T_{ij} + \rho B_i = 0$ or $\mathbf{divT} + \rho\mathbf{B} = 0$ and in this case you have to satisfy this equation somehow. Probably knowing divT does not fully determine T. Notice that $\partial_j T_{ij}$ is the divergence the row vector i of the T matrix, call it R_i. Knowing div R_i just says that vector field R_i has a certain divergence. If $\mathbf{B} = 0$, then 0 divergence, and that means to me that the "lines of R_i" are continuous and don't end on sources.

In any event, this equation is a big milestone for me. It appears as 3.33 on p 38 in Schaum's fluid dynamics book. This equation is independent of the type of continuous material. It is just $\mathbf{F} = m\mathbf{a}$ and the only limitation is that it would apply to a non-relativistic material.

Example 4.7.1 . Here we are just given a certain field T and it turns out that $\partial_j T_{ij} = 0$ for all i , so such a T is a "possible T" to have when **B** and **a** = 0 (equilibrium with no body forces). Just an example.

Example 4.7.1 . If we assume what I am calling "the gas stress tensor" $T_{ij} = -p\,\delta_{ij}$, then our $\mathbf{F} = m\mathbf{a}$ comes out being $-\nabla p + \rho\mathbf{B} = \rho\mathbf{a}$ since $\partial_j T_{ij} = -\delta_{ij}\partial_j p = -\partial_i p = -(\nabla p)_i$. I don't recall seeing that equation ever before. Does this equation have a name? [This later appears on page 382 for an inviscid fluid. It is just Navier-Stokes with $\mu = k = \lambda = 0$. It has the name Euler Equation of Motion.]

4.8 Equations of Motion in Cylindricals and Sphericals (170)

See divT.doc for details, all done. The equations especially in sphericals are amazingly ugly in the general case, the reason being that divT is a messy object.

Example 4.8.1. This related to **"the Kelvin problem"** (1848) we get later on p 290. It is a certain T tensor presented in cylindricals. The claim is that this T satisfies the Cauchy static equation with no body force. So it is a stress pattern "one could actually have" in "an infinite elastic space" with some finite force $\mathbf{F} = F\hat{\mathbf{z}}$ applied only at the origin point, called "a point load". You would think this would require some delta function action, but the web has little to say, same for Lai. So put this aside for a while. Sounds a little like a finite point charge in electrostatics. [I have Kelvin's collected papers?]

4.9 Boundary condition for the stress tensor (171)

If you evaluate T at a point on a surface of some object, as usual $\mathbf{t_n} = T\mathbf{n}$ where **n** is the normal to the surface at your point of interest. What ever is "outside" your object must be applying this force/stress to the surface. Obviously $\mathbf{t_n}$ need not be normal to the surface. This *externally applied force* is called a **surface traction**. Fine. Imagine holding a balloon in your hands and twisting it, so your fingers then apply surface tractions to the surface of the balloon. Conversely, if there is no surface traction being applied, it must be that $\mathbf{t_n} = T\mathbf{n} = \mathbf{0}$ at your surface point, so this is a boundary condition!

Example 4.9.1. (172) We are given a simple diagonal stress tensor in cylindricals having only T_{rr} and $T_{\theta\theta}$ nonzero (T_{zz}= 0). In part (a) we show that this T is a valid one for equilibrium with **B** = **a** = 0. The physical object of interest is a "thick walled cylinder" with radii r_i and r_o. It is easy to compute the "surface traction" (in this case it is normal to the surface) on either surface which is just the general radius-a result evaluated at these two points. Only T_{rr} contributes to $\mathbf{t}_n$ (see p 172 A) and if we require that t_n = -p on the inner surface and t_n = 0 on the outer, we can solved for the two unknown constants A and B present in the expressions for T_{rr} and $T_{\theta\theta}$.

As usual, I am hazy about the physical situation this implies. Maybe a thick walled tube with gas in the inside core, and vacuum on the outside. But what would hold it together? You would have to put a thin film on the boundaries to stop diffusion then maybe OK? Cannot apply to a solid. [But why did I say that. This is just a thick solid cylinder with gas inside and vacuum outside.]

Comments: (1) This T matrix provides *a* "static solution" to F = ma for a fluid (no shear force) in infinite space viewed in cylindrical geometry where for some reason there is no pressure at the ends of the cylinder. It is basically a 2D problem. (2) The Cauchy **F**=m**a** equation reads $\partial_j T_{ij}$= 0 and as we saw earlier this means $-\nabla p = 0$. Although this is not the Laplace equation, I suspect that solution is probably unique, so this probably IS the unique stress tensor T for a fluid in this situation, apart from the two free parameters A and B. (3) If you consider the thick cylinder situation, the normal anywhere on either surface is (1,0,0) if you think r,θ,z. Then **t** = T**n** gives a very simple expression for the normal stress on either surface. Then you can force the two surfaces to whatever pressures you want, and this sets A and B.

Example 4.9.2. (173) This is the thick *spherical* shell analog of the above problem. The T tensor now has three diagonal elements (only) and the last two are the same, and we have unknowns A and B in our two expressions (which involve $1/r^3$ instead of $1/r^2$). If there is pressure only on the inside, compute A and B.

4.10 The Piola Kirchhoff Stress Tensors (174). [circa 1850 I think]

Now the concept of deformation over time is added to our picture. Initial situation has subscript 0, current situation has no subscript.

(a) the first P-K tensor. [0 subscript means at flow start, no subscript at flow end]

Step 1. We first define **t**, T, d**f**, dA, **n**, dA_0, $\mathbf{n}_0$ in the obvious manner. Then,

d**f** = **t** dA // this is our usual and obvious force equation at time t

d**f** = $\mathbf{t}_0$ dA_0 (4.10.5) // this **defines** pseudo-stress $\mathbf{t}_0$ (current force, previous area)

Therefore: $\mathbf{t}_0 = (dA/dA_0)\mathbf{t}$ (4.10.6) // so the new guy is just scaled version of the old guy

Step 2. We have

t = T **n** // this is our usual thing

$\mathbf{t}_0 = T_0\,\mathbf{n}_0$ (4.10.4) // this **defines** the pseudo-stress tensor T_0

We might say we are making a micro ansatz here that there exists a matrix T_0 to make this true. This tensor T_0 has several names: (1) the **Lagrangian Stress Tensor**; (2) the **first Piola-Kirchhoff stress tensor.**

Step 3: Process the stuff above to get

$\mathbf{t}_0 = T_0\,\mathbf{n}_0 = (dA/dA_0)\mathbf{t} = (dA/dA_0)\,T\,\mathbf{n} = (1/dA_0)\,T\,d\mathbf{A}$ (4.10.7)

And our famous result is that (now embedded in tensor doc Section 5 (o))

$d\mathbf{A} = dA_0 J (F^T)^{-1}\mathbf{n}_0$ (4.10.8)

Insert this into the previous line to get

$T_0\,\mathbf{n}_0 = (1/dA_0)\,T\,\{\,dA_0 J (F^T)^{-1}\mathbf{n}_0\,\} = J\,T\,(F^T)^{-1}\mathbf{n}_0$ (4.10.9)

and since $\mathbf{n}_0$ is arbitrary, we find that

$T_0 = J\,T\,(F^T)^{-1}$ (4.10.10)

or

$T = J^{-1}T_0F^T$ (4.10.11)

So the first PK tensor T_0 can be written in terms of T and F, and in general is not symmetric.

Step 3. Now do this little bit of processing

$\mathbf{t}\,dA = \mathbf{t}_0\,dA_0$

$T\,\mathbf{n}\,dA = T_0\,\mathbf{n}_0\,dA_0$

$T\,d\mathbf{A} = T_0\,d\mathbf{A}_0$ where of course $d\mathbf{A} = \mathbf{n}\,dA$ = the usual differential vector area.

On page 176 Lai defines a certain $d\tilde{\mathbf{f}}$, then defines a certain $\tilde{\mathbf{t}}$, and finally a certain $\tilde{T}$ which is the second PK stress tensor. It can be written this way

$\tilde{T} = F^{-1}T_0 = J\,(F^{-1})T(F^{-1})^T$

and is seen to be symmetric if T is symmetric, unlike T_0. All steps were clear.

Two long examples follow on pages 177 and 178. Then on page 179 we see a matrix theorem I recall as Theorem 4B in my matrix binder, which says

$\partial|A|/\partial A_{pq} = \text{cof}(A_{pq})$

which is trivial from the usual expansion of |A|. Then we can finish it by using

$(A^{-1})_{pq} = \text{cof}(A^{T}_{pq})/\det(A) = \text{cof}(A_{qp})/\det A$

so then

$\partial|A|/\partial A_{pq} = \det A(A^{-1})_{qp}$

Now if A is a field A(**X**) then

$\partial|A|/\partial X^{i} = (\partial|A|/\partial A_{pq})(\partial A_{pq}/\partial X^{i}) = \det A(A^{-1})_{qp}\ (\partial A_{pq}/\partial X^{i})$ // sum on p and q (4.10.22)

He then shows that (4.10.23) is true, something I never ran into in tensor doc:

$S_{ik} \equiv (\partial x_i/\partial x'_k)$

$\partial_i[S^{i}_{k}/\det(S)] = 0$

4.11 Equations of Motion (179)

A very clean section. Recall the analogy of the inverse flow where x = x = x-space, x' = X = X-space = t_0.

$\mathbf{div}\mathbf{T} + \rho\mathbf{B} = \rho\mathbf{a}$ $[\text{div}T]^{i} = \partial_j T^{ij} = \partial T^{ij}/\partial x^{j}$

and end up with

$\mathbf{div}\mathbf{T}_o + \rho_o\mathbf{B} = \rho_o\mathbf{a}$ $[\text{div}T_o]^{i} = \partial^{(o)}{}_j T_o{}^{ij} = \partial T_o{}^{ij}/\partial X^{j}$ // $T_0 = J\ T\ (F^{T})^{-1}$

Where T_0 is the 1st PK tensor, aka the Lagrangian tensor. I can see how this equation might be easier to work with in an oscillation / wave problem since ρ_0 can then be the constant density of the unperturbed medium, whereas in the first equation it would be a function of time. Lai has no such comments.

4.12 Stress Power (180) **continue review here**

The math is all perfect, but words are needed. Look at the little cube. There is a flow velocity field **v**(x) active at all points in this cube. For example, the face on the right has **v** = **v**(x_1+dx_1, x_2, x_3) in our usual approximation (as much as gets the job done). The face on the right has some force on it and is moving at this velocity. Both vectors are to the right if we use $\mathbf{t}_{e1}$ so quantity $\mathbf{v} \bullet \mathbf{t}_{e1}$ will be positive if vectors are as shown and **v** is to the rightish. Pause: imagine force pushing on a friction block, force on block is same direction as velocity of the block, positive work is there done against friction by the applied force. So positive work is done by the stress force $\mathbf{t}_{e1}$ on something! This is power, rate of doing work.

But what about points inside the cube? Why don't they contribute to the power? OK, it is the outside of the cube medium that is applying the stress force, and it is doing work on the cube. The outside

medium does not touch points inside the cube so cannot do work on such points. It has to mediate everything through the surface. So we end up with this result

$$dP/dV = \rho v_i \, d_t v_i + T_{ij} \partial_j v_i \; = \; \rho \, \mathbf{v} \bullet \mathbf{a} \; + T_{ij} \, (\nabla \mathbf{v})_{ij} = \rho \, \mathbf{v} \bullet \mathbf{a} \; + \mathrm{tr}[T^T (\nabla \mathbf{v})]$$

This is the rate at which the medium does work on the cube, per unit volume, and the result is very strange looking to me! But on page 181 it is all cleared up! First,

$$(\rho dV) \, \mathbf{v} \bullet \mathbf{a} \; = (\rho dV) d_t v^2/2 \; = dm \, d_t v^2/2 \; = d_t(\, (1/2) dm \, v^2 \,) \; = d_t(KE)$$

so this first term represents any increase of the kinetic energy inside the cube caused by the work of the stress forces on the boundary of the cube. That is to say, the kinetic energy OF the cube, considered a regular particle of mass dm and average velocity v as it shrinks to a point. Yes indeedy.

The second term can be written, if T is symmetric,

$$\mathrm{tr}[T^T (\nabla \mathbf{v})] \; = \mathrm{tr}[T^T (\nabla \mathbf{v})^S] = \mathrm{tr}[T^T D] \; = \mathrm{tr}(T^T D^T) = \mathrm{tr}[(DT)^T] \; = \mathrm{tr}(DT)$$

where D (the rate of deformation tensor) is the symmetric part of F = ($\nabla \mathbf{v}$) (the deformation gradient). So this extra term tr(DT) is some increase in energy of the cube due to external stress vector work on the cube which changes its strain configuration! This is of course the potential energy of the cube but Lai does not say that. Think of the "atoms" inside the cube connected by little springs and when you change the shape of the cube, some of these springs stretch and PE changes. Very good. So bottom line:

$$dP = d_t(KE) + \mathrm{tr}(DT) dV \; = d_t(KE) + P_s dV$$

Notice that P_s is a volume power density called "stress power" which would be the total power of work done if the cube were not moving (v = 0). On the other hand, d_t(KE) is the KE of mass dm and so is not a density. Lai writes P instead of my dP. I might think of $P_s = d_t U_s$, the rate of increase of the stored potential energy density in the cube.

4.13 Stress Power in terms of the two PK tensors (181)

I checked every single line and agree with the results. Here are all three results:

$P_s = \mathrm{tr}(TD)$ D = deformation tensor $= F^S$

$P_s = (\rho/\rho_0) \, \mathrm{tr}(T_0^T \, d_t F)$ 4.13.7

$P_s = (\rho/\rho_0) \, \mathrm{tr}(\tilde{T} d_t E^*)$ 4.13.14

where in the last item

E* = Lagrangian finite strain tensor = (C-1)/2 where $C = F^T F$ from page 118 (and p 87)

Recall that P_s is the rate of change of the potential energy due to work by the strains, and it is a density per unit volume, and all we are doing here is writing it in terms of the Cauchy stress tensor, and then the two PK tensors. In the last two formulas, notice that a d_t is acting on the strain tensor, but not in the first formula. [D already is a "rate of" thing.] We are supposed to think of the T,D and T_0,F and $\tilde{T}$, E* as "conjugate pairs". Note also that in the second equation, the T object is transposed, but not in the others.

4.14 Heat flow into a cube (183)

Another excellent simple section. Assume vector field **q**(**x**) which describes heat flow, meaning heat perhaps in Joules per unit area per second, or watts per unit area. Do the usual construction with the little cube (it could of course be a 3-piped, I just call it a cube) to conclude that

dQ_c = -div(**q**) dV = total heat flowing into the cube of volume dV per second

This heat is flowing across the surfaces of the cube, and such heat flow is conductive heat flow! Radiative heating would act on the entire volume, not the surfaces, so is not described by this model. I don't think convection exists in this continuum mechanics picture. The whole particle moves as part of a larger scale convection process, perhaps, but for the particle itself, there would be only conductive and radiative, I think.

The example then supposes that **q** = $-\kappa\nabla\ \Theta$ which Lai calls the **Fourier heat conduction law**. Inserting into the above gives $dQ_c/dV = -\nabla\bullet\ (\kappa\nabla\Theta)$ and finally if κ = constant, $dQ_c/dV = -\kappa\ \nabla^2\ \Theta$. In steady state we get $\nabla^2\ \Theta = 0$ which Stak would write as $\nabla^2 u = 0$ and we have the usual "potential theory" for heat flow in steady state. This same analysis is done in a Stak appendix. [Lai uses Θ for temperature to avoid confusion with T being the stress tensor.]

Lai refers to heat flow as "the rate of heat flow". For me, the rate idea is implied by the word "flow".

4.15 The Energy Equation (184)

Well the notion of "radiative" comes up right here. Imagine a plane wave bathing the cube, and atoms in the cube then radiate in situ due to energy supplied by this incoming radiation. That in situ radiation is then treated as a local source of heat and is then part of Q_s in 4.15.1. Non-absorbed radiation passes through the cube and does not register in the heat equation.

What does 4.15.1 say? LHS is the increase in the internal stored energy, potential plus kinetic. On the right sides is work done ON the cube by the stresses (which we computed earlier), and heat flow into the cube by conduction Q_c. The final term Q_s picks up all other heat sources within the cube, such as the radiation effect just described, or perhaps some little resistor inside the cube supplied with current, etc. It is just a "source" in the general PDE Stak sense. The capital letter quantities in 4.15.1 are all totals for the cube, they are not per unit mass or per unit volume. [Note that symbol $P = P_s + d_t(KE)$.]

We then say $Q_s(E) = q_s(E/M)dm(M) = q_s(\rho dV)$ so after dV is cancelled, the last term in 4.15.5 is just ρq_s where q_s is the scalar heat generated per second per mass by sources inside the cube. The term which is tr(TD) stays as is and dV is cancelled. Same for div(q). The LHS does this

$$d_t(U) = d_t(u\ dm) = dm\ d_t u \quad \text{where } \rho dV = dm = \text{constant} \qquad 4.15.3$$

so this term ends up as $\rho dV\, d_t u$ and then again dV is cancelled so the ρ hangs on there in 4.15.5

$\rho\, d_t u = tr(TD) - div\mathbf{q} + \rho\, q_s$ u = stored energy per unit mass

Notice that the kinetic energy terms on the two sides cancelled out early on in this little discussion. The above result says that, per unit mass, the increase in stored energy in the cube equals the sum of the work done on the cube + the heat the conducts in + the heat generated within.

So this is the general statement of energy conservation in continuum mechanics! Earlier we saw the general statement of linear momentum conservation which was the F = ma Cauchy equation. Somewhere there must be lurking an angular momentum conservation equation.

4.16 Entropy (184)

Time for a little external review. My baby Reif notes of 2007 are I think enough. Perhaps we can regard the surrounding environment of a particle as a local heat reservoir at some temperature T, Lai calls Θ. If we add heat dQ to a particle, its entropy changes by dS = dQ/T as explained in the notes.

Translating to Lai, that becomes $d\eta = dQ/\Theta$. If we have a heat flow vector $\mathbf{q}$, it is associated with an entropy flow vector maybe call it $\boldsymbol{\eta} = \mathbf{q}/\Theta$. By the same argument leading to 4.14.1, perhaps we can write

$N = -div(\boldsymbol{\eta})dV = -div(\mathbf{q}/\Theta)dV$

as the rate of increase of entropy in the cube *due to* heat flow into the cube. Another source of entropy increase is due to any heat sources inside the cube, and those then contribute q_s/Θ.

I am not doing this exactly, but the result is 4.16.2 which says

$\rho d_t \eta \geq -div(\mathbf{q}/\Theta) + \rho(q_s/\Theta)$

Lai says very little in this tiny 1/3 page section where we make our first contact with this thermo concept of entropy, always a shock for the student who "hasn't done it for a while" (in my case, 5 years). Some questions come to mind:

(1) is the inequality due to the fact that it is an equality only for a reversible process?

(2) why is there no term like tr(TD) in the energy equation? That term is work done by the stress forces acting on the cube. That is an energy term.

The index of Lai suggests that this topic plays little role in the rest of the book, we shall see. I could redo the above more carefully in terms of "per unit mass" or "per unit volume" or "total for the cube".

Page 185 ends with a small 1D example where we do a boundary problem using the 1D Laplace equation and we quickly find the temperature distribution in our little "rod". If we are then sitting in steady state, meaning temperature is stable everywhere (heat is flowing though!), we have no entropy change going on within the rod, and we assume no heat sources. Just plugging things in produces the conclusion $\kappa \geq 0$ and this implies that in the Fourier conduction formula, heat flows from higher T to lower T, something we seem to know about.

4.17 Helmholtz Energy (186)

My same baby Reif notes talk about H = E-TS as one of the functions of interest, the Helmholtz energy. Lai writes this as $A = u - \Theta\eta$ where we are now on a per mass basis. Lai then restates the previously stated entropy inequality with η replaced by A. We end up with a pretty meaningless result 4.17.4 which I think they just want to "wedge" into their book, I know the desire.

The example is OK to equation 4.17.7. I think Lai is saying: this inequality must be true for any values of quantities D_{ij} and $d_t\Theta$. One possible value for these two quantities is 0, and then 4.17.8 applies and the rest follows. This example is less interesting to me than the previous one.

4.18 Integral Formulations of stuff in this chapter (187)

Lai puts the four "principles" into blob-integral form, very enlightening. I just call each principle by its "conservation of" name. In each case, Lai *starts* with a an integral statement of the conservation principle, which involves volume and surface integrals of a finite size blob. He then uses the divergence theorem to convert the surface integrals to volume integrals. Then, once everything is a volume integral, he applies it to a differential volume and then equates the integrands and comes up with the differential statement of the same conservation law. These differential statements agree with the laws we have already derived. In the case of angular momentum, the only conclusion I think is that T_{ij} must be symmetric. Lai and Schaum use different approaches to the math for the mass conservation idea, and I outline both below. The Lai approach involves the tricky "squirmy integral" concept, whereas Schaum does not.

(I) Conservation of Mass (187)

The Schaum approach. Lai is unclear for me, so I digress to page 27 of my Schaum book. At some instant in time t we have some matter inside a "control volume" V(t) which is bounded by a "control surface" $\sigma(t)$, these are just names. Both the actual blob volume and surface are changing in time. You then let the flow run a tiny time dt and there is now a new "dotted" surface as shown, but the control stuff is still as it was. There is a velocity field **v** and it of course carries mass. The control volume is fixed in space as the motion occurs. The control volume loses some mass which flows from region A into exterior region B, and gains mass which flows in from outside region A and into region C which lies inside A. Note that C is inside A, whereas B is outside A. By how much does the mass in the control volume change? The answer is this:

$$\text{d(mass in control volume)/dt} = -\int_{c.s.} (\rho\mathbf{v})\bullet\, d\mathbf{A} = -\int_{c.s.} \rho(\mathbf{x},t)\, \mathbf{v}(\mathbf{x},t)\bullet\, d\mathbf{A}$$

where $\rho\mathbf{v}$ is the mass current in mass/area. Now which "view" are we using for these functions? I think it is what I show, things are functions of **x** and not of **X**, because we are integrating over the current **x**. So the above equation can be written

$$d_t\left(\int_{c.v.} \rho(\mathbf{x},t)\, dV\right) = -\int_{c.s.} \rho(\mathbf{x},t)\, \mathbf{v}(\mathbf{x},t)\bullet\, d\mathbf{A}$$

But since the control volume is fixed, I think d_t can be moved in on the LHS so we get

$$\int_{c.v.} d_t\rho(\mathbf{x},t)\, dV = -\int_{c.s.} \rho(\mathbf{x},t)\, \mathbf{v}(\mathbf{x},t)\bullet\, d\mathbf{A}$$

On the left, mass increases in the control volume due to local mass density increases. Now x is a conventional integration variable (non-squirmy see later), and we really mean d_t to act only on the t argument of ρ, so write

$$\int_{c.v.} \partial_t \rho(\mathbf{x},t)\, dV = -\int_{c.s.} \rho(\mathbf{x},\mathbf{t})\, \mathbf{v(x,t)} \bullet d\mathbf{A}$$

Now we can use the divergence theorem on the RHS to say

$$\int_{c.s.} \rho(\mathbf{x},\mathbf{t})\, \mathbf{v(x,t)} \bullet d\mathbf{A} = \int_{c.v.} \text{div}\, \{\rho \mathbf{v}\}\, dV$$

Then we end up with

$$\int_{c.v.} \partial_t \rho(\mathbf{x},t)\, dV = -\int_{c.v.} \text{div}\, \{\rho \mathbf{v}\}\, dV \qquad \text{// Lai 4.18.4 second equality}$$

Then since this must be true for any control volume no matter how small, we get

$$\partial_t \rho = -\text{div}\, \{\rho \mathbf{v}\}$$

and we recover the differential mass continuity formula. [way #2, see p 99 notes for Ch 3 !]

How does Lai express this idea? OK, I had to go off at this point and write a doc called "Time derivatives of differentials and Squirmy Integrals.doc" and this made the Lai stuff all make sense. His approach is a little different from that of Schaum in that he doesn't use a control surface, just a control volume, and no need for the divergence theorem therefore. The main conclusion of my doc is this:

$$I(t) \equiv \int_{v_m} \widetilde{f}(\mathbf{x},t)\, dV(t)$$

$$\Rightarrow\ d_t I(t) = \int_{v_c} d_t[\widetilde{f}(\mathbf{x},t) dV(t)] = \int_{v_c} \{\partial_t[\widetilde{f}(\mathbf{x},t)] + \text{div}\,(\widetilde{f}\,\widetilde{\mathbf{v}})\}\, dV(t)$$

which Lai applies to f = ρ. This was a hump I had to get over, and I can now continue on. Lai's point is that by considering for an arbitrary finite moving volume V_m that the mass stays in the volume, if you shrink that volume down to a point you get the usual continuity equation for mass current.

(II) The Conservation of Linear Momentum (188)

Here we start by writing down $\mathbf{F} = d_t\mathbf{P}$ for a finite blob. The total stress force is $\int \mathbf{t}\, dS$ on the surface of the blob, for example, and of course $\mathbf{t} = T\mathbf{n}$ and then this integral is ready for the divergence theorem and all integrals are on the control volume V_c and just as with pass conservation, out pops the Cauchy equation of motion in its local sense.

Notes added 6.30.12. So this is a direct derivation of the Cauchy equations from linear momentum conservation, maybe pay more attention here. A key step is using "the divergence theorem" in a slightly new way. The theorem in the original form from integral theorems.doc says

$$\int_V dV\,(\text{div } \mathbf{A}) = \int_S d\mathbf{S}\bullet\mathbf{A} = \int_S dS\,\mathbf{n}\bullet\mathbf{A} = \int_S dS\,\mathbf{A}\bullet\mathbf{n}$$

or

$$\int_V dV\,\partial_j A_j = \int_S dS\,A_j n_j$$

Now suppose we take for our vector field A one of the rows of T,

$A_j \equiv T_{ij}$ where i is fixed, so we A_j is then just the ith row of T

Then the above divergence theorem says

$$\int_V dV\,\partial_j T_{ij} = \int_S dS\,T_{ij}\,n_j$$

which we can write in vector notation as

$$\int_V dV\,(\mathbf{div}\,\mathbf{T}) = \int_S dS\,T\mathbf{n} = \int_S dS\,\mathbf{t}$$

So *this* is how **div T** appears in Lai's thread. In 4.18.9 we have $d_t\mathbf{P} = \int_S dS\,\mathbf{t} + \int_V dV\,\rho\,\mathbf{B}$ = the sum of the forces on the blob, and we see how when we use the divergence theorem to convert the surface to a volume integral, we get (**div T**) appearing. Then all three are volume integrals and we just equate the integrands as Lai shows and we end up with the Cauchy equation of motion.

(III) The Conservation of Angular Momentum (189)

Lai refers to this as "the principle of the moment of momentum" which seems a bit old-fashioned. His starting equation here is to say that $\boldsymbol{\tau} = d_t\mathbf{L}$ and he writes down all the torques $\boldsymbol{\tau}$ on the blob relative to some arbitrary origin. A torque is of course **r** x **F** which is called "a moment". The surface torque is the obvious $\int_{CS} \mathbf{r} \times \mathbf{t}\,dS$ and there is then a similar body torque which is a CV integral. This is all very well done by Lai and crew, and we end up with the conclusion that T_{ij} must be symmetric. As before, this assumes there are no "body moment" effects due to non central body forces etc etc.

(IV) The Conservation of Energy (189)

This is handled in a similar fashion. Note the constant use of $d_t(\rho dV) = d_t(dm) = 0$. In each case, he starts with the entire blob, and there are in general volume and surface contributions to "whatever it is". He

converts them all to volume contributions, and then extracts the differential statement of the "principle" of interest.

(V) The non-Conservation of Entropy (190)

Again we connect the differential and integral forms of the "entropy inequality".

Appendix 4.1 (191)

This section provides details of the "maximal shear stress" problem posed in Section 4.6. I appreciate the fact that the authors have included this detail, but I won't read it now, I am happy with the result.

Problems: there are a whopping 46 problems! I think I could do all these problems like ducks in a row, but it would take several hours. Most are just drills in computing things. So as usual, I skip them for now. If you were teaching a course, you would make the kids do at least some of these where they have to get the right answers which are not given.

Comment: This is the farthest I have ever been in this subject. I can see that Chapter 7 is another chapter on "theory" in which we will revisit the general "principles". I think this Reynolds transport thing is just my generalized notion of arbitrary flow currents of whatever you are dealing with. This is the same Reynolds as in the Reynolds number: Osborne Reynolds, Irish, a fluid mechanics guy.

But first, we are going to do two huge subjects in the next two chapters:

Chapter 5: The Elastic Solid

Chapter 6: Newtonian Viscous Fluid (ie, "fluid dynamics")

I hope to learn to do some real-world calculations in both these fields.

Comment 6.30.12. Well, I have now finished Chapters 5 and 6, and I did learn a lot. The time has come now to resume the theory discussion so I am now starting Chapter 7. To that end, I just reviewed all of the above notes for Chapter 4, and it seems to make sense. This included a read through of my squirmy integrals support doc which is needed to understand the Lai method of doing certain things.