Lai Chapter 5 Meta Notes: Elastic Solids PhL 1.5.12

Introduction. 2

5.1 Mechanical Properties (201) -- experiments with steel 3

5.2 The linearly elastic solid (204). 3

PART A: THE *ISOTROPIC* SMALL-DEFORMATION LINEAR ELASTIC SOLID 4

5.3 The *isotropic* linearly elastic solid (207). 4

5.4 The five parameters (two independent) (209). 4

5.5 Equations of Motion (213). 5

5.6 The Navier Equations (215). 5

5.7 The Navier Equations in Curvilinear Coordinates (215). 5

5.8 Superposition (218). 6

A.1 PLANE ELASTIC WAVES. 6

5.9 Plane Irrotational Waves [aka dilatation waves] (218). 6

5.10 Plane Equivolumnal Waves [e = 0] (221). 6

5.11 Internal Reflection of Plane Elastic Waves (226). 6

5.12 Vibration of an Infinite Plate (226). 7

A.2 SIMPLE STATICS SITUATIONS (231). 7

5.13 Simple Extension (231) 7

5.15 Torsion of a Non-Circular Cylinder (239) 8

5.16 Torsion of an Elliptical Cylinder (239). 8

5.17 Prandtl's Formulation of the Torsion Problem (242) 9

5.18 Torsion in a Rectangular Bar (246). 9

5.19 Pure Bending of a Beam (247) 9

A.3 PLANE STRESS AND PLANE STRAIN SOLUTIONS (250) 10

5.20 Plane Strain Solutions (250) 10

5.21 Rectangular Beam Bent by End Couples (253) 11

5.22 Plane Stress Problems (254) 12

5.23 Cantilever Beam with End Load (255). 12

5.24 Simply Supported Beam under Uniform Load (258). 13

5.25 Slender bar demonstration of St Venant (260). 13

5.26 Conversions between plane strain and plane stress solutions (262) 13

5.27 Polar Coordinates (264). 13

5.28 Stress Distribution Symmetrical about an axis (265). 14

5.29 What displacement structure is implied by the previous section (265) 14

5.30 Thick walled cylinder with pressures in and out. (267) 14

5.31 Bending of a annular curved beam (268) **15**
5.32 Initial Stress in a Welded ring (270) **15**
5.33 2D Polar atomic forms for Airy Functions **15**
5.35 Hole in Plate under Pure Shear (276) **16**
5.36 Wedge Loaded at Apex (277) **17**
5.37 The Flamant Problem (277) **17**
A.4 3D ELASTOSTATIC POTENTIAL THEORY (279) 17
5.38 Fundamental potential functions for elastostatic problems (279). **17**
The Ten Examples **18**
5.39 Delta Force at center of full infinite 3D med. (290). [Kelvin Problem] Example 3 ..... **19**
5.40 Delta Force at face center of *half*-infinite 3D med. (293). [Boussinesq Prob] Example 4 . **19**
5.41 Distributed normal stress on *half*-infinite 3D medium (296). **20**
5.42 Thick spherical shell with inner and outer pressures (297). Example 5 ..... **20**
5.43 Spherical Hole in 3D axial stressed material (298). Example 8 ..... **20**
5.44 The cylindrical indenter problem (300). Examples 4 and 10 ..... **20**
5.45 The general and spherical indenter problems (302) **20**
Appendix 5A.1 Solving the integral equation used in Section 5.45 on general indenters. **20**
Part A Problems **20**
PART B: THE *ANISOTROPIC* LINEAR (SMALL DEFORMATION) ELASTIC SOLID (319) 21
5.46→5.55 Hooke's Law for anisotropic cases (aka constitutive equations) (319) **21**
PART C: THE *ISOTROPIC* LARGE DEFORMATION OF ELASTIC SOLIDS 21
5.56 Change of Frame (334) **21**
5.57 Constitutive Equation for Elastic Medium under large deformation (338) **24**
5.58 Constitutive Equation for Isotropic Elastic Medium (340) **26**
5.59 Simple Extension of an incompressible isotropic elastic solid (340) **27**
5.60 Simple Shear of an incompressible isotropic elastic block (343) **27**
5.61 Bending an incompressible isotropic rectangular bar (344) **28**
5.62 Torsion and tension of an incompressible isotropic cylinder (344) **28**
Appendix 5C.1 (p349) . The most general form of F in T = F(B) for isotropic solid. **29**
Part C Problems (p 351, Problem 5.104 through 5.113) **29**

Introduction. Chap 1 did the math, Chap 2 did the kinematics of strain like E_{ij}, Chap 3 did stress T_{ij}, and Chap 4 did the conservation principles. Now in Chap 5 we explore Hooke's law,

$F_i = -ku_i$ where $u_i = x_i - X_i$, => $E_{ij} = -\delta_{ij}$ $\partial_a^{(x)} u_b = -1$

which appears in the form (k above = spring constant, C below = **elasticity tensor**)

$T_{ij} = C_{ijab}\, E_{ab}$ where $E_{ab} = (\partial_a^{(X)} u_b + \partial_b^{(X)} u_a)/2 = (\nabla \mathbf{u})^S_{ab}$

Recall that a prescribed displacement field $u_b(\mathbf{X})$ results in a known strain field E_{ab} as shown. For the 3D spring case, the prescribed u_b field is very simple, so E_{ab} is very simple, again as shown. In general at a point in space strain has 9 components $E_{ab}(x)$ and so does stress $T_{ij}(x)$. If material is homogeneous, then C_{ijab} is a set of 81 constants!

An equation which relates T to E as shown above provides a *model* for the medium of interest and is called a **constitutive equation**.

We shall here only be interested in the linear part of the hysteresis curve, and that defines the subject of elastic solids, the title of Chapter 5. Plan of this chapter:

Part A: isotropic
Part B: anisotropic
Part C: non-linear (large deformation)

Here are the strain-stored-energy equations for the two cases (no derivation yet)

$U(u) = (1/2)ku^2$ $F = -dU/du = -ku.$
$U(E) = (1/2)\, C_{ijab}\, E_{ij}E_{ab}$ $T_{ij} = \partial U/\partial E_{ij} = C_{ijab}\, E_{ab}$

5.1 Mechanical Properties (201) -- experiments with steel

For axial tensing $\varepsilon_a \equiv \Delta\ell/\ell$ and we find that, analogous to F = -ku, we have $T_{33} = [E_Y]\, \varepsilon_a$ so **Young's Modulus** plays the role of spring constant except instead of u (an absolute elongation), we use ε_a which is a dimensionless fractional elongation. The result is that E_Y has the same units as T_{33} which is psi or mP. Here T_{33} is positive and so is ε_a.

The same experiment results in $\varepsilon_d = \Delta D/D$ (negative) for rod diameter, and $\nu = (-\varepsilon_d)/\varepsilon$ is called **Poisson's Ratio** and is on the order of magnitude of 1. Certainly $\nu > 0$ since the rod is not going to expand in cross section when you stretch it. Later we learn that in fact $\nu \leq 1/2$.

A hydrostatic experiment sees $T_{33} = k\, e$ where $e = \Delta V/V$ and e is the **dilatation**. For this experiment, T_{33} and e are both negative. k is called the **bulk modulus**.

A final experiment concerns torquing a rod, see raw notes where μ is a **shear modulus**.

Since ε_a is typically only .001, we are allowed to use the E strain tensor in place of E* or F. Recall that E is the infinitesimal strain tensor defined as $E = (\nabla \mathbf{u})^S$ on page 87. The general idea is that for all solids, not just steel, if you elongate more than a very small amount, you get into the plastic region and we are no longer doing elastic solids. For this same reason, roughly $\partial_i^{(X)} = \partial_i^{(x)}$.

[So now using E everywhere, and we are in the infinitesimal regime until section C below!]

5.2 The linearly elastic solid (204).

In the opening raw notes, I try to see how many of the 81 constants are really independent. Lai claims there are 21 but I was not quite above to verify that, more info is needed.

In Example 5.2.1 Lai reminds us that $D_{ij} = (\nabla^{(x)}\mathbf{v})^{S}_{ij}$ was the "rate of deformation tensor" and in our elastic solid approximate world he shows that $D_{ij} = d_t E_{ij}$. Part b shows that $P_s = d_t U$ so that all the stress power goes into stored energy.

Example 5.2.2 talks the U potential energy function a bit more, but still no derivations. But I do agree that if $T_{ij} = C_{ijab}\, E_{ab}$, then $C_{ijab} = \partial T_{ij}/\partial E_{ab}$.

PART A: The *Isotropic* Small-Deformation Linear Elastic Solid

5.3 The *isotropic* linearly elastic solid (207).

By considering the most general rank-4 isotropic tensor, Lai shows that for an isotropic elastic solid, one must have $C_{ijk\ell} = (\lambda\delta_{k\ell}\delta_{ij} + 2\mu\delta_{ik}\delta_{j\ell})$ so the 81 or 21 constants are reduced to just 2 constants λ and μ called the **Lamé constants**. We then get

$T_{ij} = \lambda \mathrm{tr}(E)\delta_{ij} + 2\mu E_{ij}$	with tr(E) = e from earlier work.
$T_{ii} = (3\lambda+2\mu)\,\mathrm{tr}(E)$	// relation for diagonal elements
$2\mu E_{ij} = T_{ij} - \lambda \mathrm{tr}(T)\,\delta_{ij} 1/(2\mu+3\lambda)$	// inversion of the above
$2\mu E_{ii} = [(1-3\lambda)/(2\mu+3\lambda)]\, T_{ii}$	// relation for diagonal elements
$\mathrm{tr}(T) = (2\mu+3\lambda)\mathrm{tr}(E) = (2\mu+3\lambda)e$	// from first line above

In Example 5.3.1 Lai shows that matrices T and E in this case have the same eigenvalues, eigenvectors, principle axes, and all that stuff. If one is diagonal, so is the other. The second Example derives the inversion formula shown above.

5.4 The five parameters (two independent) (209).

We now look back at our steel experiments armed with our isotropic model knowledge and we learn new things. First, from our axial tensing experiment: (I give fast proofs in the raw notes)

State of uniaxial stress: $T_{ij} = \{ T_{11}, \text{all else } 0 \}$

$e_a = E_{11}$ so $E_Y = T_{11}/E_{11} = \mu(3\lambda+2\mu)/(\lambda+\mu)$

$e_d = -E_{22}$ so $\nu = -\varepsilon_d/\varepsilon_a = -E_{22}/E_{11} = (1/2)\lambda/(\mu+\lambda)$

Recall from Chap 2 how we interpreted the diagonal elements of E_{ij} just as shown above! So now we have the connection between the historically famous Young Modulus E_Y and Poisson Ratio ν and our generic isotropic parameters λ and μ. Here are the inversions:

$\mu = (1/2)E_y/(\nu+1)$ and $\lambda = -\nu E_Y / (2\nu^2+\nu - 1) = -\nu E_Y / [(2\nu - 1)(\nu+1)]$

5.4.14 Lai never states this one

State of simple shear: $T_{ij} = \{ T_{12} = T_{21}, \text{all else } 0 \}$

$T_{12} = \tau = G\,\Delta\theta$

$G = \mu$ = shear modulus

and here we see that we can interpret the Lamé parameter μ as the shear modulus G, supporting a symbol name used earlier in the chapter.

State of hydrostatic stress: $T_{ij} = T_{11}\,\delta_{ij}$ => $tr(T) = 3T_{11}$ //$T_{11} = -p < 0$

$T_{11} = k\,e = k\,tr(E) = k\,tr(T)/(2\mu+3\lambda) = k3T_{11}/(2\mu+3\lambda)$

=> $1 = 3k/(2\mu+3\lambda)$ => $k = (2\mu+3\lambda)/3 = \lambda + (2/3)\mu$

$k = \lambda + (2/3)\mu$ = bulk modulus

We now have 5 constants λ, μ, E_Y, ν, k but only two are independent.

If you pick a pair to be the independent pair, the other three are functions of those two, see **Table** on page 212. Since E_Y and ν are always finite values, the Table shows that $0 \le \nu \le 1/2$. In the limit $\nu \to 1/2$, we get $k \to \infty$ and solid is then incompressible and in that limit $\mu = E_Y/3$ it happens.

Table on page 213 shows the 5 parameters for various elastic solids. Rubber is incompressible and takes those values ν = 1/2 and k = ∞. I give a little explanation of this in the raw notes.

5.5 Equations of Motion (213).

We already knew from **F** = m**a** that $\rho\mathbf{B} + \mathbf{div}\mathbf{T} = \rho\mathbf{a}$. In our small displacement approximation model, we have that $\mathbf{a} = \partial_t^2\mathbf{u}$ and $\rho = \rho_0$ so we end up with $\rho_0\partial_t^2\mathbf{u} = \rho_0\mathbf{B} + \mathbf{div}\mathbf{T}$ where T = C E. The term divT contains 2nd order spatial derivatives of u_i so we end up with a system of three linear PDE's where variables are t and x_i, and where derivatives are 2nd order in both. In general, this is very complex PDE system. A candidate solution **u** is called "a possible motion" if it satisfies the above equation or motion. An actual solution has to also satisfy various boundary conditions. If some candidate **u** fails to satisfy the above, you can rule it out *a priori* regardless of the BC's.

5.6 The Navier Equations (215).

If we use the isotropic $T_{ij} = \lambda tr(E)\delta_{ij} + 2\mu E_{ij}$ and from this compute divT, we get the Navier-Cauchy equation which says

$$\rho_0\partial_t^2\mathbf{u} = \rho_0\mathbf{B} + (\lambda+\mu)\,\mathrm{grad}(\mathrm{div}\,\mathbf{u}) + \mu\nabla^2\mathbf{u} = \rho_0\mathbf{B} + (\lambda+\mu)\,\mathbf{G} + \mu(\mathbf{G}-\mathbf{V}) = \rho_0\mathbf{B} + (\lambda+\mu)\,\nabla e + \mu\nabla^2\mathbf{u}$$

Since e = div **u**, you can write grad div **u** = ∇e as shown. In tensor doc I deal with

$$\nabla^2\mathbf{u} \equiv \mathrm{grad}(\mathrm{div}\,\mathbf{u}) - \mathrm{curl}\,(\mathrm{curl}\,\mathbf{u}) = \mathbf{G} - \mathbf{V}$$

so the (λ+μ) term is in fact the first term in $\nabla^2\mathbf{u}$, and I know quite a lot about that term. Even in the isotropic case with infinitesimal assumptions, this is still a *very* messy equation. At least it is linear, and at least it is 2nd order in spatial derivatives. Page 215 shows these equations in Cartesians.

5.7 The Navier Equations in Curvilinear Coordinates (215).

Lai writes two sets of equations here. For cylindricals and then for sphericals he writes

$T = \lambda\, \mathrm{div}\mathbf{u}\, 1 + \mu\, [(\nabla \mathbf{u}) + (\nabla \mathbf{u})^{T}]$ // expression for T

$\rho_0 \partial_t^2 \mathbf{u} = \rho_0 \mathbf{B} + (\lambda+\mu)\, \mathrm{grad}(\mathrm{div}\, \mathbf{u}) + \mu \nabla^2 \mathbf{u}$ // Navier equation

It took me about 7 days to have Maple compute all this stuff in these two systems, but of course now I can easily compute the above in any system I want.

5.8 Superposition (218).

The usual claim is well stated here concerning superposition for a linear PDE. What is not stated is that the boundary conditions on the PDE must also be of a corresponding superposed nature.

A.1 Plane Elastic Waves.

5.9 Plane Irrotational Waves [aka dilatation waves] (218).

If $E_{ij} = \begin{pmatrix} E_{11}(x_1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ we find that $T_{ij} = \begin{pmatrix} (\lambda+2\mu)E_{11} & 0 & 0 \\ 0 & \lambda E_{11} & 0 \\ 0 & 0 & \lambda E_{11} \end{pmatrix}$ and the Navier equation simplifies to say $\rho_0 \partial_t^2 u_1 = (\lambda+2\mu)\partial_1^2 u_1$ which is a 1D longitudinal wave equation in the 1 direction with speed $c_L^2 = (\lambda+2\mu)/\rho_0$. A possible motion is $E_{11}(x_1) = \alpha \sin(kx_1-\omega t) + \beta\cos(kx_1-\omega t)$ and **Example 5.9.1** then computes various aspects of this solution given certain boundary conditions at $x_1 = 0$ (first BC is a sine displacement there, the second BC is a sine traction there). For this kind of wave, it turns out that the dilatation e oscillates, but the spin tensor $\mathbf{W} = 0$ hence the name "irrotational" (E is symmetric).

5.10 Plane Equivolumnal Waves [e = 0] (221).

For this kind of wave, e = 0 so $\nabla e = 0$ so Navier = wave equation. The displacement candidate form is $u_2(x_1)$ which forces $e = \mathrm{div}\, \mathbf{u} = 0$. Obviously this is a transverse shear wave (travels in 1 direction) and now the speed is just $c_T^2 = \mu/\rho_0$ which is slower (in phase velocity) than c_L. **Example 5.10.1** does a sample solution for this kind of wave. **Example 5.10.2** provides a fancier shear wave example where we have $u_3(x_1,x_2)$ with fixed profile in x_2. This is a waveguide type solution, see raw notes for details. All three spatial dimensions are involved, wave moves still in the 1 direction at an intermediate speed. In **Example 5.10.3** Lai gets the L and T waves into standard form allowing for arbitrary direction $\mathbf{e}_n$. The L wave has **u** in this direction, while the T wave has u in some $\mathbf{e}_t$ direction. Wave amplitude is taken as ε to suggest small amplitude. In messy **Example 5.10.4** the reader is asked to write out u for each of the three waves shown in the p 225 picture, where two are T and one is L. This is a big setup for the reflection analysis to come in the next section.

5.11 Internal Reflection of Plane Elastic Waves (226).

One example is worked out here in full detail: reflection at an internal boundary of an elastic solid where the incoming plane wave is a T wave with normal in the plane of paper. By requiring that the total surface traction t = 0 on the outside face of the boundary, the problem can be completely solved. One finds a

usual mirror type reflected T wave, but the big surprise to me is a second L type reflected wave at some Snell-like angle (and of course no refraction since there is no medium outside the boundary). This angle is determined by an index $n = c_T/c_L = (\mu/[\lambda+2\mu]^{1/2} \leq 1$. If the incoming T wave had polarization out of the plane of paper, there is no additional reflected L wave. Finally, if the incoming is an L wave, you get two reflected waves again, one the L at the usual angle, and the other a T at some Snell angle. In this case, I think in all cases for straight-on incidence you get 100% reflection and no third wave.

5.12 Vibration of an Infinite Plate (226).

You can think of this situation as 100% reflection and standing waves, but things are solved in the usual manner with various BC's: **u** = 0 on right face, perhaps **u** = sine or **t** = sine on the left face. **Example 5.12.1** does L wave and **5.12.2** does T wave. The k vector gets quantized so there are resonance frequencies ω_ℓ. If your driving frequency ω hits a resonance, amplitude blows up in this undamped model.

So this concludes our small amplitude waves exploration of elastic solids, a nice little introduction. These waves are dynamic solutions of the Navier-Cauchy equation. I think the small amplitude is just to make Hooke's law reasonable, the higher terms don't matter much. Well, it also justifies using the E tensor as strain characterization, and also justifies things like $\rho \approx \rho_0$ in some places and $\mathbf{x} \approx \mathbf{X}$.

We are now going to enter the world of the Civil Engineer I suspect. Beams and bending and all those great things. Lots of trade terms start appearing, like a "prismatic bar" which means an extruded shape. Lots of "moments" will appear, the word torque is never used.

A.2 Simple Statics situations (231).

5.13 Simple Extension (231)

Here Lai formally treats the "**state of axial stress**" of a prismatic bar (also known to civil engineers as "simple extension") as a proper static solution of the static Navier equation $\partial_j T_{ij} = 0$. One way to write his solution is $u_1 = (\sigma/E_Y)\,(x_1-x_{1c})$ and $u_2 = -(\nu\sigma/E_Y)\,(x_2-x_{2c})$ where I have adjusted constants so that $u_1 = 0$ at the center of the bar, and $u_2 = 0$ on its center line. Then u_1 shows the increasing axial stretch as you go either way, and u_2 shows the transverse shrinkage as you move off the center line, showing that the bar gets thinner as you pull on its ends. We find the proper Poisson Ratio ν.

Along the way, Lai takes the diagonal E_{ij} and "integrates it" to find the u_i using a very interesting functional form method reminiscent of my Stackel paper methods. But he gets "extra terms" so the general solution does not agree exactly with what I show above. It turns out that these extra terms are rigid translations (could also be rotations in a general problem). I prove a **theorem** that such **rigid body extra terms in a solution u have an *antisymmetric* ∇u and therefore contribute nothing to E**. An example is that in my solution above, the center point of the bar stays fixed, but if you required the left end center point instead to be fixed, you would add an overall bar lengthwise translation to one solution to get the other one. I show in detail that ∇u is antisymmetric for a rigid rotation and vanishes for a translation.

Saint-Venant's Principle [1855] is stated and I have some great web pictures to demonstrate it. An example is a hydraulic chuck that obviously does not apply a uniform normal traction on the end face of a bar, but away from the chuck it is as if that were the case.

5.14 Torsion of a Circular Cylinder (234)

An impressive section. Lai first makes the "twisted stack of coins" ansatz for **u** as in 5.14.1 where the twist angle is assumed some $\alpha(x_1)$ with x_1 along the cylinder. With x_1 along the bar, this ansatz **u** has u_2 and u_3 components (shear only, $u_1 = 0$). From this he computes E and then T which have this form

$$E_{ij} = \begin{pmatrix} 0 & E_{12} & E_{13} \\ E_{21} & 0 & 0 \\ E_{31} & 0 & 0 \end{pmatrix} \qquad T_{ij} = \begin{pmatrix} 0 & T_{12} & T_{13} \\ T_{21} & 0 & 0 \\ T_{31} & 0 & 0 \end{pmatrix}$$

In order to satisfy static Navier $\partial_j T_{ij} = 0$, the angle function $\alpha(x_1)$ must be linear $= \alpha' x_1$. Lai shows there are then no tractions on the bar sides, and he computes the torque (moment) on the end faces from the surface traction there resulting from the above equations. The moment of inertia $I_p = \pi a^4/2$ of a disk is involved here. He then treats the twisting moment M as the "given" and in terms of M, ℓ and radius a the problem is completely solved. Next, he considers the problem of diagonalizing matrix T_{ij} because then you can know the maximum normal and shear force in the bar using our developed methods. The eigenvectors of T are the special directions discussed earlier, and are 45 degree planes in this case. Maximum normal and shear stress is at the outer edge, and we expect an elliptical crack (they say helical) to form if you "twist the cucumber".

In final Example 5.14.2 he notes that you could also have a sine solution for $\alpha(x_1)$ and then you would have "torsion waves" in the bar, which are of course entirely shear waves in nature. So here Lai gives me enough detail that would allow me to do a practical torsion problem with a bar.

Notice that the "coins" remain planar in the above solution since $u_1 = 0$.

5.15 Torsion of a Non-Circular Cylinder (239)

When the cross section is not simply a disc, things are more complicated. In addition to getting rotated relative to neighbors, the "coins" now get identically warped out of their starting planes according to some "**warping function**" $u_1 = \varphi(x_2,x_3)$ (same for all x_1 along bar). Given the cross section shape, it turns out that $\varphi(x_2,x_3)$ is the solution of a 2D Laplace equation Neumann problem where $\partial_n\varphi$ is prescribed by the requirement that there be no surface traction on the bar sides! The twist angle function $\alpha(x_1)$ still comes out being linear. Again, we are solving static Navier $\partial_j T_{ij} = 0$. This problem is called **St-Venant's Torsion Problem**. I compare this to a membrane drumhead problem. The finally solution for **u** is as shown in 5.15.1 so the only problem is to find the warping function.

5.16 Torsion of an Elliptical Cylinder (239)

This is an application of the previous section. Lai "guesses" the Laplace solution $\varphi = Ax_2x_3$ for the warping function and then finds A that makes it all work. He computes the end face traction and states the problem solution in terms of the twisting moment M_t. He goes on to diagonalize T and finds that the location of maximum stress occurs on the cylinder surface at the *minor* axis radius (where there is less

flex available), so that is where it will crack. These are excellent examples! Notice that α' falls out as in 5.16.10, so we know α and thus we have the complete solution **u** based on 5.15.1.

5.17 Prandtl's Formulation of the Torsion Problem (242)

The twisted bar problem we have seen above has only T_{12} and T_{13} being non-zero, so we can think of these forming a 2D vector in the cross section plane $\mathbf{Q} = (Q_2,Q_3) = (T_{12},T_{13})$. Prandtl's idea is to write the components of this vector in terms of a "potential" ψ such that $T_{12} = +\partial_3\psi$ and $T_{13} = -\partial_2\psi$. This is similar to the notion in 2D (x_2, x_3) electrostatics that $\mathbf{E} = -\nabla V$, except the 2,3 components are swapped and there is a minus sign, which means basically that the vector **Q** with respect to potential ψ is rotated 90 degrees relative to **E** with respect to potential V (see raw notes comments on this at end of section). The upshot is that vector **Q** will be *tangent* to the lines of constant ψ, which says that the shear stress in a cross section is tangent to ψ at any point, a very useful fact. Furthermore, Prandtl shows that ψ must satisfy a 2D Poisson equation (similar to V) with a certain constant source term as in 5.17.5. So in Prandtl's world, the St-Venant's Torsion Problem reduces to solving a 2D Poisson equation in ψ driven by a negative constant, so it is just like a 2D drumhead where the drum volume is a vacuum so the membrane goes down with max deflection in the central region and ψ = 0 on the rim. The membrane contours tell you where the shear stress vector points at any point. So the plan here is to first solve for ψ, then take the above derivatives and you then known (T_{12},T_{13}), and then work backwards from that to get E and finally **u**. Notice that ψ is the same function at any value of x_1 along the bar. For general cross section, you will presumably end up with u_1 being the same warping function as found by the Neumann/Laplace method discussed above.

5.18 Torsion in a Rectangular Bar (246)

Here Lai uses the Prandtl method for a bar of rectangular cross section. Lai expands ψ in $\cos(kx_2)$ as a complete set in the x_2 direction and uses Stak's partial eigenfunction method to determine the coefficients. We get a simple ODE for these coefficients $F_n(x_3)$ and final ψ is shown 5.18.10. I am very familiar with this kind of analysis so I did not derive all the equations. Lai goes on to compute from ψ the max shear stress and the end moment M (both these are assigned as Problems). For a very thin rectangle, an approximate solution is noted. As with the ellipse, cracks will first form at the middle of the *wide* side which is like the minor axis of the ellipse.

5.19 Pure Bending of a Beam (247)

In the simple axial load problem we know there is only T_{11} and all other $T_{ij} = 0$. Imagine morphing the end surface tractions keeping them normal but allowing them to vary in the x_3 vertical direction in some simple manner to cause an end couple which will maybe bend the bar down. It turns out that the static Navier solution in this case is this end-applied $T_{11}(x_3)$ but at any cross section, and $T_{11} = Mx_3/I_{22}$ where I is a certain geometric property of the bar cross section. Along the way, Lai notes that T_{11} has to be linear in x_2 and x_3 in order to satisfy the "compatibility equations" for E_{ij}, and this leads to his ansatz $T_{11} = ax_3$ for our case. From this T_{11} solution he then computes E_{ij} and finally **u** as shown in 5.19.14. I have drawn a little picture showing this displacement **u**, quite reasonable. You can then ask about how the centroid line varies and the result is $u_3(x_1) = Mx_1(\ell - x_1)/(2E_YI_{22})$ with max deflection in the center by an

amount $u_{3,max} = -M(\ell/2)^2/(2E_Y I_{22})$. Recall that $\varepsilon_a = T_{11}/E_Y$ for the axial load problem, so E_Y is a measure of axial stretchability. Larger Young means less stretchable, so pure bending bends less. You could then think of large E_Y as meaning more stiffness against bending. For steel, E = 200 GPa, but for rubber E = .002 GPa table p 213. In order for a beam to bend, it must stretch on one side and compress on the other side.

A.3 Plane Stress and Plane Strain Solutions (250)

5.20 Plane Strain Solutions (250)

"Plane strain" is a classification pigeonhole for certain static elastic problems. I found the presentation to be not clean, despite the fact that the authors did a complete rewrite on this section since 3rd edition. But I think the following ideas are part of the "plane strain" scenario:

(1) The E and T matrix for the elastic solid take the following form:

$$E_{ij} = \begin{bmatrix} E_{11} & E_{12} & 0 \\ E_{21} & E_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad T_{ij} = \begin{bmatrix} T_{11} & T_{12} & 0 \\ T_{21} & T_{22} & 0 \\ 0 & 0 & T_{33} \end{bmatrix} \qquad \text{where } T_{33} = \nu(T_{11} + T_{22}) \Rightarrow E_{33} = 0$$

where it is assumed that all E_{ij} and T_{ij} are functions only of x_1 and x_2.

(2) Think of the solid as being a rectangular solid aligned with the 1,2,3 axes. Think of a "slice" through this solid at some fixed value of x_3. For a bar, this might be an axial slice, or it might be a longitudinal slice, depending on how you label your axes. The fact that $E_{33} = 0$ is directly connected with the fact that $u_3(x_1,x_2) = 0$ for a plane strain solution. This says that any x_3= constant cross section does not warp into a potato chip, but rather remains planar. Also, the story in each cross section is the same! Notice that the stress T_{33} in any slice is the same and must be present to prevent this potato chip warping! Thus, the end cases in the 3 direction must have "frictionless walls" to support the T_{33} which appear there.

(3) the upper-left quad of T_{ij} elements can be represented as derivatives of an **Airy function φ** in this way

$$T_{11} = \partial_2^2 \varphi$$
$$T_{22} = \partial_1^2 \varphi$$
$$T_{12} = -\partial_1 \partial_2 \varphi = T_{21}$$

and any function φ which satisfies the **biharmonic equation** $\nabla^4 \varphi = 0$ provides a viable plane strain solution. The above forms for T_{ij} imply, via Hooke, certain forms for E_{ij}. It is the requirement then that these derived E_{ij} forms satisfy the compatibility conditions that results in $\nabla^4 \varphi = 0$.

(4) If you consider this possible stress solution with $T_{33}(x_1, x_2)$,

$$T_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & T_{33} \end{bmatrix}$$

then if you compute the form that E_{ij} then has, and if you require that E_{ij} to satisfy the compatibility equations, you find that T_{33} must have the form $T_{33} = \alpha x_1 + \beta x_2 + \gamma$ (it is linear). In this case, T_{33} is of course the same anywhere in the body. A special case of this is the simple axial stress situation. From p 210 Hooke's Law, the above T matrix implies a diagonal E matrix of this form:

$E_{11} = (-\nu/E_Y)T_{33}$
$E_{22} = (-\nu/E_Y)T_{33}$
$E_{33} = (1/E_Y)T_{33}$

(5) In a plane-strain solution, one must have $T_{33} = \nu(T_{11} + T_{22})$ in order to make $E_{33} = 0$. If we then insert this value of T_{33} into Hooke's Law on page 210 and write things in terms of ν and μ, we get

$E_{11} = \{ (1-\nu)T_{11} - \nu T_{22} \}/(2\mu)$ // this is the form of the "plane strain solution"
$E_{22} = \{ (1-\nu)T_{22} - \nu T_{11} \}/(2\mu)$
$E_{33} = 0$
$E_{12} = T_{12}/(2\mu)$
$E_{13} = T_{13}/(2\mu)$ = assumed to be 0
$E_{23} = T_{23}/(2\mu)$ = assumed to be 0

5.21 Rectangular Beam Bent by End Couples (253)

We already solved this problem in Section 5.19, but here Lai wants to solve it in a different way which is quite peculiar. The beam width is now the x_3 direction, so our "plane" of interest is a longitudinal vertical slice through the beam. The width of the beam is now the uninteresting direction since we think the action in each vertical longitudinal slice will be exactly the same. But we need some "frictionless walls" now on the sides of the beam to maintain the required T_{33} so those longitudinal slices won't warp! Lai is going to obtain the bent beam as Problem 3 in a superposition of Problem 1 + Problem 2.

Problem 1 has those require frictionless x_3 side walls and will be a plane strain problem. He assumes a certain Airy φ and computes from it the T_{ij} matrix which has only T_{11} and T_{33}. The assumed φ results in $T_{11} = 6\alpha x_2 = T_{11}(x_2)$, so this T_{11} is the same in any axial slice (x_1 =constant) of the beam. In particular, we have $T_{11} = 6\alpha x_2$ at the end faces of the beam, and this implies a couple M at each end which Lai computes. Meanwhile, $T_{33} = \nu(6\alpha x_2)$ which we note is in fact linear.

Problem 2 is a problem with a T_{ij} matrix having only $-T_{33}$ which we just noted is linear. Problem 2 we found has a uniform T_{33} stress everywhere. When we add Problem 1 + Problem 2, we end up with Problem 3 which has no frictionless walls and end couples and this is supposed to align with our bent beam problem. My concern here is that when we remove this T_{33} , those longitudinal slices might warp. Problem 1 had no E_{33} but Problem 2 has $E_{33} = (1/E_Y)T_{33} = (1/E_Y)\nu(6\alpha x_2) = E_{33}(x_2)$. But we know that $E_{33} = \partial_3 u_3$ so there will be some $u_3 = kx_3x_2$ so $E_{33} = \partial_3 u_3 = kx_2$. This says that the warping is proportional to x_3 , so maybe if the beam is "thin" we can ignore this warping. Lai has nothing to say here about this issue.

5.22 Plane Stress Problems (254)

This situation is very similar to the plane strain game, but now $T_{33} = 0$ which forces $E_{33} \neq 0$ so we have

$$E_{ij} = \begin{bmatrix} E_{11} & E_{12} & 0 \\ E_{21} & E_{22} & 0 \\ 0 & 0 & E_{33} \end{bmatrix} \qquad T_{ij} = \begin{bmatrix} T_{11} & T_{12} & 0 \\ T_{21} & T_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{where } E_{33} = -(\nu/E_Y)(T_{11} + T_{22})$$

and now the T stress matrix is "plane" instead of the E matrix. Lai claims that this situation arises only when you have a plate or disk which is very thin in the x_3 direction with no end-face tractions. Since $T_{13} = 0$ then on the faces, and since the disk is thin, we assume $T_{13} = 0$ inside as well (same for T_{23}).

We assume the same relation between the four T_{ij} and the Airy function φ as in the plane strain case. As before, we compute the E_{ij} from these T_{ij} . In the plain strain case we set $T_{33} = \nu(T_{11} + T_{22})$, but in the current plane stress case we set $T_{33} = 0$, so our E_{ij} equations are now *different*! We find now that

$E_{11} = 1/[2\mu(\nu+1)] \{ T_{11} - \nu T_{22}\}$ // this is the form of the "plane stress solution"
$E_{22} = 1/[2\mu(\nu+1)] \{ T_{22} - \nu T_{11}\}$
$E_{33} = -(\nu/E_Y)(T_{11} + T_{22})$
$E_{12} = T_{12}/(2\mu)$
$E_{13} = T_{13}/(2\mu)$ = assumed to be 0
$E_{23} = T_{23}/(2\mu)$ = assumed to be 0

When these E_{ij} forms are forced to meet the compatibility requirements, we get the same result as before which said $\nabla^4\varphi = 0$, and we get another condition that E_{33} (and hence $T_{11}+T_{22}$) must be linear in x_1 and x_2. Lai makes these two points: (a) if E_{33} is linear, then solution can apply to large x_3 widths. (b) if E_{33} is not linear, solution is approximately good for something very thin in x_3 direction.

5.23 Cantilever Beam with End Load (255)

Our first shot at this famous problem! He starts with a proposed $\varphi = \alpha x_1 x_2^3 + \beta x_1 x_2$ which of course does solve $\nabla^4\varphi = 0$. He computes from this the four upper left T_{ij}. Requiring no top and bottom tractions makes β a multiple of α. This proposed φ results in a left-end down traction which we can interpret as our load on the beam P. So in terms of this P, we now know α and β and then everything is then known.

If we place this problem into the "plane strain" pigeonhole, then we need to have $T_{33} = k\, x_1 x_2$ in order to get $E_{33} = 0$. Since T_{33} is not of the desired linear form, it cannot trivially be removed by simple superposition, so T_{33} stays put in our plain strain solution. He then computes the upper left quad of E_{ij} values from the φ-derived T_{ij} and these are shown in 5.23.11 and this is a "plain strain" solution and it is valid for any beam width b.

On the other hand, if we go with a "plain stress" solution, we need $T_{33} = 0$ and this forces $E_{33} = -(\nu/E_Y)(T_{11}+T_{22})$ as noted above. If the plate is very thin in the 3 direction, we can ignore any u_3 displacement that this E_{33} tries to induce.

The discussion continues within **Example 5.23.1 (257).** Lai proposes certain $u_{1,2}$ solutions which in fact generate the desired upper left E_{ij} quad. Presumably we just set $u_3 = 0$ and ignore it. The proposed u_i tell us how the beam bends! The results agree with known literature.

5.24 Simply Supported Beam under Uniform Load (258)

Lai treats this as a plane stress problem. The top surface uniform compressive load is -p. He assumes a certain φ form with 5 constants, then applies boundary conditions to set all those constants. He computes the quad of T_{ij} and then stops, not bothering to determine the E_{ij} and the u_i displacements. I quote a result from the web showing what the center line does.

5.25 Slender bar demonstration of St Venant (260)

This is an excellent section! The plan is this: take a thin rectangular bar and apply a **knife-edge** axial compression to both ends. We know that if we apply a *uniform* compression, T_{11} will be constant inside the bar and T_{11} = F/A. But here we are applying P in a highly non-uniform way at the edges. Imagine a stick of one-by held between two chisels pressing in. Lai shows in this section that far from the ends, you do in fact get a uniform F/A, thus providing a real-world demonstration of St-Venant's Principle.

This section has lots of fancy math which, fortunately, is old hat to me after my Stakgold Basic Training stint. I easily verified every step.

5.26 Conversions between plane strain and plane stress solutions (262)

In the above meta notes we see E_{ij} equation sets for the plain strain and plain stress solutions. It happens that there is a nice trick which converts one equation set into the other:

$\nu \rightarrow \nu/(1+\nu)$ converts strain to stress equations

$\nu \rightarrow \nu/(1-\nu)$ converts stress to strain equations

These same changes transform not only E_{ij} but also the solution u_i. We have seen above an example where the plain strain solution was valid for any beam width, but the plain stress was valid only for a very small width, so you cannot just assume that both solutions are globally valid.

5.27 Polar Coordinates (264)

• First of all, Lai writes the static Navier equations in terms of T_{ij} for polar coordinates. This means we need $(\text{div}T)_i = \partial_j T_{ij}$ in polars. But we had this for cylindricals on page 170 so just read if off as shown.

• Next, what happens to the Airy rules? This subject resulted in a whole Lai support doc called "Lai problem 5.71" followed by a 5+ day flurry of tensor doc updates. Tensor doc now has Appendix E (h) which talks in general about tensor expansions on the unit basis vectors $\hat{e}_n$ and includes this Lai application as an example. So we can now just quietly check off the equations (5.27.3) after this storm of activity. Lai does claim that you could avoid all that activity and just accept 5.27.3 as a good guess, then show it works in 5.27.1,2 which are the Navier static equations (but I did not do this).

Lai then quotes the relations between E and T which we already found earlier. But he silently and slyly changes the meaning of T_{ij} from Cartesian to polar, as if Hooke's Law were "covariant". I need to check on this! // I did so, and show this in tensor doc at the very end of App E (h). The T_{ij} to E_{ij} equation is covariant with respect any *local* rotation, so it is true just as stated in terms of things like T_{rr} and E_{rr} . Lai did not mention this small technical detail and just quotes the equations at the bottom of page 264.

5.28 Stress Distribution Symmetrical about an axis (265)

My drawing shows a little particle in a cylindrical framework. "Symmetrical" includes the idea that the stress pattern has to be the same if you reflect the particle in a plane through it and the cylinder axis, so this rules out $T_{r\theta}$ and $T_{\theta r}$ shear stress components. You could have shear T_{zr}, but I think we are still in the plane strain and plane stress scenario. We have a slice of the bar and we are just expressing things x_1,x_2 in terms of r,θ so we still have a T matrix of the shape before and we have just altered the upper left quad of elements. We still have $T_{rz} = 0$ and $T_{\theta z} = 0$ by assumption. We might have some T_{zz} at some point, but not in a **plane stress** situation. So our only non-zero stress components are T_{rr} and $T_{\theta\theta}$. Symmetrical means that these do not vary with θ, so we then have $T_{rr}(r)$ and $T_{\theta\theta}(r)$.

So that was the first step. We now attempt to represent these two guys in their Airy Function form which in polars was 5.27.3 where the ∂_θ term now goes away. This Airy form is OK if it solves $\nabla^4\varphi = 0$ as we know. This is written now in cylindrical coordinates but nothing varies in z or θ, so $\nabla^4\varphi$ is pretty simple as shown in 5.28.3. Since this is a 4th order ODE in variable r, there must be 4 independent solutions and they are all trivially listed off in 5.28.4, each with a constant A,B,C,D. Then the Airy formulas give T_{rr} and $T_{\theta\theta}$ as in 5.28.5.

Conclusion: with cylindrical symmetry and our plane stress framework, a static stress pattern must have this very restricted form! In any azisym problem, all we have to do is find the coefficients.

5.29 What displacement structure is implied by the previous section (265)

In other words, given the restricted form found in the previous section that T_{ij} can have in a cylindrical azisym static situation,

$$T_{ij} = \begin{bmatrix} T_{rr} & 0 & 0 \\ 0 & T_{\theta\theta} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

what does that say about the restricted form that E_{ij} and u_i can have? From Hooke, Lai first finds that E_{ij} has the same form as T_{ij} above, and then he does a very fancy integration of E_{ij} to get u_r and u_θ, and the most general allowed form is shown in 5.29 17 and 18. This is an excellent section. He has to integrate a pair of coupled first-order PDE's and does this quite well in a clever way. He then shows that the integration terms f(θ) and g(r) are connected with rigid body motions and we can ignore them. He is showing us the "atomic forms" are for this kind of problem. In a separate document, I verify Lai's rigid body motion claims.

5.30 Thick walled cylinder with pressures in and out. (267)

The problem here is a finite-thick-walled cylinder like PVC pipe with pressure p_i inside and p_o outside. Using the previous section results, Lai computes T_{rr} and $T_{\theta\theta}$ in terms of p_i, p_o and radii a and b. We find that $T_{rr} < 0$ which means the pipe material is under radial compression for any p_i and p_o. For $p_i > p_o$, we get $T_{\theta\theta} > 0$ (I think) so pipe wants to rip apart in the tangential direction and mostly at the outer surface. I think it would be extremely difficult to crush a pipe with $p_i < p_o$.

5.31 Bending of a annular curved beam (268)

We start with an unstressed beam that is part of an annulus, sort of a Roman arch. How does this beam bend down if we apply end couples as shown? Lai fits this into our azisym form and obtains solutions for T_{rr} and $T_{\theta\theta}$ as shown in 5.31.11. The result is expressed in terms of the beam radii b > a, the end couple M, and N which is a function of a and b. Logs make a strong appearance in the result.

5.32 Initial Stress in a Welded ring (270)

Start with a flat annular curved washer that is almost a full ring. Apply a bending moment M to both ends to cause a deflection at the gap of $u_\theta = \alpha r$ and weld it into a stressed annular ring. We can compute M from the previous problem. Once welded, there is some $T_{\theta\theta}(r)$ in any cross section of the ring. If you integrate $T_{\theta\theta}(r)r$ across *any* mathematical dotted cross section, you will get M as your bending moment, and if you integrate just $T_{\theta\theta}(r)$ you will get 0 so the ring is doing "pure bending". This is an azisym problem because there is no reason for there to be a $T_{r\theta}$ type stress in the ring.

5.33 2D Polar atomic forms for Airy Functions

OK, suppose you do NOT have an azisym situation and you are faced with $\nabla^4\varphi(r,\theta) = 0$ which must be solved to find an Airy function for a plane strain or plain stress problem solution in polar coordinates. If the equation were instead just $\nabla^2\varphi(r,\theta) = 0$, we would try to find a set of "harmonics" or "atomic forms" with which we could construct a "Smythian form" solution with some coefficients to be found from BC's. I know that for this 2D Laplace situation the atomic forms are these

$[\, r^n, r^{-n}],[\sin(n\theta),\cos(n\theta)]$ n = 1,2,3..... and $A + B \ln r$ for n=0 // Stak V1 p 92

But now we are asking about the "atomic forms" for $\nabla^4\varphi(r,\theta) = 0$ which is a different PDE. Perhaps the atoms are called "biharmonics", but I will just call them "atomic forms". Clearly, the 2D Laplace atomic forms will also be *among* the atomic forms for $\nabla^4\varphi(r,\theta) = 0$, but there will be some new ones as well. The new ones it turns out are these:

$[\, r^{n+2}, r^{-n+2}],[\sin(n\theta),\cos(n\theta)]$ n = 2,3.....

and special cases for n = 0 and n = 1!

How do these new forms arise? We start with the full blown 2D polar coordinates $\nabla^4\varphi$ and assume the trig forms for θ and then $\nabla^4\varphi=0$ becomes just 5.33.1 which is a 4th order ODE in r where n appears as a constant. Lai assigns problem 5.74 to show that r^{n+2} and r^{-n+2} are the new solutions, but I never did that piece of work, I accept it still.

The upshot of doing lots of fiddling is this: for n = 2,3.... the atomic forms are 5.33.4 and there are 8 of them. For n = 0, there are 5 as shown in 5.33.9. And for n = 1, there are 10 atoms as in 5.33.14.

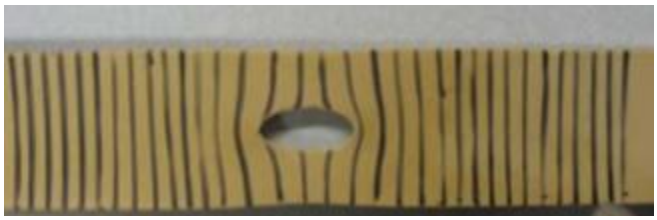
For each of these three cases, we can of course compute T_{rr}, $T_{\theta\theta}$ and $T_{r\theta}$ from the Airy formulas. Lai does all the painful work for us in a table on page 272.

Here is then the general plan: you are given some polar-coordinates plain stress or plain strain problem which has some boundary conditions. You want to come up with the right Airy function. You must construct that Airy function from the totality of available atomic forms, and each form will have some unknown constant for each value of n. Somehow, the problem and the BC's narrow things down and determine all the constants and you end up with your Airy $\varphi(r,\theta)$.

In Example **5.33.1** we are given a set of 4 BC's on T_{rr} and $T_{r\theta}$ which force n = 2. That is to say, we look on page 272 and see that if we want to satisfy the BC that $T_{rr} \sim \cos(2\theta)$ at r=b, we must select from the bottom of the table and the n=2 cosine atoms work. These same atoms also work for the $T_{r\theta} \sim \sin(2\theta)$ BC. So we have a set of four atomic forms for n=2 with cosine, and our four BC's set the four constants as shown. This example is a Lai "setup" for a more practical problem soon to come!

5.34 Hole in Plate under Tension σ (274)

Another very impressive section. Look at the picture. Our boundary conditions are first that there are no tractions on the inner edge of the hole r=a (5.34.1) and that at a large radius b according to St Venant things are normal so we have just $T_{11} = \sigma$ Cartesian which converts to (5.34.3) using the fancy matrix thing. So here we are using my tensor doc Appendix E (h) derivation of how a rank 2 tensor expanded onto unit vectors transforms! So we have four BC's. Lai now constructs the solution to the hole problem as Problem 3 of a superposition of Problem 1 and Problem 2. Problem 1 is the azisym PVC pipe problem with $p_i=0$ and $p_o = -\sigma/2$. Problem 2 is the setup thing we just did at the end of the previous section which had those n=2 atoms. When you add these solutions, you get the Problem 3 solution for the hole, as stated in 5.34.8 for T_{rr}, $T_{\theta\theta}$, and $T_{r\theta}$, each a function of r,θ since Problem 3 is not azisym. I think we might view this as a plane-stress solution with a thin material, or a plane-strain one with a thick material. The $T_{\theta\theta}$ function describes the tangential stress on particles relative to hole center, and when evaluated at the bottom of the hole we get the famous result that $T_{\theta\theta}(\max) = 3\sigma$, called stress concentration, showing how making a hole dramatically weakens a material in this manner.



5.35 Hole in Plate under Pure Shear (276)

This repeats the above problem except now the pre-drilled material rather than being under axial stress with $T_{11} = \sigma$ is under simple shear stress with $T_{12} = T_{21} = \tau$, see picture. He does the same matrix conversion on this to get BC's at r=b on T_{rr} etc. For the solution, Lai tries the sin(2θ) atom for n=2 with four constants C_i and applies the four BC's to get the answer. His answer has a rare error which I will

eventually report. This problem seems less useful than the previous one, but demonstrates application of the atoms method.

5.36 Wedge Loaded at Apex (277)

Another winner, see picture p 278. He makes an ansatz that $T_{\theta\theta} = 0$ on the wedge sides. No surface tractions on same gives $T_{r\theta} = 0$. Thus, thus 5.36.1 is four BC's.

You then do a force balance deal. His picture is very helpful and I agree with 5.36.2 which are done at some arbitrary radius r.

He then rolls out an n=1 proposal from his ample tool kit (n=1 general case on page 272) and we have two constants to find. The barred $B_5 = 0$ due to the vertical force balance equation, while the left right balance relates P to B_5. The final result is T_{rr} only, 5.36.7. The ansatz is justified.

In this toy problem, the wedge is infinite in the radial direction, so we never have to worry about what happens on the right surface. The solution shows $T_{rr} \sim 1/r$ and peaks at $\theta = 0$ as you might expect. I like this problem.

5.37 The Flamant Problem (277)

We just set $\theta = \pi/2$ in the previous problem and we have a 2D point force acting on 2D half space. This would be a knife edge acting on a half plane. The solution is a simple limit of the previous problem and is extremely simple. This type of wedge problem is associated with Flamant's work in 1892.

Alfred-Aimé Flamant (born in Noyales , Aisne , October 31, 1839 - disappeared during the war from 1914 to 1918) was a French engineer, former disciple of Barre de Saint-Venant [1] and chemistry professor Charles Leon Durand- Claye.

He was a French civil engineer and buddy of Saint-Venant and Joseph Valentin Boussinesq, another big name in this field, but more so in the fluid realm. This one was a "hydraulic engineer". Boussinesq did however do the 3D version of the Flamont Problem and it will show up on page 293 where in fact it is called the Boussinesq Problem.

And so ends this very nice section on more or less 2D plane stress type problems. It was very good. Notice that we have not really done any truly 3D problems.

[meta notes to this point on 5.11.12, after a Cod trip: raw notes 49 pages, meta notes 16 p, 4X]

A.4 3D Elastostatic Potential Theory (279)

5.38 Fundamental potential functions for elastostatic problems (279).

If you write the displacement u_i in terms of the two potentials Φ (scalar) and $\boldsymbol{\Psi}$ (vector)

$$\mathbf{u} = \boldsymbol{\Psi} - 1/[4(1-\nu)]\, \nabla(\mathbf{x}\bullet\boldsymbol{\Psi} + \Phi)$$

and if you insert this u into the static Navier equation,

$\mu/(1\text{-}2\nu)\ \nabla e + \mu\nabla^2\mathbf{u} + \mathbf{B} = 0$ $\quad e = \nabla\bullet\mathbf{u}$ $\quad \lambda+\mu = \mu/(1\text{-}2\nu)$ $\quad \lambda = 2\mu\nu/(1\text{-}2\nu).$

you find that the Navier equation is satisfied if the potentials satisfy these little Poisson-like equations

$\nabla^2\ \boldsymbol{\Psi} = -\mathbf{B}/\mu$ $\quad$ and $\quad \nabla^2\Phi = +\ x_n B_n/\mu$

where of course ∇^2 on the left is that vector Laplacian monster in non-Cartesians. If there is no body force, which is often the case, then the potentials need only satisfy two Laplace-like equations

$\nabla^2\ \boldsymbol{\Psi} = 0$ $\quad$ and $\quad \nabla^2\Phi = 0$

The left one is not really the Laplace equation except in Cartesians.

So this is a very stunning fact I think. If you are looking for candidate solutions to some problem, what you really want to do is find Laplace-satisfying potentials. Of course as usual $E = (\nabla\mathbf{u})^S$ and T comes from Hooke. The path from $(\Phi,\Psi) \rightarrow u \rightarrow E \rightarrow T$ is a bit indirect. Of course you could use the u equation above to write E in terms of the potentials and then Hooke to write T in terms of them.

Due to the similarity with electrostatics, this subject is called elastostatics.

An alternate method uses different potentials:

$2\mu\mathbf{u} = -4\ (1\text{-}\nu)\ \boldsymbol{\psi}\ + \nabla(\mathbf{x}\bullet\boldsymbol{\psi} + \varphi)$

$\nabla^2\boldsymbol{\psi} = 0$ $\quad \nabla^2\varphi\ = 0$

Lai then gives **"ten examples"**, the first of which is not really an example. All of the examples are here because they exhibit a "form" which will be applied in some real-world problems in sections following these ten examples. They are "setups". This way he does the dry math first and gets it out of the way.

The Ten Examples

Example 1. **(280)** Lai derives an equation which I did anyway on my own.

Example 2 (280) (Cartesian) What if $\boldsymbol{\Psi} = 0$ and you have only Φ? Then it really is exactly like electrostatics and $-2\mu\mathbf{u}$ plays the role of the electric field. You have then $2\mu\mathbf{u} = \nabla\Phi$, $\mu E_{ij} = \partial_{ij}\Phi$, $e = \text{div}\ \mathbf{u} \sim \nabla^2\Phi = 0$ and so $T_{ij} = \partial_{ij}\Phi$. This last relation is very similar to that for the Airy potential in plane problems, but the Airy has a minus sign for T_{12} whereas this thing here does not have that minus sign. So it is NOT the Airy potential.

Example 3 (281) (cylindrical, azisym). What if $\boldsymbol{\psi} = \psi_z(r,z)\ \hat{\mathbf{z}}$ and the other 3 potentials vanish? Here we are in cylindricals with azisym. In this and all the cases that follow, Lai computes $\mathbf{u}$, then E_{ij}, then $e = E_{ii}$, and finally T_{ij}. Of course you have to use the equation for $E_{ij} = (\nabla\mathbf{u})^S{}_{ij}$ in its appropriate

curvilinear coordinate system. On the other hand, since e = tr(E) we don't need any fancy coordinate work for this, and the same for T_{ij} in which e appears. For example, e is a scalar under the local rotation x to x" as in tensor doc.

Example 4 (282). What if $\varphi = (1-2\nu)f(r,z)$ and $\boldsymbol{\psi} = \psi_z(r,z)\,\hat{\mathbf{z}}$ where $\psi_z(r,z) = \partial_z f(r,z)$? [Lai uses φ for f] . As just noted, Lai writes out **u**, then E_{ij}, then $e = E_{ii}$, and finally T_{ij}. He omits E_{ij} in his summary at the start, but has it in the work section.

Example 5 (284)(sphericals with R,β,θ in place of r,θ,φ; radial only). What if $\psi = \psi(R)\hat{\mathbf{R}}$, $\varphi = \varphi(R)$? He finds the usual results. Only u_R exists and T_{ij} is diagonal.

Example 6 (285) (spherical and cylindricals, azisym) What if $\boldsymbol{\psi}$ =0, $\varphi = \varphi(r,z)$ = $\hat{\varphi}(R,\beta)$? Example 2 above did this case in Cartesian, here we do it in both sphericals and cylindricals.

Example 7 (286) (spherical, azisym) What if $\boldsymbol{\psi} = \psi_z(R,\beta)\,\hat{\mathbf{z}}$ **?** A slightly unusual case because this is spherical coordinates but **ψ** points in the z direction.

Example 8 (286) (spherical and cylindrical) What if $\boldsymbol{\psi} = Bz\,\hat{\mathbf{z}}$, $\varphi = A(z^2-r^2/2)$. With the correct constants A,B this gives a uniform T_{zz} field. In cylindricals this example is the superposition of examples 3 and 6, so he can just crib those T_{ij} results. And for sphericals, this example is a superposition of examples 7 and 6. Two constants here are A and B, same in both coordinate systems.

Example 9 (287) (Cartesian/spherical, a very specific example) First, we are given two specific scalar functions φ_1 and φ_2 in Cartesians, and Lai shows both satisfy Laplace. Second, Lai writes φ_2 in sphericals and then what if we use this as $\hat{\varphi}$ in Example 6? He cribs the T_{ij} from that example.

Example 10 (288) (cylindricals) What if $\varphi(r,\varphi,z) = z^* \ln(\sqrt{r^2+z^{*2}} + z^*) - \sqrt{r^2+z^{*2}}$ with $z^*=z+it$? Lai shows that such φ satisfies Laplace, and derives various properties of such a φ for later use. I spent some time trying to see how this φ might be written in terms of cylindrical atoms, without success.

5.39 Delta Force at center of full infinite 3D med. (290). [Kelvin Problem] Example 3

Using the Example 3 results, Lai shows that if you put a force $\mathbf{F} = \delta^{(3)}(\mathbf{r})\,F_z\hat{\mathbf{z}}$ at the origin of an infinite 3D isotropic medium, you can know everything including displacement **u** and stress T. ψ_z = A/R is all the potential you need. He does this in cylindricals. This is very much like the point charge canonical problem in electrostatics which has potential V = 1/R.

5.40 Delta Force at face center of *half*-infinite 3D med. (293). [Boussinesq Prob] Example 4

This is example 4 but with a specific f(r,z) = C ln(R+z). This potential turns out to solve the B problem where you apply that same $\mathbf{F} = \delta^{(3)}(\mathbf{r})\,F_z\hat{\mathbf{z}}$ but now half the medium is removed (the half with z < 0). We plug in this f and out comes "everything" as usual. In an exercise Lai converts the stress from cylindrical to Cartesian. This problem is the 3D version of Flamant above.

5.41 Distributed normal stress on *half*-infinite 3D medium (296).

Here we treat the previous problem as a sort of Green's Function solution and then integrate that against some distributed finite force distribution on the half-space surface. As an example, we consider this force to be a uniform normal stress in the shape of a disk and we solve that problem for displacement $u_z(r=0,z)$ only. This force is "like" an indenter, but $u_z(r,z)$ here will not be constant in r, so it is then an indenter of some strange radial shape -- that required to create a uniform normal stress. The simple cylindrical shaped indenter is coming soon.

5.42 Thick spherical shell with inner and outer pressures (297). Example 5

This is the 3D version of our PVC pipe problem, and I call it the tennis ball problem. Uses a simple potential $\psi = BR\hat{\mathbf{R}}$ and $\varphi = A/R$. As usual, Lai assumes the potential and then shows it works.

5.43 Spherical Hole in 3D axial stressed material (298). Example 8

This is the 3D analog of a 2D problem we solved earlier. Lai solves it and we obtain again a "stress concentration" effect, but it is not as strong as in the 2D case. After we write down the T_{zz} stress in sphericals, Lai comes up with something to add to it which causes no surface tractions on the hole surface.

5.44 The cylindrical indenter problem (300). Examples 4 and 10

Lai treats the problem and gives a complete solution. u_z has arcsin shape outside the indenter.

5.45 The general and spherical indenter problems (302)

The general problem is solved in terms of a Sneddon-like auxiliary function f(t). If you specify some indenter radial shape w(r), you can find f(t) and then you can find everything. The sphere with a << R is treated as an example.

Appendix 5A.1 Solving the integral equation used in Section 5.45 on general indenters.

I skipped this for now. It might be Sneddon like, it might not. I know I can understand it if I need to.

Part A Problems

He has been saving them up, we have 90 problems here, grouped by subsection:

Part A, opening section
Part A.1
Part A.2
Part A.3
Part A.4

General Comment: This is "the general theory" of 3D elastostatics, the four potentials stuff. Lai has boldly provided solutions to about 7 canonical 3D problems, with all the math done. We are always leaning on work done earlier, such as Hooke. Cylindrical and spherical coordinates are essential.

PART B: The *Anisotropic* Linear (Small Deformation) Elastic Solid (319)

5.46→5.55 Hooke's Law for anisotropic cases (aka constitutive equations) (319)

Lai returns to this general topic after a long break. The 9x9 world is really 6x6 since E and T are symmetric. This 6x6 C matrix is called the **stiffness matrix** with either C_{abcd} or C_{ij} elements. Lai shows that it is symmetric and positive definite, and so is its inverse $S = C^{-1}$ called the **compliance matrix**. I did the math details of all this. In a symmetric 6x6 there are only 21 unique elements.

The above is all for the fully anisotropic case. He then treats three specialized cases where there is symmetry. In each case, the C matrix has zeros outside of a certain block form. The lightest symmetry is called monoclinic, one plane of reflection symmetry and then we are reduced to 13 elements and the block form is 4x4+2x2 as on page 324. If you have two perp such planes of symmetry, you are then called orthotropic and you are then 4x4+1+1+1 in block form as page 325 with 9 elements. Finally, if you are 2D isotropic which he calls transversely isotropic, you have this same form but only 5 elements. He then takes the final limit to full 3D isotropic and recovers the μ λ situation with only 2 elements.

Example 4.57.1 deals with thermal expansion which is a rank-2 version of the rank-4 C business discussed above.

The final sections talk about how engineers like to think of these various unique constants. In general they are formulated in terms of Young moduli in various directions, and Poisson ratios in various pairs of directions, and few more exotic constants called coupling constants. I am unsure whether any of this stuff will be used in the rest of the book, so I didn't study it too hard, I get the point.

PART C: The *Isotropic* Large Deformation of Elastic Solids

Note that "large deformation" just means elastic theory outside of the "infinitesimal" region where we were allowed to use the E strain tensor. It does not mean we are doing any kind of plastic theory. An application of large deformations would be pulling a rubber band.

5.56 Change of Frame (334)

Preliminary Comment Added 9.27.12. Suppose x'-space and x-space in tensor doc with $x^\mu = (\mathbf{x},t)$ as coordinates are related by this 4x4 matrix transformation F of $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ which says $x'^\mu = F^\mu{}_\nu\, x^\nu$ with

$$F = R_{4x4} = \begin{pmatrix} R(t) & 0 \\ 0 & 1 \end{pmatrix}$$

where R(t) is a *time-dependent* 3x3 rotation matrix. In this case we have $\mathbf{x}' = R(t)\ \mathbf{x}$ and $t' = t$ as the transformation written in two parts. Since $x' = R\ \mathbf{x}$, we conclude that x transforms as a vector under this 3x3 rotation transformation, even though it is time dependent. But now consider,

$\dot{\mathbf{x}}' = R\,\dot{\mathbf{x}} + \dot{R}\,\mathbf{x}$ or $\mathbf{v}' = R\,\mathbf{v} + \dot{R}\,\mathbf{x}$

Our first conclusion is that velocity **v** *does not* transform as a tensorial 3-vector under rotation R(t) because we have that extra term $\dot{R}\,\mathbf{x}$. It seems very likely that **F** = m**a** as an equation of motion won't be covariant under the transformation F stated above (we have two different F objects here). Thus, in the rotating frame of x'-space, although we do have m' = m, we will not have **F**' = m**a**' . As confirmation of this conjecture, we have "frames doc" which tells us that in fact we have **F**' = m**a**' + fictitious forces. So in this little transformation example, which does not include translations, we still find the idea that the velocity of a particle **v** fails to be a tensorial vector under $R_{3x3}(t)$ and **F** = m**a** is non-covariant.

This entire section is a perfect example of my tensor doc notion of a covariant equation. Any law (any equation) of non-relativistic continuum mechanics must show Galilean invariance. Both sides of any equation must have the same tensor nature with respect to rotations in particular. When something transforms as the tensor which it appears to be, rather than saying the object is a true tensor, Lai says that the object is "objective". An objective vector and rank-2 tensor must transform this way

$\mathbf{V}^* = Q\mathbf{V}$ // my usual vector rule where my R = Q

$A^* = QAQ^T$ // the usual rank-2 rule in matrix form, devel notation.

where Lai uses * to denote an object as it appears in what I always call "the primed frame" or x'-space.

Before getting around to the subject of covariant equations, Lai talks a lot about these Galilean transformations and whether specific objects are objective or not. So we shall return soon to the covariant equation issue, but now we annotate Lai's general math comments.

What exactly *is* the "underlying transformation" Lai is implying? I conjecture this:

$\mathbf{x}^* = Q(t)\mathbf{x} + \mathbf{c}(t)$
$t^* = t$

where we transform from 4 variables to 4 starred variables. Thus, $x^\mu = (\mathbf{x},t)$. The above is not a linear transformation, and it can be written this way

$$\begin{pmatrix} \mathbf{x}^* \\ t^* \end{pmatrix} = \begin{pmatrix} Q(t) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} + \begin{pmatrix} \mathbf{c}(t) \\ 0 \end{pmatrix} \qquad R(\mathbf{x},t) = R(t) = \begin{pmatrix} Q(t) & 0 \\ 0 & 1 \end{pmatrix}$$

where Q(t) is a 3x3 rotation matrix.

What is the "linearized" R matrix tensor doc we would assign the above F? We can see that

$d\mathbf{x}^* = Q(t)d\mathbf{x}$
$dt^* = dt$

so the corresponding R matrix would be exactly the one shown above. Since the time part of the transformation is decoupled from the space part (R is in block form), we will never be worrying about the

time part of the transformation. BUT since time is in the picture as shown, time derivatives are going to be affected! When we write $R(x^{\mu})$ in tensor doc, that comes out here being R(t) with no dependence on **x**.

When Lai says that d**x** is an "objective vector", he means (d**x**,dt) is a true vector under the above transformation, but he never worries about the dt part. He just says that the vector quantity d**x** is "objective", whereas the vector **x** is not objective due to the **c**(t) term. That is, we don't have **x*** = Q(t)**x** for a *general* Galilean transformation.

More comparisons with tensor doc:

• In tensor doc I talk about just the 3x3 world with d**x*** = R(**x**)d**x** where R does not depend on a time variable, so in that world **v** ≡ d**x**/dt transforms as a tensorial vector under rotations. In that tensor doc example, the primed frame is simply rotated relative to the original, and both frames are not moving in time. But here with R(t), we get instead

$$\mathbf{v}^* = Q\,\mathbf{v} + \dot{Q}(\mathbf{x}-\mathbf{x}_0) + \dot{\mathbf{c}}(t) \qquad = Q\,\mathbf{v} + \text{two extra pieces} \qquad \text{so non-objective}$$

• In tensor doc, the quantity (**∇w**) transforms as a true rank-2 tensor under static rotations.

$$(\nabla \mathbf{w})_{ij} = \partial_j w_i \qquad \text{// which really means } \partial^j w^i$$

providing that **w** is a vector. In other words, both w^i and ∂^j transform as vectors, so the combination transforms as a rank-2 tensor. This is the case when R does not depend on **x**, meaning it is a vector with respect to global rotations. If R = R(x), then we found in tensor doc that (**∇w**) is not a true vector due to extra terms. Well, here we have R = R(**x**,t) = R(t) and so we do in fact get some extra terms, but not all the terms we get for a general R(**x**,t) . Lai shows that

$$(\nabla^* \mathbf{v}^*) = Q(\nabla \mathbf{v})Q^T + \dot{Q}Q^T \qquad = Q(\nabla \mathbf{v})Q^T + \text{one extra piece} \qquad \text{so non-objective}$$

Lai and I like to think of this transformation we being implemented by a flying camera platform which is the S* coordinate system. This thing can move and even accelerate arbitrarily. However, there is one initial condition that we require of our camera flight plan: At some starting time t_0, our platform is perfectly aligned with the x-space coordinate system (at least in a rotational sense), so $Q(t_0) = 1$. This time t_0 is the *same* "initial time" that makes the **x** and **X** coordinates align from the much earlier Lai flow discussion. Recall that $\mathbf{x}(t_0) = \mathbf{X}$, the initial material coordinate for a flow. Thus, all the X coordinates in a blob are at time t_0 and at that time Q = 1 and so d**X*** = d**X** . This is very important. At this one time, these vectors are the same.

The result is this important fact: F* = Q(t)F where all three objects are matrices.

Based on this fact that F* = QF, it immediately follows that

$$C^* = C \qquad \text{where } C = F^T F = \text{right Cauchy-Green}$$

Proof: $C^* = F^{*T}F^* = (QF)^T(QF) = F^T Q^T Q F = F^T F = C$.

Also based on this fact that F* = QF we get

$B^* = QBQ^T$ where $B = FF^T$ = left Cauchy-Green

Proof: $B^* = F^*F^{*T} = (QF)(QF)^T = QFF^TQ^T = QBQ^T$.

This then is the first objective rank-2 tensor encountered in this section ! B_{ij} transforms as a true rank-2 tensor under transformation F described above! Immediate corollaries are:

$e = (1 - B^{-1})/2$ is objective Eulerian strain tensor

$E = (C-I)/2$ is not objective Lagrangian strain tensor

5.57 Constitutive Equation for Elastic Medium under large deformation (338)

We have various stress tensors and various strain tensors, and Lai tries various combinations to come up with a covariant replacement for our old infinitesimal Hooke's law that T = CE. But right away he makes a postulate: The Cauchy stress tensor T must be objective. That is, he requires that

$T^* = QTQ^T$

Why does he assume this? We know that a little area will be invariant under any moving camera position, but what about the force part of T? Consider a spring pulling on a scale in some contraption and the scale registers a force of 1 lb. This scale reads the same no matter what the camera platform does. So somehow if you regard an elastic solid as a system of such springs all tightened to some amount, you would expect that the stress tensor T_{ij} at least in some internal sense is the same to all camera observers even when the camera is violently moving in time. A counterexample is provided by a falling billiard ball. The camera can be in the rest frame of the falling object and in that frame there is no force and no acceleration as seen by the camera. So this is a counterexample in the sense that the "force" for frames S and S* are then not the same in this case, but the spring forces in the first case are the same.

Here is a little wiki piece addressing this subject:

Cauchy stress tensor [edit]

The Cauchy traction vector t is related to the Cauchy stress tensor σ at a given point x by the outward normal to the surface n such that: $t = \sigma n$. The Cauchy traction vector for another observer can be simply written as $t^* = \sigma^* n^*$, where t and n are both **objective** vectors. Knowing that, one can write:

$$\begin{aligned} & t^* = \sigma^* n^* \\ \Rightarrow \quad & Qt = \sigma^* Qn \\ \Rightarrow \quad & Q\sigma n = \sigma^* Qn \\ \Rightarrow \quad & \sigma^* = Q\sigma Q^T . \end{aligned}$$

This demonstrates that the Cauchy stress tensor is **objective**.

So this "proof" assumes that any traction force vector **t** (which is a force vector on an area) is objective. Maybe all this stuff relates to static tractions like my spring scenario.

OK, so we accept the requirement that $T^* = QTQ^T$ and T is objective. Once we make this assumption, many things follow. He first shows that (based on $T^* = QTQ^T$), for those P-K stress tensors,

$T_0^* = QT_0$ // first kind

$\tilde{T}^* = \tilde{T}$ // second kind // recall from above that $C^* = C$

This at once suggests that a law of the form

$\tilde{T} = f(C)$

would be a viable constitutive equation since $\tilde{T}$ and C transform in the same way (same f in both frames). The equation might be

$\tilde{T} = a + bC + cC^2 + ...$ or $\tilde{T} = \exp(-5C)$.

where a,b,c are some constants. You can see at once that either of these equations is "covariant" because it has the same form in the * frame. These are basically scalar equations. These tensors "look like" rank-2 tensors, but in fact each of their elements is just a scalar. I think this is why Lai says this $\tilde{T} = f(C)$ is a viable candidate for the general anisotropic case, though he never does anything with such an equation.

The other covariant option is this one

$T = f(B)$

since T and B both transform objectively. But in this case, things are more complicated. Our previous example might not work:

$T^* = f(B^*) = f(QBQ^T) = QTQ^T$

Then for example

$T^* = \exp(-5B^*)$ $\Rightarrow QTQ^T = \exp(-5\, QBQ^T)$?

But if we expand exp in a power series, we are OK. So in this case, it seems to me that any f is OK as long as it is a power series in B.

Now why is this case considered isotropic only , whereas the previous case is anisotropic? These are all questions Lai never asks. In the previous case, f(C) is a function of a scalar argument in effect, but in the current new case f(B) is a function of a tensor argument which implies there is a notion of direction I think. I don't have this nailed very well, but I think that is the underlying difference. So since direction counts in this case, when we say $f^* = f$, we are restricting ourselves to isotropic.

As we shall see, the requirement that $f(QBQ^T) = QTQ^T$ restricts the function f to being basically a quadratic in B with some coefficients that might depend on space. This of course does meet my polynomial idea just mentioned.

Back in the infinitesimal theory world we had $T_{ij} = C_{ijk\ell}\, E_{k\ell}$ and I think T and E are true tensors and so is C, so this equation is already "covariant" .

In continuum mechanics, the notion that any equation must be covariant with respect to time-dependent Galilean transformations is called "**the principle of material indifference**".

5.58 Constitutive Equation for Isotropic Elastic Medium (340)

Lai shows, and I agree, that the most general isotropic form of T = f(B) are either of these:

$T = a_0 I + a_1 B + a_2 B^2$	where a_o are functions of the eigenvalues of symmetric B.
$T = \varphi_0 I + \varphi_1 B + \varphi_2 B^{-1}$	where φ_o are functions of the eigenvalues of symmetric B.

where the coefficients are functions of the "invariants" of the matrix B, which is the same as saying that the coefficients are functions of the eigenvalues of B (B is symmetric!), with different functional form. The proof of the above general forms involves the Cayley-Hamilton theorem, so finally I get to see that thing used somewhere. Details appear in the appendix on page 349. No higher powers of B appear because if they did, they could be reduced to lower powers by the C-H theorem and then their contributions are absorbed into the constants shown.

If it turns out that you have just $T = a_1 B$ with a_1= a constant, this special case is a **Hookean solid** for obvious reasons. Much simpler that the T = CE infinitesimal tensor business.

All Lai's subsequent examples are going to involve incompressible solids (like rubber). In this case, the function φ_0 shown above is renamed -p, and we know that $I_3 = \det(B) = 1$, because earlier in the book we showed that in fact 1 = detF = detC = detB in this case. So you only have I_1 and I_2 of interest. This leads to a certain planar-like (Airy-like) potential theory with a potential function A. Since $I_3 = \Lambda_1\Lambda_2\Lambda_3$ where the Λ_i are the eigenvalues of B, we have $\Lambda_1\Lambda_2\Lambda_3 = 1$ which is called the **isochoric condition**. In the first example, it happens that $\Lambda_i = \lambda_i^2$ where λ_i are stretch factors, so get $\lambda_1\lambda_2\lambda_3 = 1$ as well.

Along the way I have learned that a certain 1965 book by **Truesdell and Noll** is the sort of Bjorken and Drell of continuum mechanics. They are superstars of their field (but of course I never heard of them).

Lai ends this section with **four "examples"** of incompressible materials being deformed in a non-infinitesimal way. Each example has certain ingredients which I will try to outline in general here:

[I presume he does *incompressible* examples since these are easier problems to solve compared with compressible solid materials!]

(1) There is some set of equations that relates coordinates x to coordinates X. It is as if the rubber stuff "flowed" from one shape at time t_0 (material coordinates X) into another shape at time t. The initial shape is one where things are all relaxed, there is no stress or strain. In our earlier infinitesimal world, we were usually given equations for displacement u_i since we could compute E_{ij} from u_i in a simple way.

(2) In each example, coordinates x are specified in a certain curvilinear coordinate system, and X are specified in some other curvilinear coordinate system. Thus, the machinery of Section 3.29 which I

studied at length (and wrote up in tensor doc) comes into play. I did all the theoretical and Maple work on this, and what you get is the F, F^{-1}, B, B^{-1}, C, C^{-1} for all combinations of x and X being one of the three basic curvilinear systems of Lai interest. Lai then can just quote the B and B^{-1} results. The elements of these matrices are certain derivatives which you just compute from the equations mentioned in (1) above.

(3) This, in each example we know B and B^{-1} so we then know $T = -pI + \varphi_1 B + \varphi_2 B^{-1}$. In terms of the unknowns φ_1 and φ_2 we can then compute anything we want to know about the deformed rubber solid. If there are surfaces of no surface traction, then those provide boundary conditions. We find in the case of the twisted cylinder (think long power eraser refill) that there is some normal force at the ends and also a "twist couple" which makes the thing be twisted. In the case of the bent rectangular eraser, we find a different kind of couple M which is bending the thing. In other words, Lai processes things in the same manner he always uses to find the resulting solution T_{ij} stress pattern in the deformed object.

(4) In some of the examples, the unknowns φ_1 and φ_2 are constants and they must be determined for a given rubber-like material by doing experiments. No interpretation is provided for these constants really. Since μ and λ isotropic constants were only relevant to the infinitesimal theory, we don't expect that φ_i are simply functions of μ and λ. They are basically "new properties". In other examples, the φ_i are functions perhaps of radius in the twisted cylinder eraser. In no case does Lai talk about these unknowns, he just stops the example.

(5) along the way, the Cauchy equation of motion is noted (static form) and we want to make sure that the T_{ij} obtained solve this equation. In some examples the Cauchy equations are in fact used in part to determine constants.

We can now look at the four examples, each lives in its own section. The rubber model he uses here is more or less known as the **Mooney-Rivlin model**, done in the 1940's.

5.59 Simple Extension of an incompressible isotropic elastic solid (340)

(1) deformation equations are stated in 5.59.1, very simple, but Lai has typos! The λ_i are the three "stretch" factors for our rectangle of rubber that we stretch axially.

(2) both coordinate systems are Cartesian. No picture is provided.

(3) The B matrix is very simple and diagonal

(4) final result is 5.59.6 and it gives the axial stress T_{11} in terms of φ_1 and φ_2 and stretch λ_1. I think everything is a constant throughout this deformed solid, so the φ_i are probably just constants here. You could measure them by doing some experiments.

(5) Since all T_{ij} are constants, there is no Cauchy equation issue at all. It is happy.

5.60 Simple Shear of an incompressible isotropic elastic block (343)

(1) deformation equations are stated in first line of page 343, very simple. This is a rare equation set that has no number! There is a twist factor K instead of stretch factors λ_i as in the previous example.

(2) both coordinate systems are Cartesian. We get a little picture.

(3) The B matrix is very simple but is now non-diagonal. B is just obtained from FF^T in situ.

(4) final result is 5.60.7 for the entire T_{ij} matrix and I think the T_{ij} are again independent of position inside the sheared squished eraser. In this problem the φ_i he says are constants of space, but are functions of K. Lai computes the surface tractions on the slanted faces needed to hold things in position. It turns out there is both a normal traction and a shear traction required. He finds a little relation between the T_{ij} components that is independent of the φ_i.

(5) the T_{ij} are again constants, so no Cauchy equation issue.

5.61 Bending an incompressible isotropic rectangular bar (344)

(1) deformation equations are stated as first line of the section on page 344, again with no equation number.

(2) the X coordinate system is Cartesian, while the x one is cylindrical. We get a nice picture.

(3) The B matrix is diagonal but now some diagonal elements are functions of space coordinate r.

(4) results for T_{ij} (cylindrical) are stated in 5.61.4 and 5 in terms of potential A. Boundary conditions are no normal surface traction on the curved surfaces.

(5) The cylindrical coordinate version of the Cauchy equation is made use of to show that p = p(r) and to find relations between the constants in this problem (r_1, r_2, $\alpha = 1/c$, a, b, β), such as $\alpha^2 = r_1 r_2$. As just noted, the A potential method is used in this example as well, and he finds that A is the same on the two curved surfaces, but he does not know what A(r) does in between. The bending couple that holds things in the deformed position is computed as an integral of A(r) and this is the only unknown. Well, in this problem K is another unknown integration constant. See 5.61.19.

5.62 Torsion and tension of an incompressible isotropic cylinder (344)

(1) deformation equations are stated in 5.62.1, very simple. The λ_1 and λ_3 are "stretch" factors and K is a twist factor, all taken as constants. It seems that this long eraser will get thinner and longer or vice versa (to maintain volume since incompressible). In fact yes, since $\lambda_1^2\lambda_3 = 1$.

(2) both coordinate systems are Cylindrical. No picture is provided.

(3) The B matrix is not diagonal and is the messiest one of all examples so far.

(4) the T_{ij} are shown in 5.62.4 and following. Lai then computes the normal force N at the end of this twisted eraser, and also M which is the torquing force holding it twisted. Both N and M are integrals involving the unknowns φ_i which in this problem are functions of r (or R if you like). Lai then obtains some simple results in the limit that K is very small (but I guess we are still doing large deformation) The term "torsion stiffness" is used for the ratio M/K.

(5) The Cauchy equations (cylindricals again) are used to show p = p(r) and to evaluate the N integral.

Appendix 5C.1 (p349) . The most general form of F in T = F(B) for isotropic solid.

See raw notes on this subject. C-H theorem is used. The form was quoted above.

Part C Problems (p 351, Problem 5.104 through 5.113)

And so ends this monstrous Chapter 5 on Elastic Solid Continuum Mechanics, about 150 pages. My meta notes here are about 28 pages, a 3x compression from the 90 pages of raw notes.