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Introductory remarks. We first "did the math" in Ch 1. Then strain kinematics in Ch 2. Then stress in Ch3, then in Ch 4 the 5 general "principles" of continuum mechanics: conservation of mass, linear momentum, angular momentum, energy, and the entropy inequality. What has been missing is a relationship between stress and strain provided by some kind of model for the medium of interest. The model we will use for an elastic solid is basically Hooke's law which says F = -k(X-x) = -ku, where u is displacement. In this simple case 1D world, the infinitesimal E strain tensor $(\nabla u)^{\mathbf{S}}$ is just $\partial_{\mathbf{x}} u_{\mathbf{x}} = 1$. The model tells you what force is required to produce a certain state of strain. This particular model is linear, and of course that is always where one wants to start. Later one might add corrections using perturbation theory. This Hooke's law linear model for a continuous medium takes the form $T_{ij} = C_{ijk\ell}\, E_{k\ell}$ where this fancy rank-4 tensor C is what replaces Hooke's simple constant k. There are *a priori*, $3^4 = 81$ elements of this C tensor!

This chapter has three parts. In Part A, we will assume the medium is isotropic as well as linear. In part B, we allow for an anisotropic medium, still linear. Finally, Part C looks at a particular non-linear situation which is isotropic but incompressible.

The "model" provides the "constitutive equations", accent on "const", like F = -kx. The term "elastic" means that, if you apply and then remove some stress (called loading then removing the loading), the medium goes back to where it started. It has not undergone some permanent change of any kind.

5.1 Mechanical Properties (201) -- experiments with steel

Lots of terms of the trade appear here: a bar is "tensed" by an "axial load" P (force, not pressure). The rod is seen to "elongate" by a fractional amount $\Delta\ell/\ell$ called $\varepsilon_{\mathbf{a}}$ (a for axial). The first graph on p 202 shows a sort of hysteresis diagram for doing this test. If you keep your axial load small enough, you stay in the elastic region of the curve shown, which region is linear. If you apply too much load, the bar permanently stretches and when you remove the load, you find you stretched it a bit in its rest position. This effect is called plastic deformation (as opposed to elastic). Wiki explains that plastic deformation creates dislocations which in turn make the medium "stronger" against more such plastic deformation, and the material is said to be "**work-hardened**", but that is a topic for another day.

Clearly we only care for measurement $P/A_0 = \sigma$ which is the axial stress (force per area) which is used to tense the rod. Notice that σ is used for "stress" now for the first time in this book. We expect to find then a linear Hooke's law in the form σ = constant x $\varepsilon_{\mathbf{a}}$ similar to F = kΔx. The constant here has the strange name "**Young's modulus**" written $E_{\mathbf{Y}}$ so we then have $\sigma = E_{\mathbf{Y}}\, \varepsilon_{\mathbf{a}}$. Since $\varepsilon_{\mathbf{a}}$ is dimensionless, this $E_{\mathbf{Y}}$ has dimensions of stress which is pascals or psi etc.

An interesting example is given. Take a steel bar with area 5 in^2 (diameter 3"). Now create a platform and hang from this a set of 27 Ford Expeditions at 5550 lbs each for a total axial load of 150,000 pounds.

The bar will stretch by a mere 1/10th of 1%. Lai does not tell us what load would take you close to the plastic deformation limit, I put this on a list of questions for now. A point of this example is that for steel at least, and probably for other solid construction materials, ε_a will be very small, and that means we can use the E strain tensor (recall it is the infinitesimal one of page 87 and $E = (\nabla u)^S$.)

Now when you put that axial load on your bar, you will also find that the cross sectional area gets reduced under the load. The fractional change in "lateral dimensions" is ε_d, d for diameter. I turns out (no theory yet) that ε_a and ε_d are in the same ball park. Since $\varepsilon_d = \Delta D/D$, it will be negative, and so we define $\nu = (-\varepsilon_d)/\varepsilon_a$ and this then is not too far from unity. This ratio is called **Poisson's Ratio** and for steel it is about 0.3.

Lai then moves to another "thought experiment". Take a cube of steel and lower it down into the ocean. This of course puts a hydrostatic normal stress on all faces of the cube. You might expect that the volume of the cube might decrease. At the Challenger Deep, the pressure is 16,155 psi = 111 MPa. The stress tensor is $T_{ij} = \sigma\, \delta_{i,j}$ and we use our same symbol σ here for the normal stress, though in this experiment it pushes in instead of pulls out. The volume "dilatation" is $e = \Delta V/V$. Lai claims that experiment shows that $\sigma = ke$ is Hooke's law in this case and k is **"the bulk modulus"**.

Two things to ponder in this case. You would think e would be negative, but there is no sign appearing in this equation. Again, this modulus has dimensions of stress or pressure. For steel Lai's table shows that $k = 20 \times 10^6$ psi. So for steel at Challenger Deep, $e = \sigma/k = 16155/20\text{x}10^6 = 806\text{e-}6 \sim 10^{-3}$ which is similar to the small number of the axial load experiment.

The second thing to ponder: you might think that you could get σ = constant $\Delta x/x$ for each of the three dimensions, and then you could get $e \sim (\Delta x/x)^3 = \text{const} \times \sigma^3$, but that is not what happens. So this question goes on the list as well.

The third experiment is to torque the bar and see by what angle θ it twists. In this case, the parameters are more complicated and one writes

$$M_t = \mu\, (\pi r^2/2)\, \theta/\ell$$

where I think M_t is the applied torque at the bar surface (think one end tied down, you torque the other end with your torque wrench). μ is called the "**shear modulus**". Lai has never used the word "torque". You can see that this experiment does involve tangential stress on each particle of the bar rather than normal stress. Presumably later we will see the origin of this formula.

5.2 The linearly elastic solid (204).

We study here the relationship $T_{ij} = C_{ijk\ell}\, E_{k\ell}$. Although T and E are in general fields, functions of position in the medium (Lai would call this "the material") , C is not a field, it is independent of position because we are assuming our medium is homogeneous! So C is a tensor of constants!

Theorem: suppose $A = X_{ij}Y_{ij}$ where Y is symmetric. Does this say that X_{ij} must be symmetric? I would say NO, but it does say that only the symmetric part of X is involved in making A. Just so, in the above case one can say

$$T_{ij} = C_{ijk\ell}\, E_{k\ell}. = (1/2)\, [C_{ijk\ell} + C_{ij\ell k}]\, E_{k\ell} = C_{ijk\ell}{}^{(SR)}\, E_{k\ell}$$

which is symmetrized on the right pair of indices. For fixed i,j, there are only 6 unique constants in this object, so we have then 9 * 6 = 54 constants. If T_{ij} is also symmetric, the same argument reduces us to a total of 6*6 = 36 constants (called "material coefficients"). C is the **elasticity tensor**, by the way.

On page 205 we get the mystery zinger which claims there exists some potential energy function U such that U = U(E), and

$$T_{ij} = \partial U/\partial E_{ij}$$

If that were true, we then have more information:

$$C_{ijk\ell}\, E_{k\ell} = \partial U/\partial E_{ij}$$

but I don't see why this implies as they claim that $C_{ijk\ell} = C_{k\ell ij}$. Given that this is true, how many conditions is this that we don't already have? It is only a condition if ij ≠ kℓ . But it is only a new condition if it tells us something we don't already know. At first, there are 9 ij values and so 8 kℓ values. But only maybe 5 of those kℓ values are something new. I defer this counting problem till later! The claim is that you have 15 new conditions here so you go to 36-15 = 21 constants. This should be in the problems somewhere.

Example 5.2.1. This is not really an example, it is mainline text.

Part (a) shows that

$$d_t E_{ij} = D_{ij} \qquad \text{where } D_{ij} = (\nabla^{(x)} v)^S{}_{ij}$$

where you have to realize what E and D are. The derivatives in E are with respect to X_i, but those of D are with respect to x_i. The result is valid only for small deformations such that $\partial u_i/\partial X_j << 1$. So now the name "rate of deformation tensor" for D seems awfully reasonable, if you think of E as the deformation.

Part (b) claims to show that. if you assume a U function as in 5.2.8, then $P_s = d_t U$. This just says that the stress power all goes into increasing the stored internal energy U. This is pretty reasonable, but it is not a derivation of 5.2.8.

Example 5.2.2.

Part (a): OK, here is what I was trying to do above. Go back to

$$T_{ij} = C_{ijk\ell}\, E_{k\ell}.$$

and then you can isolate C by saying

$$C_{ijk\ell} = \partial T_{ij}/\partial E_{k\ell}$$

Now again if you assume 5.2.8 $T_{ij} = \partial U/\partial E_{ij}$, you conclude that $C_{ijk\ell} = C_{k\ell ij}$. So this gives part of the answer. But I still don't have the 21 number.

Part (b): Again assuming 5.2.8, we can conclude that

$$U = (1/2)\, C_{ijk\ell}\, E_{ij} E_{k\ell}$$

which reminds us of Hooke's thing $U = (1/2)\, k\, (\Delta x)^2$.

The idea I guess is that U is going to be quadratic in E (the strain) to first order. I will find out later why this must be true. Once true, there must be a coefficient tensor C, and then all else follows. Why couldn't there be a linear term in the strain? True, Hooke's law does not show such a term, and it would perhaps have a sign problem. I am missing some Goldstein fact here. If you have some system of particles and springs, you are going to have stored energy being quadratic in deformations. [$Fdx = kdx^2$?]

Suppose we define U according to $d_t U = P_s = T_{ij} D_{ij}$. In part (a) of this example, we show that $D_{ij} = d_t E_{ij}$. So we then have $d_t U = T_{ij}\, d_t E_{ij} = C_{ijk\ell}\, E_{k\ell}\, d_t E_{ij}$. This takes us not quite there. I can get to this point:

$$d_t U = C_{ijk\ell}\, E_{k\ell}\, d_t E_{ij} + C_{k\ell ij}\, E_{ij}\, d_t E_{k\ell}$$

but I can't get the RHS to be a perfect differential without assuming 5.2.9.

PART A: The *Isotropic* Linear Elastic Solid

Where have we assumed "solid" I wonder? I guess it is hard to imagine an elastic fluid that restores to its previous shape. I don't think Lai has defined a "solid". Perhaps something that has a well-defined shape that is stable against small forces, unlike a free blob of water or air or plasma.

5.3 The *isotropic* linearly elastic solid (207).

What is an "isotropic tensor" of any rank? Perhaps

$$A'^{abc\cdots} = R^a{}_{a'} R^b{}_{b'} \ldots\ldots A^{a'b'c'} = A^{abc}$$ // yes, R = rotation.

It looks the same under any rotation matrix R. Well, the Cartesian metric tensor has this property

$$g'^{ab} = R^a{}_{a'} R^b{}_{b'}\, g^{a'b'}$$

and that is identified with δ^{ab}. For rank-3, only ε_{abc} is an isotropic tensor says Wolfram. There is an amazing little theory that goes with this subject. Here is a little PDF blurb

3.8 Isotropic Tensors

An *isotropic tensor* is one whose components are the same in all frames, i.e.,

$$\boxed{T'_{ijk\ldots} = T_{ijk\ldots}.}$$

We can classify isotropic tensors up to rank four as follows:

Rank 0: All scalars are isotropic, since the tensor transformation law states that $T' = T$ for tensors of rank zero.

Rank 1: There are no non-zero isotropic vectors.

Rank 2: The most general isotropic second rank tensor is $\boxed{\lambda\delta_{ij}}$ where λ is any scalar, as proved below.

Rank 3: The most general isotropic third rank tensor is $\boxed{\lambda\epsilon_{ijk}.}$

Rank 4: The most general isotropic fourth rank tensor is

$$\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk}$$

where λ, μ, ν are scalars.

So OK, here is some Lai support on rank 4.

So assuming the relationship $T_{ij} = C_{ijk\ell}\, E_{k\ell}$ (which already assumes we are doing infinitesimal work only), since E is symmetric, and since C must have the triple form shown above, Lai shows that T must have this form

$T = \lambda tr(E)1 + 2\mu E$ $\qquad$ $tr(E) = e$, recall

$T_{ii} = \lambda e\delta_{ii} + 2\mu E_{ii} = \lambda e3 + 2\mu e = (3\lambda+2\mu)e$ $\qquad$ => $\qquad$ $e = tr(T)/(3\lambda+2\mu)$

where λ and μ are the **Lamé constants**. [Also Lamé coefficients, so a 2nd set of such coefficients associated with Lamé , the other being the h'_n as reported in tensor doc.] They are constants because λ,α,β are constants, and that is because C is a tensor of constants, and that is so because the medium is uniform (homogeneous).

In any event, what you see above is an F = -kx type equation in its most general form for an elastic solid with infinitesimal motions. T is related to E by this equation, so it is "**the constitutive equation**" for isotropic linear elastic solids. It is still true that $T_{ij} = C_{ijk\ell}\, E_{k\ell}$. This relation which involves in general at least 21 constants, has been reduced to 2 constants for the isotropic case, those constants being here called λ and μ. This is an impressive reduction.

$T_{ij} = \lambda tr(E)\delta_{ij} + 2\mu E_{ij} = \lambda E_{kk}\delta_{ij} + 2\mu E_{ij} = \lambda E_{kn}\delta_{kn}\delta_{ij} + 2\mu\delta_{ik}\delta_{jn}E_{kn}$

$= (\lambda\delta_{kn}\delta_{ij} + 2\mu\delta_{ik}\delta_{jn})\, E_{kn} = (\lambda\delta_{k\ell}\delta_{ij} + 2\mu\delta_{ik}\delta_{j\ell})\, E_{k\ell}$ $\qquad C_{ijk\ell} = (\lambda\delta_{k\ell}\delta_{ij} + 2\mu\delta_{ik}\delta_{j\ell})$

Note also from above that

$tr(T) = \lambda tr(E)3 + 2\mu tr(E) = (2\mu+3\lambda)tr(E) = (2\mu+3\lambda)e$ $\qquad$ 5.3.17

Example 5.3.1. Using the above equation, he shows that T must have the same eigenvectors as E and of course if E is diagonal, then so is T, so they have the same principle axes.

Example 5.3.2. Just a way to write the inversion of the above equation (solve it for E).

$2\mu E = T - \lambda tr(T)\, 1/(2\mu+3\lambda).$

$2\mu E_{ij} = T_{ij} - \lambda tr(T)\, \delta_{ij} 1/(2\mu+3\lambda).$

$2\mu E_{ii} = T_{ii} - \lambda T_{ii} 3/(2\mu+3\lambda) = [(1-3\lambda)/(2\mu+3\lambda)]\, T_{ii}$

5.4 The five parameters (two independent) (209).

Page 210. We now have a nice little "model" for our material, that is to say, we have a constitutive equation that describes it. Lai is now going to show you can reconsider our thought experiments so as to replace these constants μ and λ by the experimental numbers E_Y and ν, which we recall are Young's Modulus and Poisson's Ratio. Then the above equation can be $E = E(T; \nu, E_Y)$.

Consider then our <u>axial stress test</u> of the steel bar. It seems clear that in essence, the T matrix acting on any element in the bar is, to zeroth order, just T_{11} and all other components of T_{ij} are basically 0. And the main result is going to be that E_{11} is the stretch in this direction, so we can identify (we are using our earlier knowledge that E_{11} is the fractional stretch in the 1 direction)

young $= E_Y = T_{11}/E_{11} = \mu(3\lambda+2\mu)/(\lambda+\mu)$ $\qquad e_a = E_{11}$

<u>proof:</u> $T_{11} = \lambda e\delta_{11} + 2\mu E_{11} = \lambda e + 2\mu E_{11}$, but if no other stress, then
$e = tr(T)/(3\lambda+2\mu) = T_{11}/(3\lambda+2\mu)$ so then get $T_{11} = \lambda T_{11}/(3\lambda+2\mu) + 2\mu E_{11}$
which says $(3\lambda+2\mu)T_{11} = \lambda T_{11} + 2\mu(3\lambda+2\mu)E_{11}$ or $2(\lambda+\mu)T_{11} = 2\mu(3\lambda+2\mu)E_{11}$
or $T_{11}/E_{11} = \mu(3\lambda+2\mu)/(\lambda+\mu)$. QED.

Recall that we also had a waist tightening on this bar, and that would be

$e_d = -E_{22}$ $\qquad \nu = -\varepsilon_d/\varepsilon_a = -E_{22}/E_{11} = (1/2)\lambda/(\mu+\lambda)$

<u>proof:</u> Since no other stress, $2\mu E_{22} = 0 - \lambda tr(T)\, \delta_{22} 1/(2\mu+3\lambda) = -\lambda T_{11}/(2\mu+3\lambda)$
so $-E_{22}/E_{11} = -E_{22}/T_{11} * T_{11}/E_{11} = \{\lambda/2\mu(2\mu+3\lambda)\} * \{\mu(3\lambda+2\mu)/(\lambda+\mu)\} = (1/2)\lambda/(\mu+\lambda)$ QED

So we can take these two constants either as μ, λ or as E_Y and ν:

$E_Y = \mu(3\lambda+2\mu)/(\lambda+\mu)$
$\nu = (1/2)\lambda/(\mu+\lambda)$ => $2\mu+2\lambda = \lambda/\nu$ => $\mu = \lambda(1/[2\nu] - 1) = (\lambda/2\nu)(1-2\nu)$
$\lambda = 2\mu\nu/(1-2\nu)$

When the T matrix has the one element T_{11} at any point within a material, Lai calls that a **uniaxial stress state.**

You can then write the constitutive equation in terms of E_Y and ν instead of λ and μ and he claims that the result is the set of equations 5.4.8 through 5.4.13. Notice that we use the axial tensile test idea to make the above identifications, then we can replace the constant pair in *any* situation, not just that tensile test.
Maple tells me that

$\mu = (1/2)E_Y/(\nu+1)$ and $\lambda = - \nu E_Y / (2\nu^2+\nu - 1) = - \nu E_Y / [(2\nu - 1)(\nu+1)]$
5.4.14 Lai never states this one

I now agree with all equations on page 210 and the first two on page 211.

A simple shear stress state arises when you have say $T_{12} = T_{21} = \tau$ and all other elements are 0. That is to say, of the 6 parameters of T_{ij}, five are 0 and one is τ. On page 89 Lai showed that an off diagonal element like $2E_{12}$ is the *angle* that the 1 and 2 axes are deformed from their 90 degree rest state by a torque stress of the type T_{12} (just think of that rectangle bending into a 2-piped). In this situation, our Hooke's law F = -kx takes the form $T_{12} = \tau = G\,\Delta\theta = G\,(2E_{12})$ and the effective k for this situation is called G, named the **shear modulus**. I presume this is the same thing as μ on page 203 which we measure from a torqued rod experiment, but Lai does not comment, so hold on this.

Well, he then shows that G = the Lame constant μ, and I agree, so we are halfway there. I presume he will clear this up in an example soon.

Next, we review the **hydrostatic stress** experiment which related pressure σ to k and e, but we have e in 5.4.2 and this gives $k = \lambda + (2/3)\mu$.

proof: $T_{11} = k\,e = k\,tr(E) = k\,tr(T)/(2\mu+3\lambda) = k3T_{11}/(2\mu+3\lambda)$
=> $1 = 3k/(2\mu+3\lambda)$ => $k = (2\mu+3\lambda)/3 = \lambda + (2/3)\mu$.

So to summarize, there are μ,λ,E_Y,ν,k which is 5 interrelated symbols, only 2 of which are independent.

μ	E_Y = Young	$G = \mu$ = shear modulus
λ	ν = Poisson	$k = \lambda + (2/3)\mu$ = bulk modulus

Table on page 212 and Example 5.4.1 The rows of the page 212 table list off which two of the five parameters you want to regard as independent, then you can read off expressions for the other three.
[Note that (5,2) = 5!/3!2! = 10, hence 10 rows in the table.] The Young E_Y for any material is a finite number since any material will stretch in our bar experiment. The ν value is also always finite (ansatz then confirm). So the row with E_Y, ν on the left is a good one to look at. One thing you see is that you

must have $\nu \leq 1/2$ in order to avoid $k < 0$ which would say something expands under hydrostatic pressure which is pretty hard to imagine. Another thing you see is that as $\nu \to 1/2$ from below, you get $k = +\infty$ and $\lambda = +\infty$. The first means that your substance cannot be compressed under hydrostatic pressure! In this situation also we have $\mu = E_Y/3$. So the limit $\nu = 1/2$ represents an *incompressible* isotropic elastic solid.

Table on page 213. This lists 7 metals, glass, and rubber. The metals all have E_Y in the 70-220 GPa range. Remember that large E_Y means small axial stretch. The range is only a factor of 3, with aluminum at the low end being the most stretchable and steel being the least. Glass is a little lower than aluminum. Normally I don't think of glass as stretchable without breaking, but I guess it is.

The most dramatic entry is rubber! As we know, it is very stretchable, so much so that $E_Y \leq .004$ on the same GPa scale. The claim made at the site below is that rubber is liquid-like in its structure and is therefore basically as incompressible as water, and that is why you see $k = \infty$ in the table! This then goes with $\nu = 1/2$ for Poisson. Lai makes not a single comment about the rubber entry, maybe later.

http://www.imechanica.org/node/10589 Discussion of why rubber is so incompressible!

Phil observation: Imagine stretching a rubber band. If incompressible, the volume does not change. Suppose in your (small) stretch, you have $\Delta L = L\varepsilon_a$, and $\Delta R = -R\varepsilon_d$ in diameter. The area is

$$A = \pi R^2 \quad => \quad dA = 2\pi R dR \quad => \quad dA/A = 2\pi R dR/(\pi R^2) = 2dR/R = -2\varepsilon_d$$

You could write then that

$$\Delta V = +\Delta L \times A - L\,\Delta A = L\varepsilon_a A + LA2\varepsilon_a = LA(\varepsilon_a + 2\varepsilon_d) = V(\varepsilon_a + 2\varepsilon_d)$$

If the volume stays the same, then $\Delta V = 0$ and you must have $\varepsilon_a = -2\varepsilon_d$ so Poisson $\nu = -\varepsilon_d/\varepsilon_a = 1/2$.

5.5 Equations of Motion (213).

We are cranking along here at full speed. Take the **F** = m**a** equation obtained earlier (Cauchy). Show that for small motions, we can set $\rho = \rho_0$ and $\mathbf{a} = \rho_0\, \partial_t^2 \mathbf{u}$ to get 5.5.10. In a moment we will install our isotropic expression for T_{ij}, but first: a "possible motion" is **u** which is self-consistent. That is, assume a certain **u**, compute from it E_{ij} as top p 215, compute T_{ij} next as on next line, then see if it works in 5.5.10. If it works, then **t** = T**n** tells you what stress the surface must have to make that motion.

5.6 The Navier Equations (215).

OK, here we go. Start with the isotropic T in terms of E relation with μ and λ for T_{ij}. Then using this, compute $\partial_j T_{ij}$ to get as I show in 5.6.2. Install this into Cauchy F = ma to get (5.6.4). Write out the three equations if you like bottom page 215, or write in vector notation as top page 216. Wow, we have arrived at the Navier equations. Wiki calls this the **Navier-Cauchy equations**, while the Navier-Stokes apply to fluid flow which we have not obtained yet.

Note: a lot of literature refers to T_{ij} as σ_{ij}, the stress tensor. And you see $\partial_j T_{ij}$ as $\sigma_{ij,j}$.

<table>
<tr><td rowspan="4">Continuum mechanics
The study of the physics of continuous materials</td><td rowspan="2">Solid mechanics
The study of the physics of continuous materials with a defined rest shape.</td><td colspan="2">Elasticity
Describes materials that return to their rest shape after an applied stress.</td></tr>
<tr><td>Plasticity
Describes materials that permanently deform after a sufficient applied stress.</td><td rowspan="2">Rheology
The study of materials with both solid and fluid characteristics.</td></tr>
<tr><td rowspan="2">Fluid mechanics
The study of the physics of continuous materials which take the shape of their container.</td><td>Non-Newtonian fluids</td></tr>
<tr><td colspan="2">Newtonian fluids</td></tr>
</table>

Contributions [edit]

Navier formulated the general theory of elasticity in a mathematically usable form (1821), making it available to the field of construction with sufficient accuracy for the first time. In 1819 he succeeded in determining the *zero line* of mechanical stress, finally correcting Galileo Galilei's incorrect results, and in 1826 he established the elastic modulus as a property of materials independent of the second moment of area. Navier is therefore often considered to be the founder of modern structural analysis.

His major contribution however remains the Navier–Stokes equations (1822), central to fluid mechanics.

His name is one of the 72 names inscribed on the Eiffel Tower.

Claude-Louis Navier, French, 1785-1836.

5.7 The Navier Equations in Curvilinear Coordinates (215).

The vector equation of motion is 5.6.9 which says (I call this the Navier equation written in either of these forms, but the official Navier equation is the PDE shown on the second line)

$$\rho_0 \ddot{\mathbf{u}} = \rho_0 \mathbf{B} + \mathbf{div}\mathbf{T}$$

$$\rho_0 \ddot{\mathbf{u}} = \rho_0 \mathbf{B} + (\lambda+\mu)\nabla e + \mu\nabla^2 \mathbf{u} \qquad e = \mathrm{div}\mathbf{u} = \mathrm{tr}(E)$$

Now what is this middle term saying? It is not div grad, it is grad div! Have I ever seen such a thing? Yes! It is the first term of the vector Laplacian,

$$\text{✡}\,\mathbf{B} \equiv \mathrm{grad}(\mathrm{div}\,\mathbf{B}) - \mathrm{curl}\,(\mathrm{curl}\,\mathbf{B}) = \nabla(\nabla\bullet\mathbf{B}) - \nabla\mathrm{x}(\nabla\mathrm{x}\mathbf{B}) = \mathbf{G} - \mathbf{V}$$

and I compute it in my tensor doc Section 13 as the object **G**.

$$\mathbf{G} = \partial'^{n}\{ (1/\sqrt{g'}\,)\,\partial'_{i}\,(\sqrt{g'}\;B'^{i})\}\;\mathbf{e}_n \qquad \mathbf{B} = B'^{n}\mathbf{e}_n$$

So this is something Maple has to compute for me. The last term is in fact the vector Laplacian, so maybe better to write as

$\rho_0 \ddot{\mathbf{u}} = \rho_0 \mathbf{B} + (\lambda+\mu)\ \text{grad}(\text{div}\ \mathbf{u}) + \mu \nabla^2 \mathbf{u}$

I want confirmation that we are looking at the vector Laplacian here. Yes it really is. But in his expansions, Lai is going to leave the middle term as is, namely $(\lambda+\mu)\nabla e$ and then he just writes the ∇ in various systems. Not clear why he fully expands the last term, but not the middle term, but OK. So

$\rho_0 \ddot{\mathbf{u}} = \rho_0 \mathbf{B} + (\lambda+\mu)\nabla e + \mu \nabla^2 \mathbf{u}$

and then all I have to is compute $\nabla^2 \mathbf{u}$ in general systems, which I know how to do.

Meanwhile, when he says **"Hooke's Laws"**, this is what is meant:

$T_{ij} = C_{ijk\ell}\, E_{k\ell}$ // in general

$T_{ij} = \lambda \text{tr}(E)\delta_{ij} + 2\mu E_{ij}$ // isotropic $\qquad E_{ij} = (1/2)[\,\partial_i u_j + \partial_j u_i]$

So write this last as

$T_{ij} = \lambda e \delta_{ij} + \mu[\,\partial_i u_j + \partial_j u_i]$ // Hooke's Law

and again he is going to leave e in the equation "as is". Maybe write as

$T = \lambda e\, 1 + \mu\,[(\nabla \mathbf{u}) + (\nabla \mathbf{u})^{\mathbf{T}}] \qquad e = \text{div}\mathbf{u} = \text{tr}(E)$

and I already know how to write the objects in the brackets from page 60 and my tensor doc verification.

Let's do this first and let's have Maple spit out this second term since I already have most of it. First, for cylindricals we get (order is r,θ,z)

$$\begin{bmatrix} 2\left(\frac{\partial}{\partial r} u_r\right) & -\frac{u_\theta}{r} + \frac{\frac{\partial}{\partial \theta} u_r}{r} + \left(\frac{\partial}{\partial r} u_\theta\right) & \left(\frac{\partial}{\partial z} u_r\right) + \left(\frac{\partial}{\partial r} u_z\right) \\ -\frac{u_\theta}{r} + \frac{\frac{\partial}{\partial \theta} u_r}{r} + \left(\frac{\partial}{\partial r} u_\theta\right) & 2\frac{u_r}{r} + 2\frac{\frac{\partial}{\partial \theta} u_\theta}{r} & \left(\frac{\partial}{\partial z} u_\theta\right) + \frac{\frac{\partial}{\partial \theta} u_z}{r} \\ \left(\frac{\partial}{\partial z} u_r\right) + \left(\frac{\partial}{\partial r} u_z\right) & \left(\frac{\partial}{\partial z} u_\theta\right) + \frac{\frac{\partial}{\partial \theta} u_z}{r} & 2\left(\frac{\partial}{\partial z} u_z\right) \end{bmatrix}$$

and this agrees with p 216 (5.7.1) and (5.7.2). And for spherical, Maple says

$$\begin{bmatrix} 2\left(\frac{\partial}{\partial r}u_r\right) & -\frac{u_\theta}{r}+\frac{\frac{\partial}{\partial\theta}u_r}{r}+\left(\frac{\partial}{\partial r}u_\theta\right) & -\frac{u_\phi}{r}+\frac{\frac{\partial}{\partial\phi}u_r}{r\sin(\theta)}+\left(\frac{\partial}{\partial r}u_\phi\right) \\ -\frac{u_\theta}{r}+\frac{\frac{\partial}{\partial\theta}u_r}{r}+\left(\frac{\partial}{\partial r}u_\theta\right) & 2\frac{u_r}{r}+2\frac{\frac{\partial}{\partial\theta}u_\theta}{r} & -\frac{\cos(\theta)u_\phi}{r\sin(\theta)}+\frac{\frac{\partial}{\partial\phi}u_\theta}{r\sin(\theta)}+\frac{\frac{\partial}{\partial\theta}u_\phi}{r} \\ -\frac{u_\phi}{r}+\frac{\frac{\partial}{\partial\phi}u_r}{r\sin(\theta)}+\left(\frac{\partial}{\partial r}u_\phi\right) & -\frac{\cos(\theta)u_\phi}{r\sin(\theta)}+\frac{\frac{\partial}{\partial\phi}u_\theta}{r\sin(\theta)}+\frac{\frac{\partial}{\partial\theta}u_\phi}{r} & 2\frac{u_r}{r}+2\frac{\cos(\theta)u_\theta}{r\sin(\theta)}+2\frac{\frac{\partial}{\partial\phi}u_\phi}{r\sin(\theta)} \end{bmatrix}$$

and every item agrees with Lai.

Checking the Navier equations will take more work. // I had to spend 6 days doing vector Laplacian work, and we now have Appendix I. Here then is my check for page 217 top in **cylindricals**

$$-\frac{B_r}{r^2}+\frac{\frac{\partial}{\partial r}B_r}{r}+\left(\frac{\partial^2}{\partial r^2}B_r\right)-2\frac{\frac{\partial}{\partial\phi}B_\phi}{r^2}+\left(\frac{\partial^2}{\partial z^2}B_r\right)+\frac{\frac{\partial^2}{\partial\phi^2}B_r}{r^2}$$

1 2 3 4 5 6 p 217 (5.7.4) for $(\nabla^2 u)_r$ // Lai uses θ for φ

$$2\frac{\frac{\partial}{\partial\phi}B_r}{r^2}+\frac{\frac{\partial^2}{\partial\phi^2}B_\phi}{r^2}+\left(\frac{\partial^2}{\partial z^2}B_\phi\right)-\frac{B_\phi}{r^2}+\frac{\frac{\partial}{\partial r}B_\phi}{r}+\left(\frac{\partial^2}{\partial r^2}B_\phi\right)$$

1 2 3 4 5 6 p 217 (5.7.5) for $(\nabla^2 u)_\varphi$ // Lai uses θ for φ

$$\left(\frac{\partial^2}{\partial z^2}B_z\right)+\frac{\frac{\partial^2}{\partial\phi^2}B_z}{r^2}+\frac{\frac{\partial}{\partial r}B_z}{r}+\left(\frac{\partial^2}{\partial r^2}B_z\right)$$

1 2 3 4 p 217 (5.7.6) for $(\nabla^2 u)_\varphi$ // Lai uses θ for φ

Now let's go to the bottom of page 217 and do this for **sphericals**, making adjustments as needed.

The r equation:

$$-2\frac{B_r}{r^2}+2\frac{\frac{\partial}{\partial r}B_r}{r}+\left(\frac{\partial^2}{\partial r^2}B_r\right)-2\frac{\cos(\theta)B_\theta}{r^2\sin(\theta)}-2\frac{\frac{\partial}{\partial\theta}B_\theta}{r^2}-2\frac{\frac{\partial}{\partial\phi}B_\phi}{r^2\sin(\theta)}+\frac{\frac{\partial^2}{\partial\phi^2}B_r}{r^2\sin(\theta)^2}+\frac{\cos(\theta)\left(\frac{\partial}{\partial\theta}B_r\right)}{r^2\sin(\theta)}+\frac{\frac{\partial^2}{\partial\theta^2}B_r}{r^2}$$

1 2 3 4 5 6 7 8 9

Now we need to compute Lai's first term which turns out to be my first three terms:

$\partial_r(r^{-2}\partial_r(r^2u_r)) = \partial_r(r^{-2}[r^2\partial_r u_r + 2ru_r]) = \partial_r{}^2u_r + 2\partial_r(r^{-1}u_r) = \partial_r{}^2u_r + 2\, r^{-1}\partial_r u_r - 2r^{-2}u_r = 3+2+1$

Next, compute Lai's second term, which it turns out to be my last two terms:

$r^{-2}S^{-1}\partial_\theta(S\partial_\theta u_r) = r^{-2}S^{-1}[\, S\partial_\theta{}^2u_r + C\partial_\theta u_r)] = r^{-2}\partial_\theta{}^2u_r + r^{-2}\cot(\theta)\, \partial_\theta u_r = 9 + 8$

Next compute Lai's second last term

$-2r^{-2}S^{-1}\partial_\theta(Su_\theta) = -2r^{-2}\partial_\theta u_\theta \;\; -r^{-2}S^{-1}Cu_\theta) = -2r^{-2}\partial_\theta u_\theta \;\; -2r^{-2}\cot\theta\, u_\theta = 6 + 4$

So only my terms 5 and 7 are unaccounted for. And they are the ones he shows there. So I have now **verified 5.7.14.** In retrospect, I have marked the scalar Laplacian terms.

The θ equation:

Here I have to play with his α term this way:

$\alpha = r^{-2}\partial_\theta(\, S^{-1}\partial_\theta(S\, u_\theta)) = r^{-2}\partial_\theta(\, S^{-1}[Cu_\theta + S\partial_\theta u_\theta\,]) = r^{-2}\partial_\theta(\, \cot\theta u_\theta) + r^{-2}\partial_\theta{}^2u_\theta\,])$

$= r^{-2}(\partial_\theta\cot\theta)u_\theta + r^{-2}\cot\theta\partial_\theta u_\theta + r^{-2}\partial_\theta{}^2u_\theta$

So I can now write HIS 5.7.15 this way

$\partial_r(r^{-2}\partial_r(r^2u_\theta)) + r^{-2}(\partial_\theta\cot\theta)u_\theta + r^{-2}\cot\theta\partial_\theta u_\theta + r^{-2}\partial_\theta{}^2u_\theta$

$(1/r^2\sin^2\theta)\partial_\varphi{}^2u_\theta + (2/r^2)\partial_\theta u_r - (2/r^2)\cot\theta\csc\theta\, \partial_\varphi u_\varphi$

But $\partial_\theta\cot\theta = -\csc^2\theta$ so his way becomes:

$\partial_r(r^{-2}\partial_r(r^2u_\theta)) - r^{-2}\csc^2\theta\, u_\theta + r^{-2}\cot\theta\partial_\theta u_\theta + r^{-2}\partial_\theta{}^2u_\theta$
1 5α 2α 3α
$(1/r^2\sin^2\theta)\partial_\varphi{}^2u_\theta + (2/r^2)\partial_\theta u_r - (2/r^2)\cot\theta\csc\theta\, \partial_\varphi u_\varphi$
4 6 7

Meanwhile, here is MY result:

$(\text{✡}B)_\theta = \nabla^2(B_\theta) - (1/r^2)\,[\csc^2\theta\, B_\theta - 2\,\partial_\theta B_r + 2\cot\theta\csc\theta\partial_\varphi B_\varphi\,]$
5 6 7
$\nabla^2u_\theta = (1/r^2)\partial_r(r^2\partial_r u_\theta) + (\cot\theta/r^2)\partial_\theta u_\theta + (1/r^2)\,\partial_\theta{}^2u_\theta + (1/r^2\sin^2\theta)\partial_\varphi{}^2u_\theta$
1 2 3 4

So his α term contains 2 of the scalar lap terms, it is a messy huge shuffle but I have now verified his form.

The φ equation:

First, compare:

$$(\nabla\times B)_\theta = \nabla^2(B_\theta) - (1/r^2)\,[\csc^2\theta\, B_\theta \underbrace{- 2\,\partial_\theta B_r}_{6} + \underbrace{2\cot\theta\csc\theta\partial_\varphi B_\varphi}_{7}\,]$$

$$(\nabla\times B)_\varphi = \nabla^2(B_\varphi) - (1/r^2)\,[\csc^2\theta\, B_\varphi - 2\csc\theta\partial_\varphi B_r - 2\cot\theta\csc\theta\partial_\varphi B_\theta\,]$$

so the change rule is:

in term 6: $\partial_\theta B_r \rightarrow \csc\theta\partial_\varphi B_r$

in term 7: $\partial_\varphi B_\varphi \rightarrow \partial_\varphi B_\theta$

and this agrees with what he has.

FINALLY, after about 7 days, I have verified all equations in Section 5.7. To his credit, Lai did not have a single mistake in all this stuff!

Now let's verify Lai's page 66 results in sphericals (I had problems there). OK, I did the r part on page 66 which is the same as appears in problem 2.75, who would ever write stuff this way??? I think it is just from the way he derived it. While here, what about 2.74?

5.8 Superposition (218).

The usual claim is well stated here concerning superposition for a linear PDE. What is not stated is that the boundary conditions on the PDE must also be of a corresponding superposed nature.

Comment on the PDE: This PDE 5.5.10 is "for" the displacement u_i, but it contains the "force" term $\partial_j T_{ij}$ and we have then to use in the general case $T_{ij} = C_{ijk\ell}\, E_{k\ell}$ where $E_{k\ell} = (\partial_\ell\partial u_k + \partial_k\partial u_\ell)/2 = (\nabla u)^S{}_{k\ell}$ to "close" the equation. You can see then that second space derivatives arise, and together with the time second derivative, we have the potential for a normal "wave equation", but it might not be quite such an equation depending on how the derivatives fall out. In the special isotropic case we have $T_{ij} = \lambda e\delta_{ij} + 2\mu E_{ij} = \lambda e\delta_{ij} + 2\mu(\nabla u)^S{}_{ij}$ and then that λe term causes an "extra term" to appear in 5.6.9 $\rho_0\ddot{\mathbf{u}} = \rho_0\mathbf{B} + (\lambda+\mu)\nabla e + \mu\nabla\times\mathbf{u}$ and that extra term is what makes it be "not" a wave equation in each component (assuming $\mathbf{B} = 0$ here).

A.1 Plane Elastic Waves.

5.9 Plane Irrotational Waves (218).

Example 5.9.1: Here we start by assuming a simple strain tensor $E_{11} = \varepsilon\,(2\pi/\ell)\cos(kx_1-\omega t)$ [else 0] which causes dilatation e to oscillate as shown 5.9.3. $T_{11} = (\lambda+2\mu)E_{11}$ also oscillates, and $T_{22} = T_{33} = \lambda E_{11}$ so

T is diagonal but first element differs from other two. Somehow we end up with a simple wave equation for u in 5.9.7 with a speed c_L. But how do we get to this equation from the original NC equation shown above?

terms = $(\lambda+\mu)\nabla_1 e + \mu\nabla^2 u_1 = (\lambda+\mu)\partial_1 e + \mu\partial_1{}^2 u_1 = (\lambda+\mu)\partial_1(\text{div}\mathbf{u}) + \mu\partial_1{}^2 u_1$
$= (\lambda+\mu)\partial_1(\partial_1 u_1) + \mu\partial_1{}^2 u_1 = (\lambda+2\mu)\partial_1{}^2 u_1$

and there you see that it just happens to come out. Each of the two terms makes a $\partial_1{}^2 u_1$ term and we add then and this determines the velocity. In this example u_1 is never stated, but it could be any wave solution of the wave equation I think. It's most general atomic form would be p 220 (i).

In this example, he claims $\nabla\mathbf{u}$ is symmetric, but how can that be? Well, $E_{ij} = E_{11}\delta_{ij}$ so E is symmetric, and then $\mathbf{E} = \nabla\mathbf{u}^S + \nabla\mathbf{u}^A$ so $\nabla\mathbf{u}^A = 0$. Since this last is the spin tensor $\mathbf{W}$, and since $\mathbf{W} = 0$, this type of wave is called **irrotational**. The idea is that if you track a particle in this type of wave, it is not rotating at all, it is just "breathing", getting larger and smaller as e oscillates. In fact, the particle of matter is only changing size in the x_1 direction. This is a **longitudinal** wave and it is a **dilatational** wave since e is changing.

Lesson #1: This NC equation of motion even in the isotropic cases is a lot "richer" in structure than a regular wave equation and you are going to see various kinds of wave action with different velocities and different spatial structures.

Example 5.9.1. Here we stay in the context of the previous text (certain form for E and T) and *assume* a general atomic form for the PDE solution and we add the idea of a boundary in the x=0 plane, with the medium only to the right. So now we have a boundary condition to worry about.

They first consider the case where the BC is an applied plane of displacement $b\sin(\omega t)\mathbf{e}_1$ and this fully determines the solution u_1. To make the assumed solution form work, $\omega = \omega_L$ and is not arbitrary, that is, you must drive at resonance to get the desired solution. So we are not solving an ODE here with BC's. We are assuming a particular solution form and we end up with the resonance particular solution. Fine.

They second consider the case where the BC is a plane of "surface traction" at x=0 with this same $f\sin(\omega t)\mathbf{e}_1$form. This surface traction t is related to T_{ij} and this then lets us determine the proper atomic coefficients α and β and we have the same resonance solution type, but its phase and coefficient is different. Compare (v) to (xi).

Lai is not solving the world here, he is just providing interesting toy cases. Everything is Cartesian so we don't have the added complexity of curvilinear coordinates to worry about (yet).

subdetail: p 220 Example 5.9.1. Trouble with step (vi). What is T_{11} in this example? I seem to get

$T_{11} = (\lambda+2\mu)\, E_{11}$ from 5.3.11 $E_{11} = \partial u_1/\partial x_1$ from 5.5.11

so

$T_{11} = (\lambda+2\mu)\, \partial u_1/\partial x_1$ and then all is OK.

5.10 Plane Equivolumnal Waves (221).

The context is now different, we no longer have the same E_{ij} and T_{ij} structure as in the last section. But E is still symmetric, so we are still irrotational. A particle here just moves back and forth in a **transverse** direction, it does not rotate. So, this time we assume the simplest possible sine wave travelling in the 1 direction, but u points in the 2 direction. The 3 direction plays no role at all, and $u_1 = u_3 = 0$. Now let's look again at our NC (Navier/Cauchy) equation. First

e = div **u** = $\partial_i u_i = 0$ because all we have is $u_2(x_1)$ so $\partial_2 u_2 = 0$ and the other $u_i = 0$.

So in this kind of wave, e=0! Then we have $\rho_0 \ddot{\mathbf{u}} = \rho_0 \mathbf{B} + (\lambda+\mu)\nabla e + \mu \nabla^2 \mathbf{u}$ and $= \mu \nabla^2 \mathbf{u}$ and then we have the simplest possible wave equation and velocity this time will be just $c_T^2 = \mu/\rho_o$. We get no contribution this time from that first ∇e term, so the phase velocity for this transverse wave is different, a major interesting fact. On p 222 he writes the ratio c_L/c_T and shows this is always ≥ 1 so larger phase velocity in the L wave. Both are same in that special case $\nu = 1/2$ (the rubber incompressible limit).

Example 5.10.1 . This time we assume the same kind of atomic solution with α and β and c_T. We solve the exact same problems as before: the BC is either a plane at x=0 of sin(ωt) displacement or surface traction. Solutions are stated, I skipped the details but this is the logical example to do! As before we are talking resonance solutions.

Comment: I can just imagine how people considered the vacuum to be an isotropic medium with some μ and λ. For the transverse wave just discussed, we get $\rho_0 \ddot{\mathbf{u}} = \mu \nabla^2 \mathbf{u}$. But what exactly does one use for ρ_0, the density of the ether being a vacuum? It would be very logical I think to use continuum mechanics as a model for light waves. I might find an article on this at some point.

Comment: This transverse case just considered is a **shear wave**. Your medium must be able to support $E_{12} \neq 0$, which air cannot do, so air only has the longitudinal wave. In steel, the longitudinal wave speed is about 6000 mps which is 13,400 mph. The transverse waves to about half that fast. Air longitudinal is 340 m/sec or 760 mph. Water longitudinal is 1500 m/sec, about 4x faster than air.

Example 5.10.2. This is a fancier example. Wave travels in 1 direction as before, but we have $u_3(x_1,x_2)$ with a fixed spatial profile in x_2, so now all three spatial directions are involved. For the same reason as before, we have e = 0. It seems to me that the NC equation should just be $\rho_0 \ddot{\mathbf{u}} = \mu \nabla^2 \mathbf{u}$ and in particular we get $\rho_0 \ddot{u}_3 = \mu \nabla^2 u_3 = \mu (\partial_1^2 + \partial_2^2) u_3$ so this is *not* a simple 1D wave equation. From the assumed form of u_3, we are going to find that $\partial_1^2 \rightarrow k_1^2$ and $\partial_2^2 \rightarrow p^2$ so the velocity in this case is going to be $c^2 = \mu(k^2+p^2)/\rho_0$. I think this equation might be written $\rho_0 \ddot{u}_3(x_1,x_2) = [\, \mu (\partial_1^2) + p^2 \,] u_3(x_1,x_2)$ so it is a 1D Helmholtz wave equation with this extra constant p^2 . In any event, the wave only travels in the x_1 direction at a speed that depends on both k and p. The particular x_2 shape is chosen so there are no surface tractions at $x_2 = \pm h$. So this wave might then apply to sending a wave signal through an infinitely wide slab of material of height 2h. It would be a mode for this slab considered as an acoustic **waveguide**.

This example shows the possible complexity of the set of wave solutions to the NC equation, even when e=0.

Comment: Now Lai never uses the word "acoustic", but these waves are displacement waves and I am sure if they were in the audio range and you put your ear against the end of a piece of steel, you will hear something. People used to listen to the rails to detect the coming train? They got the advantage of that high sound speed velocity in the metal.

Example 5.10.3. Here Lai is just getting things into the usual $\mathbf{k}\bullet\mathbf{x}$-ωt form where for him $\mathbf{k} = k\hat{\mathbf{k}} = k\mathbf{e}_n$ with $k = 2\pi/\ell$. He has to use ℓ for wavelength since λ and μ are those elastic parameters! He uses my script ℓ of course! The main result of this section is the basic form 5.10.8 and 5.10.10. which shows phase as $\mathbf{k}\bullet\mathbf{x}$-ωt and in one case you have $\mathbf{e}_n$ for longitudinal wave and $\mathbf{e}_t$ for transverse. They have $\omega_L$and ω_T of course. We always write $c = \ell/T = \ell f = \ell(\omega/2\pi) = (\ell/2\pi)\omega$ so $c_L = (\ell/2\pi)\omega_L$ for example. He shows k explicitly and writes the phase then as $\mathbf{k}\bullet\mathbf{x}$-ωt = $k\mathbf{e}_n\bullet x$ -kct = $k(x\bullet e_n - ct)$ where I use $c = \omega/k = T^{-1}/L^{-1} =$ L/T = velox. These two equations are very important in what is about to follow. Each one is also given an arbitrary phase η. Notice that the $\mathbf{k}\bullet\mathbf{x}$ causes various α_i trig functions to appear inside the phases.

Example 5.10.4. This is a painfully detailed example which I did in full. We need everything here when we do the reflection work of the next section. All 6 of the u_1 and u_2 equations get used.

Comment: I think he likes ε for amplitude to remind us that amplitudes must be small!

5.11 Internal Reflection of Plane Elastic Waves (226).

This section is certainly a tour de force. I did every detail except 5.11.17. Here is a summary in English. Imagine you have some internal T wave hitting an internal boundary of your material (normal $\mathbf{e}_2$), vacuum outside say, so there can be no "surface tractions" from that vacuum on the $x_2 = 0$ reflection plane. This $\mathbf{t} = 0$ rule means $Te_2 = 0$ which means all three T_{i2} (due to the total $\mathbf{u}$) must vanish on that plane. The $T_{23} = 0$ is trivial, but requiring the other two produces information laden equations 5.11,6.7, two huge expressions must vanish. Assuming that the three cosines are the same (hard to imagine any other possibility), we find that the three phases must be related as on bottom of page 227. This then leads to 5.11.11 which says many things. (a) the transverse reflected wave is just what you would think, equal angles, equal ℓ wavelength. The second reflection is a longitudinal wave at some Snell-like angle, but as they point out, we don't have refraction through the boundary, we have another reflection. The thing that plays the role in the Snell-like law is called n, and is called index, and $n = c_T/c_L = (\mu/[\lambda+2\mu]^{1/2}$. As I recall, n is also the velocity ratio in optics. We know here that $n \leq 1$ since $c_L \geq c_T$. So the main thing that pops out here is that (b) you need to have an extra L wave reflecting off at the weird Snell angle α_3 to get no surface tractions. (c) Finally, those two info-laden equations then give you the amplitude of the two reflected waves.

So this is totally new to me! I never heard reflection of waves in a solid before. They claim that if the incident T wave were polarized out of the plane of paper in the page 226 paper, then things are more usual and you don't need that extra reflected wave. On the other hand, if your incoming is an L wave, then you get two reflected waves again, but Lai does not do this case.

There is no refracted wave through the boundary because there is nothing out there to carry a wave! optics is different because you don't need anything to carry the wave! Also there is no L wave in optics.

So I can see that a huge very messy general case is looming out there. You have two different mediums on the sides of the reflection plane. Then surely there will be at least one refracted wave in

addition to the two reflected waves. This would be a very messy subject, but surely well studied and well written up somewhere, but the Lai book thankfully does not do it. Probably if both media have the same index n, nothing happens!

I wonder if waves in the earth's solid mantel are simple enough to follow these rules somewhat. The amplitude is always a few feet or less I would guess so perhaps "small". But then liquid core adds more complexity.

5.12 Vibration of an Infinite Plate (226).

One way to think of this is that if you have an internal wave straight onto the boundary plane, it probably reflects 100% and so this happens at both boundaries and they will add to make standing waves. So this thing is going to have a "mode". We have our simple wave equation in x_1 only with either c_T or c_L. He wants to assume the form 5.12.3 where you see the "standing part" out front. This form then makes things nice in that ∂_1^2 and ∂_t^2 both basically restore you with a minus sign. You will assert some BC's at the plate faces and this will quantize k and this ω. I don't see why the phrase "thickness stretch" is used for the L mode. It is L, and that is the thickness direction, yes. The T case is a shear mode. Of course you have displacement through the entire plate thickness in either case. I need to see more on what these modes look like. I think it is just reflected L and T waves, big deal.]

Example 5.12.1 (229). Take the L case, drive the left face with a sine at some ω, glue down the entire right face so it cannot move L wise (it is "fixed"). Our assumed form works and he finds all the constants and the displacement is as in p 230 (v). Every particle is in the same time phase but amplitude of the standing wave envelope varies with x_1 through the plate. There are resonant ω's as shown which result in infinite amplitude, so you could perhaps break the plate if it has no damping. In fact you would get out of the small-amplitude model and not clear what would really happen. Amplitude is of course 0 at the right plane.

Example 5.12.2 (230) Same thing but T wave. Fixed again on the right face (meaning sideways holding glue!) and it is driven in shear mode with a surface traction in this case in the $\mathbf{e}_2$ direction. Standard form works again, we find the constants, and again there are resonances which can break the plate. The traction BC is a derivative BC. We are just doing 1D wave equation with BC's here.

So this concludes our small amplitude waves exploration of elastic solids, a nice little introduction. These waves are dynamic solutions of the Navier-Cauchy equation. I think the small amplitude is just to make Hooke's law reasonable, the higher terms don't matter much. Well, it also justifies using the E tensor as strain characterization, and also justifies things like $\rho \approx \rho_0$ in some places and $\mathbf{x} \approx \mathbf{X}$.

We are now going to enter the world of the Civil Engineer I suspect. Beams and bending and all those great things. Lots of trade terms start appearing, like a "prismatic bar" which means an extruded shape. Lots of "moments" will appear, the word torque is never used.

A.2 Simple Statics situations (231).

5.13 Simple Extension (231)

Here we take an extruded 2D shape (known as a cylinder, or a prismatic bar) and do our axial load test on it with stress σ on all of each end. If 1 is the length direction, we expect $T_{11} = \sigma$ and other $T_{ij} = 0$, and all these values are constants in space! I expect no internal shear like T_{12} etc. If T_{ij} are all constants, then certainly $\partial_j T_{ij} = 0$ and if no body force and we are static, the Navier is happy. Lai shows that on the sides of the bar there are no surface tractions, and that agrees with our known BC's there.

Knowing all the T_{ij}, it is then an easy matter to compute the E_{ij} from Hooke's Law on page 210 and Lai next does that. Then he is able to deduce the three u_i from these E_{ij}. Note that E_{ij} is diagonal. We learn now for the first time that in addition to stretching in the 1 direction, the waist gets smaller in a manner proposed in our opening thought experiment section. Specifically, $u_1 = (\sigma/E_Y)\, x_1$ says that the axial displacement grows as you move down the bar from its fixed end. Makes sense to me. And the transverse direction shrinkage is proportional to distance from the "center line". But "how does it know" where the center line is for some general cross section area!! You cannot just pick any line, only one line will have zero displacement. Right now this is a huge mystery to me! How do the equations know where that center line is located? This might be related to the next mystery issue.

Noted added 4.13.12. When I first read Lai's comment about "of course a rigid body displacement field can be added" I did not grok, now I do. The next theorem shows that such a rigid adder adds an antisymmetric contribution to ∇u, but since $E = (\nabla u)^S$, such an adder does not change E, but it does change u! Now consider the solution $u_1 = (\sigma/E_Y)\, x_1$. This is the correct solution if the left end of the bar is constrained from moving, but typically both ends might be free when you pull. In this case, we can just add a translation to the above solution and say for example $u_1 = (\sigma/E_Y)\,(x_1 - x_{1c})$ where x_{1c} is the center of the bar. The added term does not affect E of course.

My confusion above was really with $u_2 = -(\nu\sigma/E_Y)\, x_2$ and "how it knows where the cylinder center line is located". It does not know of course. Again, you could write $u_2 = -(\nu\sigma/E_Y)\,(x_2 - x_{2c})$ as the correct solution in a situation where you in effect prevent the center line from moving, which would be the case for a perfectly symmetric applied stress on both ends. I replicated this observation below. All resolved.

On page 233 2/3 down, Lai is going to claim this theorem:

Theorem: For any rigid body displacement field, ∇u is antisymmetric.

I don't recall him even discussing this idea before. He does discuss the most general rigid body motion on page on page 82. We know that a rotation matrix R_{ij} has the property $R^T = R^{-1}$. Consider a small rotation so that $R = 1 + \delta R$. Then our rule on R says that $(1 + \delta R)^T = (1 + \delta R)^{-1}$. For a small matrix A, we know that $(1 + A)^n = 1 + nA +$ higher, so our last equation says that $1 + (\delta R)^T = 1 - \delta R$ which then says $(\delta R)^T = -(\delta R)$, so indeed, (δR) is antisymmetric. This is consistent with the known fact that $(J_i)_{jk} = -i\,\varepsilon_{ijk}$ for the rotation generators.

What about a translation? We know that if we translate a rigid object in some direction, all the u_i change by a constant amount, so $\partial_j u_i$ gets no adder from doing this, so E_{ij} stays the same, so T_{ij} stays the same. Our concern then is with the rotations.

We know that $F = 1 + (\nabla^X \mathbf{u})$ from p 84 and 86, so write this as $1 + \nabla\mathbf{u}$. For a pure rotation our polar decomposition says F = R and we just said that $R = 1 + \delta R$. Thus, $\nabla u = \delta R$. Since δR is antisymmetric, so is $\nabla\mathbf{u}$ for a rotation. For a translation, $\mathbf{u} \rightarrow \mathbf{u} + \mathbf{a}$ and $\nabla\mathbf{u}$ remains unchanged. A most general rigid

body motion is translate, rotate, translate (to get arbitrary rotation origin). The first and last contribute nothing to $\nabla \mathbf{u}$, so you end up with $\nabla \mathbf{u} = \delta R$ and it is antisymmetric. So I think this proves this theorem.

Example 5.13.1. Here Lai integrates the three E_{ii} = stuff diagonal elements of E_{ij}. The feel here is much like my Stackel doc where you deal with "forms" of functions. He obtains results which are more general that those shown in 5.13.4, but the extra stuff represents I think a combination of rotation and translation in the rigid body sense as per above. I did not finish all the details, but I think it is easy to show that the extra pieces generate a ∇u which is anti-symmetric and hence "just rotational/translational". I think this also explains my mystery above. Since you can add any constant to $u_2 = -(\nu\sigma/E_Y)x_2$ you can then get a new u_2 which is then $u_2 = -(\nu\sigma/E_Y)(x_2\text{-}a)$ and then the center line is now at $x_2 = a$. See inserted note above.

Page 233 continues hazily talking about centroids. Lai computes the max shearing stress and it happens in any plane whose normal is 45 degrees separated from the cylinder axis. This is just because when you take the end normal or the cross normal, there is no shearing stress at all, and it has to max somewhere and halfway seems reasonable, and of course we proved this earlier.

Lai then gets to **Saint-Venant's Principle** [1855] again hazily. He suggests that when you pull on a bar, your pull is likely not evenly distributed. I see that they use hydraulic testing machines to tense a bar, and it seems that big chucks grab the bar around the edges, so it won't then be uniform on the faces of the end of the bar (which is our usual assumption when we say $T_{11} = \sigma$). In fact, the ideal chuck provides a delta function ring around the end of the bar!



But some nice guy put up these pictures below to show demonstrate the principle: If you pull on a cloth as shown, the strain pattern is only distorted near where you are pulling and not far away. Far away things are about the same as if you did a uniform stress on the edge of the cloth. Pictures say many words. The

other two examples are also very good. I would say top right that the applied stress does "pass through the centroid".



5.14 Torsion of a Circular Cylinder (234)

A very good section. We start off making the "stack of rotated coins" ansatz (5.14.1) for the displacement field **u** with some unknown rotation angle α which is a function of coin location, ie, $\alpha = \alpha(x_1)$. From this u we compute the strain tensor E_{ij} which we find has only off-diagonal elements (but no E_{23} elements). It is then trivial to compute T_{ij} which then has the same form which is displayed top of page 236. So this is our candidate stress tensor to describe this situation, where each T_{ij} is a function of space of course as shown in 5.14.4. Lai then wants to show that this stress tensor satisfies static Navier-Cauchy $\partial_j T_{ij} = 0$. We find that it does provided $\alpha(x_1)$ is linear in x_1 meaning $\alpha(x_1) = \alpha' x_1$ in our problem, an expected result. Lai next shows (p 236) that this T_{ij} form results in $\mathbf{t} = 0$ surface traction on the lateral surface, another expected result. On the end surface, calculation of **t** shows the traction distribution that is required to cause this stack-of-coins torsion pattern. From this pattern he then computes the total torque on the bar at each end by integrating r x FdA over each end surface. This total torque **M** is called a "moment" as usual in this book. The integral here involves the moment of inertia of a disk which he calls I_p assuming a mass density of $\rho = 1$. If we regard I_p and M as "givens" in this problem, then T_{ij} is as shown 5.4.16. He notes that the moment will never be applied in this perfect manner to the bar ends. In practice, the machine will have those chucks and it will be applied to the surface near the bar ends, but our friend

Saint-Venant then says as long as we stay away from the ends, it is as if the torque were applied with the exact **t** computed in 5.14.10. The figures on pages 236 and 237 are just guides and are not referenced.

Example 5.14.1. We continue our study of the above situation. At some selected point inside the cylinder we know that T_{ij} can be diagonalized and will have certain eigenvalues λ_i and eigenvectors n_λ. The λ will of course be the diagonal values of $T'_{ij} = \lambda_i \delta_{ij}$. If we compute these three eigenvalues, we know that the largest of them is the "maximum normal force", and half the difference of the max in min will be the "maximum shearing force". In this situation, these two maxima are the same, $M_t r/I_p$. This in turn is maximal at the outer surface. This result seems intuitive for the max shear stress and less intuitive for the max normal stress.

So now we look at the details. We first compute the three eigenvalues and find $\lambda = \pm M_t r/I_p$ and 0 as the three eigenvalues, so we know what T'_{ij} looks like. Using $Tn_\lambda = \lambda n_\lambda$ we manually compute the eigenvector corresponding to $\lambda = + M_t r/I_p$ as in 5.14.18, that is, $\mathbf{n} \sim \mathbf{e}_1 - \mathbf{e}_2$. This normal defines a plane which slices through the cylinder at 45 degrees looking from the side so slices would be ellipses I guess. Any point lying on one of these slices will have the max normal force perp to this plane because this is the largest of the λ's. And of course this is maximum at r = a. Lai does not comment on the max sheer stress but we can see that this will be in the $\hat{\varphi}$ direction, max at the surface.

So how would you expect cracks to form, places where the material cannot maintain the stress? For the max normal stress, imagine that 45 degree plane which defines an ellipse of the cylinder. If a crack starts on some 45 degree plane, it will likely spread along the plane since 3D points along this plane have max normal stress inside the material AND they touch the point where the crack started. And this is most likely to happen at the surface. Thus, you should see a crack line on the surface which lies on such a tilted ellipse. Such a line they call helical, though I don't think that is quite right. I see that crack analysis is poorly understood even in 2012. [someone mentions the cucumber experiment, a good one!]

This from a Google book: The first case is that in axial loading, cracks form transversely, as if you are just ripping the material apart. In the shear case, $\hat{\varphi}$ would fall into the "transverse" case. Hard metals do the helical thing. You get the impression that crack analysis is not a simple subject:

In fracture-mechanics terms (§ 5.2.3) a crack usually grows in Mode I, that is, perpendicular to the direction of loading in a uniaxially loaded fatigue specimen. Fatigue cracks in torsional fatigue specimens, in general, may follow one of three directions. This arises because, in a cylindrical bar subjected to a torque, the planes of maximum shear stress along which microcracks develop are transverse and parallel to the bar axis, whereas the planes of maximum principal stress along which Mode I cracks grow are at 45° to the bar axis. Soft ductile metals generally give torsional fatigue fracture faces which are transverse to the bar axis, although some specimens exhibit longitudinal cracking especially at high strain amplitudes, whereas the fracture faces of specimens of harder metals follow a helicoidal path around the circumference. It would seem therefore that when a surface microcrack has developed to a certain size under a pure torsional cyclic load, it can either continue growing as a microcrack on a transverse or longitudinal plane or, if it deviates from these planes for some reason, the cyclic normal stress now acting across it will enable it to grow eventually as a macrocrack in Mode I on a plane at 45° to the bar axis; this results in a helicoidal fracture face. However, the crack front of a microcrack intersects a 45° plane at only one point, so that this change of direction is not easily achieved. Obstacles to the progress of a microcrack in the form of intermetallics, inclusions, flaws, etc., which will cause it to deviate from a transverse plane so enabling it to

You can see lots of pictures of "spiral leg fractures" on the web that no doubt are exactly this mechanism.

Example 5.14.2 Here Lai obtains a wave equation for angle function $\alpha(x_1,t)$ which I think means you can have "**torsion waves**" travelling down a bar and the velocity will be C_T which seems right since all the "action" in such a wave is transverse. I have never thought about a torsion wave before. You would have to make your chuck oscillate and thus have a driving force at one end, and the wave would go down the bar and St Venant says OK to drive it that imperfect way.

5.15 Torsion of a Non-Circular Cylinder (239)

OK, so what happens if you have some arbitrary cross section? We make the ansatz here that the "coins" are no longer going to stay flat coins, but will have displacement along the bar as well as the twisting effect of the previous section. So we throw in a **warping function** $u_1 = \varphi(x_2,x_3)$ which of course means that a coin (with strange boundary) will be warped in some way longitudinally. Lai claims that if we again consider the Navier static equation, we get the same result that $\alpha' =$ constant (I did not check this). But we get another requirement which is that φ must be a solution of the **transverse Laplace equation** 5.15.4 ! This comes from the i=1 Navier equation which "just worked" in the circular case and gave 0 since $\varphi = 0$ in that case.

Next, with this assumed φ function, Lai computes the lateral surface traction **t** as shown in 5.15.6. In the circular cross section case top page 236 we got **t** in the $\mathbf{e}_1$ direction but the magnitude was 0, so the φ function was not needed. This worked because we could write an exact simple form for the lateral normal **n**. But in the general case, **n** could be anything (in the 23 plane I guess) and the **t** result in this general case is shown top of page 240. The single condition on the 2D boundary is a classic Neumann BC for $\partial_{\mathbf{n}}\varphi$! So given the 2D shape, I think I know how to find a Laplace solution with *any* such BC from my Stakgold work (but right now I don't remember the solution method). I might write a Smythian form and

then require ∂_n of that form to meet the BC and that will determine all the coefficients. I know that the solution exists and is unique!

So we have then a general problem: For some arbitrary bar cross section, how do you find a solution of the Laplace equation which satisfies the Neumann condition at all points on the cross section surface? This problem is called the **St-Venant's Torsion Problem.**

Comment: A Dirichlet membrane problem is a drumhead with Laplace solutions -- the z displacement u_z is prescribed around the perimeter and this is usually set to constant 0, but you could of course have a drumhead with slightly non-planar perimeter. A Neumann membrane problem has a prescribed $\partial z/\partial n$ shear stress around the perimeter, where n is the normal. So you prescribe some shear surface traction around you rubber membrane and solve Laplace inside. The solution will not have $u_z = 0$ at the perimeter of course. So this is what happens to one of the "coins" in this torsion problem. It does not have $u_z = 0$ around the perimeter and it is thus non-planar after torque is applied.

5.16 Torsion of an Elliptical Cylinder (239)

So here we get an example of St Venant's problem that at least sounds solvable in closed form with a simple warping function φ. Lai write the equation of an ellipse, then compute the normal **n** at any point on the boundary. He then suggests we try $\varphi = Ax_2x_3$ as a candidate Laplace solution and we find that this warping function φ meets the St Venant Neumann BC 5.15.7 if A = α' times an obvious a/b ratio. Knowing φ, we compute T_{12} and T_{13} from 5.15.2, and from that we compute the end-cap surface traction function t, we integrate that over the end cap to get our torque M_1 . Then we use this M_t as our "given" quantity along with ellipse a and b and we find simple results for T_{12} and T_{13} as in 5.16.11 and the vector sum of these is T_2 as in 5.16.12. They key trick here is that Lai knows the right solution ahead of time and merely has to show that it works if we adjust the single constant A.

Example 5.16.1 Now as we did with the circular cylinder we diagonalize T (its diagonal elements are those eigenvalues, the principle stresses). The max normal and stress shears are again the same and equal to the max λ, just as before. In part (c) Lai shows that in fact these two stresses are maximum on the boundary of the elliptical cylinder but at the minor axis peak. This just falls out of the simple math. I think this is because, if you torque the bar, there is less play allowed in the minor lobes -- less room to stretch, so stress maxes out there. The value at the major axis lobe is reduced by ratio a/b. So expect cracks to form on the minor axis lines on the surface.

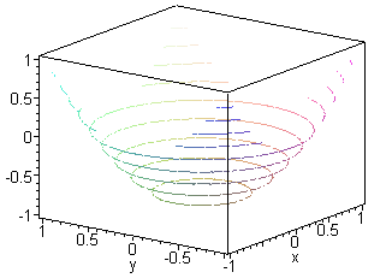
5.17 Prandtl's Formulation of the Torsion Problem (242)

Think of a little sugar cube inside the prismatic bar. The x_1 longitudinal faces are the two faces that experience shear stress and that is why we have T_{12} and T_{13}. There is no T_{11} because it is not in axial tension. If there were a T_{23}, then a side of the cube would feel a sheer force radially, say. Although that seems possible, it does not happen. On page 235 computation of E_{23} gives 0 from the nature of the u field. I do wonder why this is true for the arbitrary cross-section situation. (exercise).

So Prandtl [1875-1953, German, father of fluid dynamics, boundary layers, wing lift, so this was a minor detail of his output] suggests introducing an auxiliary function $\psi(x_2,x_3)$ (the way Sneddon did so many times). The claim is that you can get both T_{12} and T_{13} from this ψ as shown 5.17.1. This form does solve the static Navier equation, so a reasonable candidate. Not just any ψ will work, of course. By comparing this approach with the previous one, Lai shows two interesting facts about ψ. First it has to

solve the little 2D Poisson equation with constant source 5.17.5. Second, ψ = constant on the cross section boundary and you can just assume $\psi = 0$ on the boundary without losing anything. The moment integral comes out being the very simple result 5.17.13.

Example 5.17.1. Here the ψ function for the **circular cross section** case is stated. Lai shows that it gives $\psi=0$ on the boundary (obvious) and it solves the Poisson. The moment integral reproduces that we obtained earlier for this circular case. Here is a Maple plot of ψ with B = 1 and a = 1



where think of the prismatic bar being vertical. The contours are circles, and the shear force at any point is tangent to the contours. As expected for a twisted bar, the shear force vector goes in a circle around the axis. This is a little like plotting electric field lines.

Example 5.17.2. Here Lai shows that in fact the lines of constant ψ in the cross section are tangent to the shear stress at any point! At any point, $|\nabla\psi|$ gives you the magnitude of this shearing stress. This seems very strange to me. ψ is some kind of quasi-potential I would say. A very useful property.

Example 5.17.3. The membrane analogy. Suppose you put an elastic membrane on a planar frame which is the shape of your bar cross section. Then from one side you apply a constant pressure p (the other side is at vacuum). Assume the membrane has "membrane tensile force" S. By considering the force on a patch of membrane, you get the same Poisson equation 5.17.14 with a constant RHS. So you have here an "analogous problem". If you replace p/S by $2\mu\alpha'$, then the membrane problem becomes the bar problem. The slope of the membrane in the two directions tells you T_{12} and T_{13} as in 5.17.1, so the max slope of the membrane tells you where the max stress will be! Topo lines on the membrane are constant ψ contours in the bar world. This is a "visualization analogy" invented I think by our Ludwig Prandtl.

Comment: So Prandtl invents a more standard membrane problem for his ψ function with Dirichlet BC. The Neumann membrane problem is for the warping function φ and is a different problem with the same surface.

Comment: Suppose in 2D electrostatics you were to define $\mathbf{Q} = R_z(\pi/2)\mathbf{E}$ where $\mathbf{E}$ is the electric field. This says that

$$\begin{pmatrix} Q_x \\ Q_y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} -E_y \\ E_x \end{pmatrix} \qquad \Rightarrow \qquad Q_x = -E_y = \partial_y V \qquad Q_y = E_x = -\partial_x V$$

Now think of x,y as being x_2, x_3 in 5.17.1 and think of the potential V being the ψ function. You then have

$$Q_{x2} = \partial_{x3}\psi \qquad Q_{x3} = -\partial_{x2}\psi$$

and this is exactly the form of 5.17.1. But we know in electrostatics that **E** is perp to the lines of constant V, so **Q** (being rotated 90 degrees) must then be *tangent* to the lines of constant V. This then is why we get the result that vector **Q** is always tangent to the ψ lines, and vector **Q** has components which are those shear stresses. So think of **Q** as a 2D stress field acting a flat surface normal to the 1 direction.

5.18 Torsion in a Rectangular Bar (246)

Here we solve our little Prandtl Poisson equation using Stak's partial eigenfunction method. The Smythian form is shown in 5.18.3 using a complete set of cosines in the x_2 direction. It is a little like a potential theory problem where $\psi = 0$ on the boundary, but it is a Poisson problem. Installing this Smythian form into the PDE allows computation of the coefficient function Lai calls $F_n(x_3)$. Putting the pieces together, we get the series solution 5.18.10 for ψ (it certainly has a familiar look to me!). One could probably find the Green's function for this problem and just integrate that against the constant source ρ to get the solution. So your solution ends up being a constant times the spatial integral of the Green's Function. [I think, but Lai says nothing about that.]

As with the ellipse, the point of max stress is the midpoint of the long side of the rectangle. Lai comes up with sums for the max stress and for the moment M_t which are all infinite series. For a very thin rectangle, I guess the first few terms give a useful approximation for these two expressions as in 5.18.13. Lai relegates the two big expressions to the Problems.

For "lab" I played with torquing a yardstick. I found a pdf on "wood" and learned that it is anisotropic in all three directions: radial, tangential, and vertical, so wood is not a good example. Still, I would expect the ruler to crack on the center line of the wide side.

We are doing what I might call "canonical problems" in elementary elastic solid mechanics.

The membrane is in some sense an elastic solid, but it is a 2D surface solid in 3D space, and Lai has not really addressed such elastic solids. Lai is just providing the student with a "first look" at this subject, which I can see is massive in scope.

5.19 Pure Bending of a Beam (247)

It has taken me a while to grok this topic. The beam shown has nothing to do with gravity. We have a 3D bar of arbitrary cross section, horizontal view p 247 right. We imagine a "set of normal end cap tractions" which look like the little arrows I have drawn. On the right, the top part of the beam is pushed to the left, while the bottom part is pulled to the right, trying to make the beam "bow downwards". At the left end, a similar thing is happening. So this figure really shows end torques M_R in the $\mathbf{e}_2$ direction (call it then M_2) which is trying to bow the bar down. To get this torque, we would integrate the tractions over the end cap. There is also assumed to be a torque pair in the $\mathbf{e}_3$ direction trying to bend the bar out of the plane of paper and this is M_3.

Next, we make the simple ansatz that only $T_{11}(x_2,x_3)$ exists. We imagine that we have axial compression on the top, and axial stretching on the bottom, so T_{11} would then be negative on the top (medium pushes in on a particle like hydrostatic situation) and positive on the bottom, at least in a general

sense. One question is whether such a single T_{ij} component model is "compatible" and solves the static equations. The static equations say only that $\partial_1 T_{11} = 0$ in this model, which in turn says $T_{11} = T_{11}(x_2,x_3)$ which in turn says things don't vary along the bar, which is similar to the axial load test situation. Reasonable so far. Now for the first time I think, Lai uses those "compatibility equations" on page 102 to put some restrictions on the allowed form for T_{11} and that form is that it is linear in x_2 and x_3 as in 5.19.4. So with any such form, we solve the static Navier equation, AND we are compatible.

Lai then pauses to note that the lateral surface tractions are trivially zero.

Digression on moments. This digression is going to be about the nature of integrals of a tensor field in the case of linear transformations. Consider then [this has been added to tensor doc near J discussion]

$$T^{ijk\cdots} = \int_D dV\, A^{ijk\cdots}(\mathbf{x})$$

I have written a little piece to add to tensor doc section 8 which shows that under rotations, the integral of a tensor field is a tensor. Therefore, both these objects are tensors

$$I^i = \int_D dV\, \rho(\mathbf{x})\, x^i \qquad \text{// vector}$$

$$I^{ij} = \int_D dV\, \rho(\mathbf{x})\, [\, r^2\delta^{ij} - x^i x^j] \qquad \text{// rank-2 tensor}$$

The second is the usual inertia tensor for a material of mass density 1. Lai refers to these things as the second moments of the area, while I^i are the first moments, and of course he has all indices down.

So note that

$$I_{22} = \int_A dA\, [\, r^2\delta_{22} - x_2 x_2] = \int_A dA\, [\, x_2^2 + x_3^2 - x_2^2] = \int_A dA\, x_3^2$$

so you get that funny labeling situation.

Next, Lai notes that the right face normal traction distribution must be

$$\mathbf{t} = T_{11}\mathbf{e}_1$$

First, he computes the total forces this implies, and we get 0 provided we make sure our axis goes down the rod through the centroids of the end caps! This is now finally clear to me. That makes the first moments vanish. We set $\alpha = 0$ by the way because as assume that there is no total average tension in the rod, it only has those torques acting on the ends. The next big step is to compute the torques from t as shown top of page 249. We can then, as usual, invert to expression the constants α and β in terms of our torques M_2 and M_3. The result is this as in 5.19.10,

$$T_{11}(x_2,x_3) = M_2 x_3/I_{22} - M_3 x_2/I_{33}$$

and this is the main result! Since $t_1 = T_{11}$, this IS the end cap surface traction distribution, and you see it is linear in either direction about the centroidal center line, plus on one side and negative on the other, with linear shape, could not be simpler really. Also, note that for $x_2>0$ and no M_3, T_{11} is negative on the top half as conjectured above.

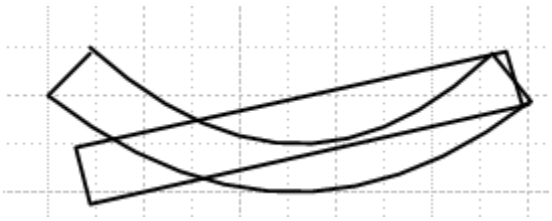
Back on page 248 in passing Lai computed the E_{ij} in terms of this T_{11} and E_{ij} is diagonal and all three diagonal elements are non-zero but two are equal. From these E_{ij}, Lai then integrates to get the displacement field **u** inside this bar! In doing this integration, he gets those usual integration constants which here are called α_i. He then shows that their contribution to the displacement field is equivalent to a rigid body displacement of some kind (as he did earlier). Since our rod has no such displacement, we can set these α_i to zero I would say. We will examine this **u** field in the example to follow.

Example 5.19.1. Certain constraints are now added to our bending rod so as to nail down all those integration constants shown in 5.19.12. These constants are α_1 through α_6. The centroid point on the left end is nailed down completely (that is, **u**(0,0,0) = 0). The centroid point on the right end cap is only allowed to move along the axis line, which says **u**(ℓ,0,0) = k $\mathbf{e}_1$. Finally, Lai adds a strange derivative condition $\partial_2 u_3 = 0$ at the left end centroid point which kills off α_1. This BC is OK but I don't really understand what it means. Applying these BC's sets all the α_i and we end up with 5.19.14. The displacement pattern inside the bar is fairly complicated still (but all just quadratic in the x_i).

He looks at a cross section slice that was at x_1=c before bending, and we get u_1 = const * x_3. This tells you that the slice which was vertical at x_1 = c is now tilted by the angle shown, which fits in with our picture of the beam being bent down as I have drawn it.

The beam is basically bent in the $\mathbf{e}_3$ direction in this example, so we naturally want to know what happens to the "center line" which means $x_2 = x_3 = 0$. From the bottom of 5.19.14, we get 5.19.17 which says that the center line describes a quadratic curve with minimum at the center. You could I suppose try to fit it with a circular curve.

Comment #1: In this example, we have $\alpha_2 \neq 0$, so there is some "**rigid body displacement**" remaining in the result. Here is my interpretation: I think the entire beam gets rotated "down" as a rigid object slightly, and then it is bent up, as in this exaggerated picture.



The rigid rotation part with just α_2 is given from 5.19.12 as

$$u_1 = \alpha_2 x_3$$
$$u_3 = -\alpha_2 x_1$$

which I think is a rotation about axis out of paper as shown above with rectangle. I have not done this in detail, but I think I can see why you might have some "rigid body displacement" as part of your answer.

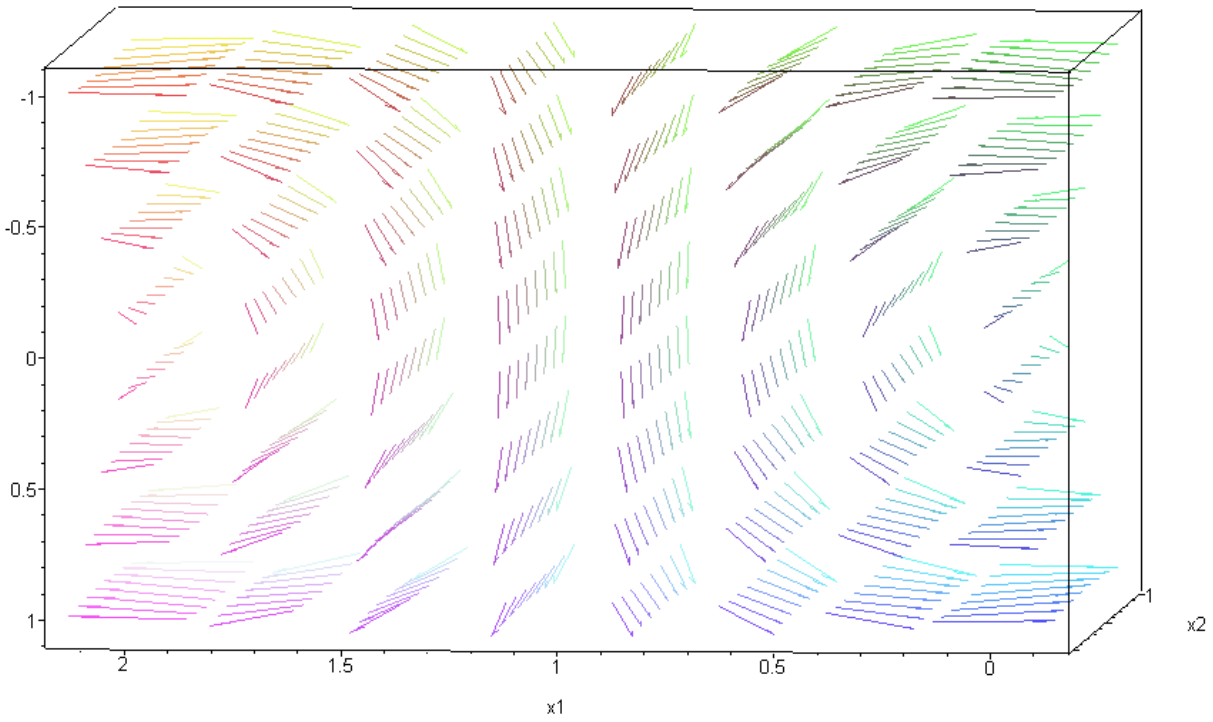
Comment #2. Why is this called **pure or simple bending**? Because it is done entirely by those "end couples", meaning you torque the bar by applying certain normal surface tractions to the ends which are indicated by a total moment M_2 (say). The traction pattern is very specific, but we know from St-Venant that those details won't matter away from the endcaps. You might use chucks again to achieve the bending.

Comment #3. Suppose I take a **massless steel bar and tie one end horizontal into cement**, and put a point mass load m on the far end. In this case, I think we can use the above model with $M_2 = mg\,(\ell/2)$ as the moment acting on each end (relative to rod center, maybe....?) Perhaps this is the above picture without that rigid rotation part, or perhaps very similar with slightly different BC's. But I think from the model above, I could compute by how much the bar would deflect! Probably Lai will treat this later. If you take into account the mass of the bar, you no longer have "pure bending" I suppose.

Note: For beam width w and height h, I find $I_{22} = (1/12)\,wh^3$.

Comment #4. How would you **plot** the displacement field 5.19.14 ?? Gradplot does not quite do what we want. I do want an arrow at each point in 3D space somehow.

Well, fieldplot3D does the trick and with $\nu = 0.3$ and $M/EI = 1$ and $\ell = 2$ I get the following arrow representation of the displacement field $\mathbf{u}(\mathbf{x})$: You see the bar "bending down mostly in the middle". This is a fascinating picture. You see how the displacement starts out "torquey" at the end caps but smoothly becomes "bent down only" in middle.



I think it would take some significant work to get things plotted nicely for continuum mechanics plots!

Comment #4. Notice that the solution obtained by Lai applies to **any cross section shape you want**! That is pretty amazing.

A.3 Plane Stress and Plane Strain Solutions (250)

5.20 Plane Strain Solutions (250)

A plane *strain* solution means the E tensor shows a "state of plane strain", which means it looks like this:

$$E_{ij} = \begin{bmatrix} E_{11} & E_{12} & 0 \\ E_{21} & E_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In terms of a sugar cube (particle) in a slice of a bar, this means that the sugar cube could have the shape of a parallelogram which is extruded dx_3 in the axial direction, so right angles are preserved between both the 1 and 2 directions and the 3 direction, so $E_{31} = 0$ and $E_{21} = 0$. Things are "the same" along the axis. Later we will allow an $E_{33}(x_1,x_2)$ and this allows that extruded particle thickness to vary over the cross section, whereas here that thickness is fixed. Another way to say this is that in the first case, $u_3 = 0$ for every particle in the slice, whereas in the second some $u_3(1,2)$ is allowed.

Comment: All these partial differential relationships are complicated, so we are constantly looking for "simple situations" which have simple solutions. We have solutions looking for problems, so to speak, and this is a simple state of strain, where we try to sort of eliminate the 3 direction as much as possible. But it is trivial to show that $E_{33} = 0$ forces you do have T_{33}, so the stress tensor has this form

$$T_{ij} = \begin{bmatrix} T_{11} & T_{12} & 0 \\ T_{21} & T_{22} & 0 \\ 0 & 0 & T_{33} \end{bmatrix} \qquad T_{33} = \nu(T_{11} + T_{22})$$

But there is more to it than said so far. We are still supposed to be thinking about a prismatic bar with x_3 now being the length direction (it used to be x_1 and later will be x_1 again!) The axial slice dimensions are of course x_1 and x_2. Another assumption of the "state of plane strain" is that the E_{ij} are functions only of these x_1 and x_2 coordinates. That means as you move down the bar, E_{ij} stays always the same in any axial slice. We are considering only situations of this type. Now when we compute the T_{ij} from Hooke's Law p 208, we end up of course with $T_{ij} = T_{ij}(x_1, x_2)$ as well.

Comment: In potential theory we can treat infinite extruded objects ("bars") as if they were 2D objects, and the 3rd dimension is truly gone from the problem. I think the above discussion shows (maybe) that you cannot do this quite so cleanly in elastic theory, though we are trying our best to do so. I think we will find that this kind of solution applies *not* if the bar is infinitely long in the extrusion direction, but infinitely thin!! Then there is no "space" for action to develop in the x_3 direction. Next item shows that across this entire thin plan, $u_3 = 0$ so yes, nothing happens.

Question #1: On page 251 Lai implies that the above set of assumptions forces u_3 = constant so in the entire bar, there is not a single point which displaces along the bar! Can I show that u_3 = constant?

Consider first $E_{33} = \partial_3 u_3$. Since $E_{33} = 0$, we know that at worst, we have $u_3 = u_3(x_1, x_2)$ just like all the other functions. Since $E_{31} = 0$, we know that $(\partial_3 u_1 + \partial_1 u_3) = 0$. But presumably u_1 is like the E's and does not depend on x_3 so we have $(\partial_3 u_1)$ and therefore $\partial_1 u_3 = 0$ so u_3 cannot depend on x_1. Similarly from E_{21} it cannot depend on x_2. So yes, I agree with characterization 5.20.4.

So given the other constraints, you could argue that $E_{33} = 0 \Rightarrow u_3 = 0$. Then you can argue

"presence in T_{ij} of $T_{33} = \nu(T_{11} + T_{22})$ as shown above " => $u_3 = 0$

The Airy Function. The static Navier equation tells us 5.20.5. The claim is now made, without proof (I will supply it here) that those static equations will be automatically satisfied if you generate your 4 basic T_{ij} from a potential-like scalar function φ which is the Airy function.

proof:

$T_{11} = \partial_2^2 \varphi$
$T_{22} = \partial_1^2 \varphi$
$T_{12} = -\partial_1 \partial_2 \varphi$

Then for the left two equations of 5.20.5:

$\partial_1 T_{11} + \partial_2 T_{12} = \partial_1 \partial_2^2 \varphi - \partial_2 \partial_1 \partial_2 \varphi = 0$ check

$\partial_1 T_{21} + \partial_2 T_{22} = -\partial_1 \partial_1 \partial_2 \varphi + \partial_2 \partial_1^2 \varphi = 0$ check

So there you have it. Now, assuming the T_{ij} has the structure shown top p 251 with the specific T_{33}, then we know ahead of time (our initial assumption) that the E_{ij} tensor will be of the plane strain type. We can if we like use the Hooke's Law equations to express these E_{ij} in terms of the Airy function, see 5.20.7.

Comment in line: we do not have to "check compatibility" for a T_{ij} stress pattern, any such pattern is allowed since T_{ij} is what appears in the Navier Cauchy equation. It is for a candidate E_{ij} that we have to check for compatibility. We have just computed some E_{ij}. Although I have not checked the math, what we find is this:

Theorem: The E_{ij} obtained by the above procedure will be "compatible" as long as the Airy function φ satisfies the 2D biharmonic equation $\nabla^4 \varphi = 0$.

Proof: accepting the compat equation 5.20.8 which says

$\partial_2^2 E_{11} + \partial_1^2 E_{22} - 2\partial_1 \partial_2 E_{12} = 0$

But 5.20.7 says

$E_{11} = k\,[\,(1-\nu^2)\partial_2^2\varphi - \nu(1+\nu)\partial_1^2\varphi]$

$E_{22} = k\,[\,(1-\nu^2)\partial_1^2\varphi - \nu(1+\nu)\partial_2^2\varphi]$ // derivatives just switched

$E_{12} = -k\,(1+\nu)\,\partial_1\partial_2\varphi$

Insert to get

$$k\,[\,(1-\nu^2)\,\partial_2^2\partial_2^2\varphi - \nu(1+\nu)\,\partial_2^2\partial_1^2\varphi] + k\,[\,(1-\nu^2)\,\partial_1^2\partial_1^2\varphi - \nu(1+\nu)\,\partial_1^2\partial_2^2\varphi] + 2\,k(1+\nu)\,\partial^2{}_1\partial^2{}_2\varphi = 0$$

$$[\,(1-\nu^2)\,\partial_2^2\partial_2^2\varphi - \nu(1+\nu)\,\partial_2^2\partial_1^2\varphi] + [\,(1-\nu^2)\,\partial_1^2\partial_1^2\varphi - \nu(1+\nu)\,\partial_1^2\partial_2^2\varphi] + 2\,(1+\nu)\,\partial^2{}_1\partial^2{}_2\varphi = 0$$

The sum of the three red term coefficients is $-(1+\nu)(2\nu-2) = -2(1+\nu)(\nu-1) = 2(1-\nu^2)$, so then $(1-\nu^2)$ is a common factor and the result, cancelling it, is

$(\partial_2^2\partial_2^2 + 2\partial_2^2\partial_1^2 + \partial_1^2\partial_1^2)\varphi = 0$ or $\nabla^4\,\varphi = 0$

or

$(\partial_1^2 + \partial_2^2)\,(\partial_1^2 + \partial_2^2)\varphi = 0$ or $\nabla^2\,\nabla^2\,\varphi = 0$

and this last form shows the origin of the name "biharmonic".

So any solution of this equation provides a φ which in turn gives us a plane-strain problem solution. Comment in line: the first time I ever saw ∇^4 was in Stakgold and I wondered where it would ever occur in physics. This is the first place I have ever seen it! The trick must be that the Airy form automagically satisfies all the other compatibility equations other than the one shown in 5.20.8. A simple example is $\varphi = \alpha x_2^3$ -- it is a solution of the $\nabla^4\varphi = 0$ equation. The key term is

$$\partial_1^2\partial_2^2\,x_2^3 = \partial_2^2\partial_1[\partial_1(x_2^3)] = \partial_2^2\partial_1[0] = 0$$

Lai shows very simply that the biharmonic equation is equivalent to this Laplace equation:

$$\nabla^2\,(T_{11} + T_{22}) = 0$$

Big Paragraph on page 251. I skipped this and now must address it. Each sentence of this paragraph is a little mystery of its own. I will now parse it:

"The T_{33} is needed to maintain zero axial strain". -- The "bar" in this discussion is along the x_3 direction so that x_3 is now the "axial direction" (this used to be the x_1 direction). So T_{33} makes $E_{33} = 0$ and we could say that $E_{33} = 0$ means "zero axial strain". That is to say, E_{33} is the "axial strain" for this interpretation of the bar orientation.

"Removal of this T_{33} will result in axial deformation" -- If you have $E_{33} \neq 0$ and $E_{33} = \partial_3 u_3$, then you are likely to get u_3 varying in the x_3 direction since $\partial_3 u_3 \neq 0$, and such variation is "axial deformation", another term would be "axial displacement" since u_3 is the axial displacement.

"Removal of T_{33} will alter the stress and strain field in the bar unless T_{33} = linear in x_1 and x_2". -- In general, if you have some real physical situation of a bar, and you suddenly take away the x_3 surface traction T_{33} (and in fact remove it everywhere), it seems likely to me that "everything may change" and this includes the four "plane" T_{ij} matrix elements and all the E_{ij} elements. This is just because all the pieces of this puzzle are interrelated by the complicated static Navier equation and Hooke's Law.

Now, Example 5.20.1 on page 252 (which we have not gotten to yet). I will first quote the result

Example 5.20.1 theorem: We now jump ahead and pick up this section right here. If you assume that $T_{ij} = T_{33}$ only, you can compute the E_{ij} from that T_{33} and you can then study the compatibility equations for said E_{ij} and you find that those equations are satisfied only if T_{33} has the form $\alpha x_1+\beta x_2+\gamma$.

Now back to the sentence parsing. Suppose the T_{33} that you suddenly "remove" does have the form shown above. We can think of a superposition problem again. The removal of T_{33} is like the application of $-T_{33}$ as a separate problem, and this 2nd problem it seems to me results in the diagonal E_{ij} matrix shown in 5.20.14. So it seems to me that, even if T_{33} has the "linear form", in the sum problem you will have *altered* at least the three objects E_{11} E_{22} and E_{33}! Therefore I think Lai's sentence is wrong. Removal of T_{33} means that we have to add these values E_{11} E_{22} and E_{33} to the 1st problem E_{ij} matrix so the E tensor (strain field) will in fact be changed. It is true that $E_{11} = E_{22}$ for the added amount.

Now can we at least say that the four main T_{ij} are not altered? Well p 208 looking at T_{11} we see that we need to add $\lambda(E_{11}+E_{22}+E_{33}) + 2\mu E_{11}$ to the existing T_{11}:

$$\lambda(E_{11}+E_{22}+E_{33}) + 2\mu E_{11} = \lambda(-2\nu/E_Y*T_{33} +1/ E_Y*T_{33}) + 2\mu(-\nu/E_Y*T_{33})$$

$$= (T_{33}/E_Y)[\ -2\lambda\nu + \lambda - 2\mu\nu\ ] = (T_{33}/E_Y)[-2\nu(\lambda+\mu) + \lambda\ ]$$

But 5.4.7 says that $2\nu(\lambda+\mu) = \lambda$ so we find that [...] $= -\lambda+\lambda = 0$. So indeed, T_{11} does not get altered! So I guess sure, if you just superpose the T matrix shown p 252 bottom, you don't alter existing T_{ij} and then the sum problem has the same four T_{ij} as problem 1.

So I would say "removal of linear-form T_{33} does not alter the existing stress field in the bar (upper four matrix elements), and alters E_{11} and E_{22} by the amount $-\nu T_{33}/E_Y$.

So I am calling this a Lai error. Lai then corrects this error in the rest of this very long sentence. So all is OK again, he acknowledges that the T_{ij} don't change but some of the E_{ij} do change.

"However, if the cylinder is long in the x_3 direction...." Without reading further, I would say that far from the ends, removal of T_{33} results in a uniform reduction of T_{33} everywhere in the center of the bar, based on our St Venant study of the axially loaded bar. So the next two Lai sentences then make sense to me. You could then replace whatever T_{33} you have at the ends with the precision T_{33} pattern which gives uniform T_{33} in the entire bar. Then if you remove that T_{33}, you don't affect the upper four T_{ij} .

"Thus, as far as in-plane stress... two kinds of problems: " The first problem has x_3 walls in which case I would see $T_{33} = \nu(T_{11}+ T_{22})$ must be present (making $E_{33} = 0$ and $u_3 = 0$ see below), but its presence if linear does not affect the four T_{ij}. In the second problem, the walls are removed so $T_{33} = 0$ at those walls and there might be some interior T_{33}. He argues for this second problem that somehow you might deduce T_{33} by superposition, but that is unclear to me. I think you would have to actually solve things to get T_{33}.

But for a long rod, out in the middle I think he is saying you would have $T_{33} = 0$. This IS confused I think, but then we have the final statement":

" In either type of problem, the in-plane stresses (those in the x_1x_2 plane) are the same". I agree.

Comment: I can see that this issue is confusing and the authors tried lots of rewrites and we end up with this somewhat muddled paragraph. The section is very different in the earlier third edition.

5.21 Rectangular Beam Bent by End Couples (253)

We already did this problem in Section 5.19. Here the problem is redone in a strange way. First, we use an Airy function in the 1-2 direction which is NOT transverse to the bar (as previously it was used). Here the x_1 is the bar axial long dimension again. Also, we assert some frictionless walls in the x_3 transverse dimension (not present in the pure bending problem). These walls support the required T_{33} to make E_{33}= 0. We solve this problem which is a "plane strain" problem (since E_{33}= 0). Then we superpose a 2nd problem with just $-T_{33}$ and in doing so we obtain that pure bending problem with an M_3 moment at the ends. So this is a demonstration of many things: (1) the Airy method applied in a plane running along the bar, which here is the 1-2 plane. (2) the notion of frictionless walls (3) the idea of superposition.

The Frictionless Wall. Imagine physically a frictionless wall on the x_3 boundary of a rectangular solid. What does this phrase "frictionless wall" actually mean? First, this wall makes a constraint $u_3 \leq 0$ at the wall, shall we say. Our solid cannot expand out through the wall. It could pull away from it, perhaps, but when we say "frictionless wall" we shall mean that $u_3 = 0$ for all points along that wall, as if the "wall" somehow holds all the particles at the boundary from moving normal to the wall. If we then reverse the above argument (loosely speaking) to argue that

frictionless wall => $u_3 = 0$ at wall => "presence in T_{ij} of $T_{33} = \nu(T_{11}+ T_{22})$ as shown above"

One could then say that this T_{33} represents a normal surface traction of the wall acting on the solid which in effect "holds the surface in place" and keeps it from moving, ie, $u_3 = 0$ on the entire wall.

If you were to "take away the frictionless wall", then this T_{33} is no longer being applied, and this means $T_{33} = 0$ and we no longer have $E_{33} = 0$ so we are no longer in a state of plane strain.

The wall is "frictionless" in that the surface of the solid is allowed to move tangentially along the wall, so that we do not requires that u_1 or u_2 be zero at this x_3= constant wall. But more directly, the frictionless wall idea means there are no shear surface tractions applied to the wall! In the 3 wall case, this would mean that T_{13} and T_{23} both = 0 at the wall, which is consistent with these shear stresses are assumed zero everywhere in the bar.

So: Forget the Airy function for now, just accept the T_{ij} shown in 5.21.4. The frictionless walls in this problem are NOT at the ends of the bar of length ℓ, they are at the top and bottom surfaces of the bar (let us say), $x_3 = \pm b/2$. I agree still that frictionless walls at those surfaces would cause $u_3 = 0$ on those surfaces, and in turn that would cause $E_{33} = 0$ and the same thing $T_{33} = \nu(T_{11}+ T_{22})$ and then we get the filled out T matrix as shown here. This bar then has normal tractions on the ends and at the "top and bottom" (x_3).

The functional form of the tractions on the end faces ($\mathbf{t} = T_{11}\mathbf{e}_1 = 6\alpha x_2\ \mathbf{e}_1$) is clearly "of the type" that will cause there to be torque trying to bend the bar in the x_2 direction (out of plane of paper in figure on page 247). This would be an M_3 torque in the symbols of the page 247 analysis. And this is the boundary that has no wall constraints. Calling this torque $M_3 = M$, we get 5.21.6. This was all "part (a)" of this discussion.

In part (b) we are supposed to take away our frictionless wall pair. This means that $T_{33} = 0$ at those walls now. My guess as to what would happen is that now $E_{33} \neq 0$ and now the bar will likely have some $u_3 \neq 0$ at those walls, and then we are no longer in a state of plane strain. Lai's discussion gets hazy at this point, and I looked at the previous version of the book and saw the problem they were having. They really want to claim this is a superposition situation but in the 4th edition they intentionally avoid using that word. In part (a) we had $T_{33} = 12\nu M_3 x_2/bh^3$. [by the way, this would seem to be applying its own torque to the bar of some sort] . We now want to superpose another "situation" in which the T_{ij} tensor has only a component $T_{33} = -12\nu M_3 x_2/bh^3$. The sum problem then has only the T_{11} of part (a) which was $T_{11} = 12M_3/bh^3$ and the sum problem then has T_{ij} as stated in 5.21.7, so I am sure this is what they are saying. Now this final T_{11} can be written $T_{11} = M_3 x_2/I_{33}$ and this is then seen to agree exactly with our previous analysis of the pure bent bar where there is only M_3, as shown in 5.19.10 p 249 (except sign is reversed). This M_3 torque at the bar ends is still trying to bend the bar in the x_2 direction.

One more item. In the pure bending analysis, we had $E_{33} = -\nu T_{11}/E_Y$. In our sum problem analysis here we have that $E_{33} = (1/E_Y)(T_{33}-\nu(T_{11}+ T_{22})) = (1/E_Y)(T_{33}-\nu T_{11}) = -\nu(1/E_Y)\ T_{11}$ so yes, we get the same E_{33}.

The conclusion: you can start with a "plane strain" problem (beam with those two x_3 frictionless walls) and you can superpose on that a simple T_{33} stress pattern problem (which negates the T_{33} wall traction, and this is equivalent to removing the walls), and then the sum problem will be the simple bending problem (which, recall, had no walls) which was not "plane strain" because it has E_{33}.

Now, it happens that in this little demonstration, the T_{11}, T_{22} and $T_{12} = T_{21}$ stresses are generated from an Airy function $\varphi = \alpha x_2^3$ which is commonly used to generate those four T_{ij} for a plane strain problem, which part (a) was.

Comment: Some day maybe I will clear up all the above confusion, but let's press on for now.

5.22 Plane Stress Problems (254)

Such a problem has just the upper left four entries in the T matrix. This is a "state of plane stress". Earlier we started this same way and then added the special T_{33} to cause $E_{33} = 0$ which got us a "state of plane strain", but here that T_{33} is not "applied". This means that $T_{33} = 0$ on the Hooke's Law formulas, so you get the E_{ij} shown bottom page 254. It then has the upper square plus E_{33} showing. We use the same Airy function representation of the upper T_{ij} square.

Next: what do the 6 compat equations have to say about this set of E_{ij} ? One of them gives $\nabla^4\varphi = 0$. Three of them together say 5.22.5 which says you must have E_{33} be linear in x_1 and x_2, and this in turn means that $(T_{11}+T_{22})$ must be so linear. The last two compat equations are OK with no further assumptions.

Conclusion: if $(T_{11}+T_{22})$ is linear, then T_{ij} describes a state of plane stress, and this works for a "body" of any width (not just something thin, say). On the other hand, if $(T_{11}+T_{22})$ is NOT linear, Lai is

going to show that your T_{ij} still describes a state of plane stress with negligible error to the extent the body is thin in the x_3 direction. Reason for this claim is not yet given.

On page 254 we get a reference to a book of Timoshenko and Goodier, Theory of Elasticity, right after a discussion of "thin things", so maybe we are supposed to go look it up? "It can be shown that errors are of order ε^2...."

5.23 Cantilever Beam with End Load (255)

OK, finally we have our problem of a rectangular cantilever of length ℓ (x_1), height h (x_2), width b (x_3) that has the right end glued to a wall and at the left end ($x_1 = 0$) it carries a shear force P. This could of course be supplied by a mass glued to the left end, or probably by St Venant, by any kind of mass load at the end. We assume in this gravity scenario that the beam itself has no mass.

The path is a bit complicated and I will give the steps in words.

First, Lai proposes a strange looking Airy function $\varphi(x_1,x_2)$ in 5.23.1 which has free parameters α and β. Remember that x_2 is in the direction of the load, while x_3 is the less interesting width of the beam. We of course compute the quad of T_{ij} from this φ as 5.23.2. The "plane" in this problem is a slice along the beam, perp to x_3.

Next, we assume that there are no tractions on the top of bottom of the beam. This condition puts a relation between α and β.

Next, we just compute the traction on the left face of the beam. It turns out that there IS in fact a traction distribution there which has a constant and an x_2^2 term, *and* this is a downward shear traction! We have this traction as a function of α . If we integrate this and call the total shear load at the beam end P, then of course we have P as a function of α, or vice versa, α as a function of P. Since P is going to be our load, we want to write everything in terms of P as in 5.23 7 and 8.

Now, in order to get into plane strain, we would have to throw in our usual T_{33} and then we have the situation shown in 5.23.10, and now we have to recompute all the E_{ij} as in 5.23.11 to account for T_{33}.

I think Lai wants now to compare this plane strain problem 5.23.10 to our original plane stress problem as in 5.23.12. Once again, the plane stress problem has E_{ij} as in 5.23.13, while the plane strain problem has E_{ij} as in 5.23.11. All the E's appear different in these two problems.

Note that T_{33} in this example is not linear, so we cannot do our "trick" to remove it.

Strangely, having arrived at these two problems, Lai does not really say what his plan is. Well, yes he does. He claims that IF the beam is thin in its width direction, THEN the plane stress problem solution is a very good approximation for our plane strain problem, ie, with good accuracy either solution is OK. I think our solution 5.23.10 is exact (the plain strain problem) if we were to apply the shear load P exactly right. So we could in theory use the E_{ij} shown in 5.23.11 and integrate to get our solution displacements u_i. Rather than do this, in the next example Lai will instead take the approximate solution 5.23.12 (the plane stress solution) and solve it instead for displacements. [There is a hard relation between T and E, so either gives the other. But E is closer to u than T is.]

Example 5.23.1 (257). In his usual time-saving manner, Lai just states the solution u_i and then shows they do in fact generate the E_{ij} quartet that we need for the plane stress solution. We just ignore E_{33} since it is wrong anyway. The term "in plane displacement" refers here to any plane in the beam with x_3 = constant, so things are functions of x_1 and x_2. The variable x_3 does not appear anywhere.

Now, if you look at an axial slice at some fixed x_1, the resulting $u(x_2)$ is not linear in x_2 (it has linear and cubic terms), so these tells you that under the load, the cross sections that start as planar rectangles end up warped into non-planar Pringle surfaces of some sort.

Once again, we have to accept Lai's claim that this solution is valid for a vertically thin beam, perhaps like that found in an I beam.

Part (b): now the right end of the beam is glued to the wall with what I guess you might call a "wall with friction" so that u_1 and u_2 must both be 0 at the right end. We also need to set the u_2 slope coming into the wall to 0 which seems somewhat reasonable to me since the bend is gentle. We then use the BC's to find b_1 and c_3 appearing in the u_2 formula and we end up with $u_2(x_1,0)$ as shown in (iv). We find that the "center line" of the beam is bent down and the formula shows a constant, a linear term, and a cubic term, so you would say the shape of the bend was "a cubic". A major interest item is how much the beam is deflected right at the load end, and that result is $u_2(0,0) = -P\ell^3/3E_Y I$ where $I = bh^3/12$. The vertical beam thickness is h, and $I = bh^3/12$, so if you double the beam thickness, you deflection → 1/8. But if you double the length, deflection increases by factor of 8, so these two actions would offset each other. The load P gets spread out more if you double width b, so deflection goes to half.

Now this is supposed to be for a vertically thin beam. But it seems to me that you could form a wide beam by just superposing a bunch of thin ones. There are no surface tractions on the sides recall. So maybe you could argue that this shape applies also to a wide beam, despite Lai's comments.

The solution shape was this:

$$u_2(x_1,0) = -P/(6EI)\ \{ -x_1^3 + 3x_1\ell^2 - 2\ell^3 \}$$

If you measure from the other end, call it $x = (\ell - x_1)$ then we have

$$\{\} = -(\ell-x)^3 + 3(\ell-x)\ell^2 - 2\ell^3$$

Maple tells us

```
f := -(L-x)^3 + 3*(L-x)*L^2 - 2*L^3;
```

$$f := -(L-x)^3 + 3(L-x)L^2 - 2L^3$$

```
expand(%);
```

$$-3Lx^2 + x^3$$

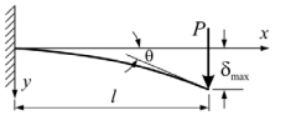
```
factor(%);
```

$$x^2(-3L+x)$$

so $\{\} = x^2(x-3\ell)$ and we then have (E means E_Y, and I is p 256 center moment = $bh^3/12$.

$$u_2(x) = -P/(6EI) * x^2(x-3\ell) = Px^2(3\ell-x)/(6EI)$$

These results agree with the first entry in my PDF beam deflection formulas table:

BEAM TYPE	SLOPE AT FREE END	DEFLECTION AT ANY SECTION IN TERMS OF x	MAXIMUM DEFLECTION
1. Cantilever Beam – Concentrated load P at the free end			
	$\theta = \frac{Pl^2}{2EI}$	$y = \frac{Px^2}{6EI}(3l - x)$	$\delta_{max} = \frac{Pl^3}{3EI}$

To repeat, I found that

$$u_2(x) = Px^2(3\ell - x)/(6EI) \qquad u_2(\ell) = -P\ell^3/(3EI).$$

and both my results agree with the table entry. And the entry makes no comment that the beam has to be vertically thin.

So Lai has come through finally with his first "practical result".

5.24 Simply Supported Beam under Uniform Load (258)

Lai neglects to tell the reader what "simply supported" means. It is the picture below!

Again, the physical situation is slightly hazy. We start as usual with an ansatz Airy function φ which is a fifth degree polynomial in x_1 and x_2 with 5 unknown constants B_0 through B_4.

This thing will satisfy $\nabla^4\varphi = 0$ if $B_4 = B_3/5$ so φ is now 5.24.2. From this we can get the quad of T_{ij} values as shown (this is a plane stress situation, we ignore T_{33}). As before, x_3 is the uninteresting horizontal beam coordinate, the width of the beam (2b).

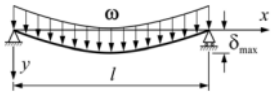
Next, we want the *bottom* of the beam to have neither normal nor shear tractions and this gives us the little pair of B_i relating equations in 5.24.4. Next, we imagine that the *top* of the beam is uniformly loaded with some pressure p pushing the beam down. This might be from a uniform mass distribution, or from air pressure with vacuum underneath, whatever. This is called a "uniform compressive load" on the top of the beam. This condition on T_{22} at the bottom gives us a little triplet relation and then we have 3 equations in 3 unknowns, and we then know B_0, B_1 and B_3 in terms of p and d. Only B_2 is not yet known.

Lai next shows that the axial force on the beam ends integrates to 0, this is T_{11} at the ends, integrated.

Next, he requires no couple at the end, and this then sets B_2 as in 5.24.6.

Just so, we end up with the stress quartet T_{ij} each in terms of p, d, ℓ.

But here Lai stops. It would be easy to find the E_{ij} using Hooke's Law and Maple. Then you would have to integrate somehow and have some BC's to get the deflection. You could then find the deflection curve of the beam center line. You might think Lai would assign this as a problem, but not so. But there is an entry in my beam table for this problem:

8. Beam Simply Supported at Ends – Uniformly distributed load ω (N/m)			
	$\theta_1 = \theta_2 = \frac{\omega l^3}{24EI}$	$y = \frac{\omega x}{24EI}(l^3 - 2lx^2 + x^3)$	$\delta_{max} = \frac{5\omega l^4}{384EI}$

so here then is a good problem for me to do at some point.

Lai solves this problem without even dealing with those shear forces at the beam ends which are holding up the beam, again there would be some St Venant going on at those ends if we really just had "simple supports".

I did not do the algebra in this section because I have done enough algebra all along. My main interest is understanding the logic.

5.25 Slender bar demonstration of St Venant (260)

First, two preliminary math details:

Detail #1 for next section. My cosine series transform says that

$\Sigma_{n=0}^{+\infty}\varepsilon_n \cos[n(\theta-\theta')] = 2\pi\,\delta(\theta-\theta')$ on $(-\pi,\pi)$ for θ ε_n = Neumann factor

So let $x_2 = \theta\,(c/\pi)$ so range of x_2 is then (-c, c). Then we have (set $x_2' = \theta' = 0$)

$\Sigma_{n=0}^{+\infty}\varepsilon_n \cos[n\pi x_2/c] = 2\pi\,\delta(\pi x_2/c)$ on (-c,c) for x_2 ε_n = Neumann factor

$= 2\pi\,(c/\pi)\,\delta(x_2) = 2c\,\delta(x_2)$

so therefore

$\delta(x_2) = (1/2c)\,\Sigma_{n=0}^{+\infty}\varepsilon_n \cos[n\pi x_2/c] = (1/2c) + (1/c)\,\Sigma_{n=1}^{+\infty}\cos[n\pi x_2/c]$

Now change n to m, and write $m\pi/c$ as λ_m to get

$\delta(x_2) = (1/2c) + (1/c)\,\Sigma_{m=1}^{+\infty}\cos[\lambda_m x_2]$ $\lambda_m = m\pi/c$ x_2 in (-c,c)

and there you have 5.25.1. (this student came to his Lai course well-prepared).

Detail #2 for next section. Proof that 5.25.5 satisfies equation top of page 261:

```
f := B1* cosh(a*x1) + B2* x1 * sinh(a*x1);
```

$$f := B1\cosh(a\,x1) + B2\,x1\sinh(a\,x1)$$

```
T3 := diff(f,x1$4);
```

$$T3 := B1\cosh(a\,x1)\,a^4 + 4\,B2\cosh(a\,x1)\,a^3 + B2\,x1\sinh(a\,x1)\,a^4$$

```
T2 := -2*a^2*diff(f,x1$2);
```

$$T2 := -2\,a^2\,(B1\cosh(a\,x1)\,a^2 + 2\,B2\cosh(a\,x1)\,a + B2\,x1\sinh(a\,x1)\,a^2)$$

```
T1 := a^4*f;
```

$$T1 := a^4\,(B1\cosh(a\,x1) + B2\,x1\sinh(a\,x1))$$

```
T1+T2+T3:simplify(%);
```

$$0$$

Thank you Maple.

This is an excellent section! The plan is this: take a thin rectangular bar and apply a **knife-edge** axial compression to both ends. We know that if we apply a *uniform* compression, T_{11} will be constant inside

the bar and T_{11} = F/A. But here we are applying P in a highly non-uniform way at the edges. Imagine a stick of one-by held between two chisels pressing in. Lai shows in this section that far from the edges, you do in fact get a uniform F/A, thus providing a real-world demonstration of St-Venant's Principle.

The knife-edge applies P per unit length in the x_3 direction which has total width 2b, so F = 2bP. This direction is into the plane of paper, but I turned the drawing into a 3D one. The St Venant answer we are looking for is that T_{11} = -F/A = -F/(4bc) = -(2bP)/(4bc) = -P/(2c). You see this answer sitting 5.25.16 along with a second term which is a sum showing the deviation from this St Venant result. As the rod is made long and thin in the c direction, the quantity $\lambda_m \ell = m\pi(\ell/c)$ becomes large even for small integers m. If you look at the sum second term in 5.25.16 you see that as $(\ell/c) \to \infty$ for $x_1 \approx 0$ (in region of rod center), all terms show expo decay due to the factor of 2 inside the denominator sinh function, and thus, as the rod becomes long, (with end size fixed) we get the T_{11} = -F/A result as St. Venant predicts.

[Lai writes St. Venant as if her were writing St. Joseph. Venant was not a Saint, he just happens to have the last name Saint-Venant. I think it is incorrect to refer to St. Venant's Principle.]

The result I think is exact to the extent the plane stress method is exact. The dimension 2b really plays no role in the problem -- parameter b does not show up in the result. The rod may be viewed as a set of planes in the x_1 direction and we have obtained a 2D solution for any one of these x_1-x_2 planes. There is not interaction between the planes.

Notice that as the rod is made thin in the c direction, the knife edge gets closer to being a uniform pressure on the ends, so the result is not surprising at all.

Technical Solution Notes: As usual, Lai proposes a form for an Airy function (in this case, 5.25.3) with some TBD "coefficients". In this case, we know that $\cos(\lambda_m x_2)$ is a complete set in the x_2 direction, so the form shown is really totally general, even with the special term separated out as shown -- this term designed to produce our desired T_{11}. The "coefficients" at this point are the functions $\varphi_m(x_1)$, and the method is Stakgold's "partial eigenfunction method", even though we are not talking eigenfunctions here. We are trying to solve a PDE which is the biharmonic. When our Airy form is inserted into the biharmonic, we obtain an ODE for φ_m shown top p 261 (fourth order!). Lai guesses a solution to this thing in 5.25.5, which I had Maple verify, and it has two constant coefficients B_1 and B_2. Lai then goes on to compute the plane quartet T_{ij}. He says nothing about any possible T_{33} etc (at this point).

Now as usual, we come to the BC's. The first is that there are no shear stresses on the ends, so this says that T_{12} = 0 at the rod ends, and this gives us one equation 5.25.10 relating B_1 and B_2 . The second is that the T_{11} function must match the knife-edge applied stress pattern, and this gives a second equation 5.25.12 relating B_1 and B_2. Lai then solves for B_1 and B_2 as in 5.25.13.

Next, he ponders possible surface tractions on the top and bottom surfaces of the rod. For shear, this involves T_{12}, but due to the $\sin(\lambda_m x_2)$ factor, this vanishes, so our solution automatically shows no shear force on top and bottom of our "plane" section. But interestingly, the *normal* traction on top and bottom does not vanish and is given by 5.25.15. So this is bad news, there is some $T_{22}(x_1)$ along the top and bottom surface in our math model, but there is none in the physical experiment because nothing is touching the rod on top and bottom. But for the same reason described above, for large ℓ/c this T_{22} term gets small and approaches the zero of the experiment. Lai remarks on p 262 that we need large ℓ/b as well to make the T_{33} issue ignorable. I don't really see how T_{33} plays out here. I know we need to have either some E_{33} or some T_{33}. Perhaps this has to do with the other compatibility equations. After all, I think the Airy form only satisfies the planar compat equations.

Comment #1: this section once again demonstrates a sort of Smythian form approach. The key is to construct an Airy function of a good form such that the resulting quartet T_{ij} can be made to solve BC's with appropriate values of constant coefficients. There is implied solution uniqueness going on here that I don't recall Lai mentioning: if you can construct a solution which meets your BC's, that solution is THE solution because the solution is unique. In this problem, there are unfortunately a lot of approximations going on, so the St Venant proof is weakened somewhat. Not all the BC's are met, and we assume the plane stress model. But the idea was a good one.

Comment #2: If you take a slender rod and do this experiment, I could imagine that with enough force, the rod will bend (and everything is still elastic). This would occur perhaps if a slight perturbation exists in the applied forces so there is a end moment twisting the bar. But this solution is ruled out here because we assume exact centered end compression.

5.26 Conversions between plane strain and plane stress (262)

We have to go back now to p 210 and rewrite the E_{11} and E_{22} equations in terms of μ and ν only. According to 5.4.14 we know that

$$1/E_Y = 1/[2\mu(\nu+1)]$$

Then our **general** starting equations are

$$E_{11} = 1/[2\mu(\nu+1)] \{ T_{11} - \nu(T_{22}+ T_{33})\} \qquad (*)$$
$$E_{22} = 1/[2\mu(\nu+1)] \{ T_{22} - \nu(T_{11}+ T_{33})\}$$

Note the relation is $T_{11} \leftrightarrow T_{22}$.

Now, in the **plane stress** case, we just set $T_{33} = 0$ to get

$$E_{11} = 1/[2\mu(\nu+1)] \{ T_{11} - \nu T_{22})\}$$
$$E_{22} = 1/[2\mu(\nu+1)] \{ T_{22} - \nu T_{11})\}$$

For some reason I don't know yet, in 5.26.2 Lai writes the same ν as $\bar{\nu}$ which is fine.

Now for the **plane strain** equations we need $E_{33}= 0$ so that from p 210

$$T_{33} = \nu(T_{11}+ T_{22})$$

Then if we start with (*) above we get

$$E_{11} = 1/[2\mu(\nu+1)] \{ T_{11} - \nu(T_{22}+ [\nu(T_{11}+ T_{22}])\}$$

$$= 1/[2\mu(\nu+1)] \{ T_{11} - \nu(T_{22}+ \nu T_{11}+ \nu T_{22})\}$$

$$= 1/[2\mu(\nu+1)] \{ T_{11} - \nu T_{22}- \nu^2 T_{11}- \nu^2 T_{22})\}$$

$= 1/[2\mu(\nu+1)] \{ T_{11}(1-\nu^2) - T_{22}(\nu+\nu^2)\}$

$= 1/[2\mu(\nu+1)] \{ T_{11}(1+\nu)(1-\nu)) - T_{22}\nu(1+\nu)\}$

$= 1/[2\mu] \{ (1-\nu)T_{11} - \nu T_{22} \}$

Now if we do the swap $T_{11} \leftrightarrow T_{22}$ noted above and T_{33} stays the same, we get

$E_{22} = 1/[2\mu] \{ (1-\nu)T_{22} - \nu T_{11} \}$

So I have now verified 5.26.1 and 5.26.2. Although we discussed plane strain and stress a lot, we never wrote these equations in this manner. To summarize

$E_{11} = 1/[2\mu] \{ (1-\nu)T_{11} - \nu T_{22} \}$ // plane strain
$E_{22} = 1/[2\mu] \{ (1-\nu)T_{22} - \nu T_{11} \}$

$E_{11} = 1/[2\mu(\nu+1)] \{ T_{11} - \nu T_{22})\}$ // plane stress
$E_{22} = 1/[2\mu(\nu+1)] \{ T_{22} - \nu T_{11})\}$

Theorem A: if you make the replacement $\nu \rightarrow \nu/(1+\nu)$ in the first equation pair, you get the second.

Proof: $(1-\nu) \rightarrow 1- \nu/(1+\nu) = 1/(\nu+1)$ // correct for first terms

$\nu \rightarrow \nu/(1+\nu)$ // correct for second terms.

Theorem B: if you make the replacement $\nu \rightarrow \nu/(1-\nu)$ in the second equation pair, you get the first.

Proof: $1/(\nu+1) \rightarrow 1/[1 + \nu/(1-\nu)] = (1-\nu)/[(1-\nu) + \nu] = (1-\nu)$ // correct for first terms

$\nu/(\nu+1) \rightarrow [\nu/(1-\nu)] * (1-\nu) = \nu$ // correct for second terms

To summarize:

$\nu \rightarrow \nu/(1+\nu)$ converts strain to stress equations

$\nu \rightarrow \nu/(1-\nu)$ converts stress to strain equations

Theorem Not Mentioned: If the above rules convert one E_{ij} form to the other, then the same rules convert one u_i form to the other.

Proof: Let the strain objects have no prime, and the stress objects have prime. Then

$E_{ij} = (1/2)(\partial_i u_j + \partial_j u_i)$ // strain
$E'_{ij} = (1/2)(\partial_i u'_j + \partial_j u'_i)$ // stress

Suppose u_i solves the first set of equations. If we *assume* that $u_i' = u_i(\nu \to \nu/(1+\nu))$, then that u_i' will solve the second equation set, since $E'_{ij} = E_{ij}(\nu \to \nu/(1+\nu))$. So we don't need to integrate these equations to solve for u_i in order to prove our Theorem Not Mentioned. All we are saying here is that the rule just shown takes *a* solution of the first equation set and generates *a* solution of the second equation set.

Example 5.26.1 (263). We now apply the Theorem Note Mentioned. It happens that the E_{ij} are stated here in polar coordinates (first time we have seen non-Cartesian coordinates since page 218, about 45 pages previous.). But the "rules" for stress ↔ strain don't care about coordinate systems! That is the point of this quiet little Example.

5.27 Polar Coordinates (264)

• First of all, Lai writes the static Navier equations in terms of T_{ij} for polar coordinates. This means we need $(\text{div}T)_i = \partial_j T_{ij}$. But we had this for cylindricals on page 170 so just read if off as shown.

• Next, what happens to the Airy rules? This subject resulted in a whole Lai support doc called "Lai problem 5.71" followed by a 5+ day flurry of tensor doc updates. Tensor doc now has Appendix E (h) which talks in general about tensor expansions on the unit basis vectors $\hat{e}_n$ and includes this Lai application as an example. So we can now just quietly check off the equations (5.27.3) after this storm of activity. Lai does claim that you could avoid all that activity and just accept 5.27.3 as a good guess, then show it works in 5.27.1,1 which are the Navier static equations. (I did not do this)

Lai then quotes the relations between E and T which we already found earlier. But he quietly suddenly changes the meaning of T_{ij} from Cartesian to polar, as if something is "covariant". I need to check on this! // I did so, and show this in tensor doc at the very end of App E (h). The T_{ij} to E_{ij} equation is covariant with respect to rotations, so it is true just as stated in terms of things like T_{rr} and E_{rr} . Lai did not mention this small technical detail and just quotes the equations at the bottom of page 264.

[I did a review starting with the waves, and am now caught up to here. This chapter for sure will need some meta notes as I did for Stak chapters. -- 4.13.12]

5.28 Stress Distribution Symmetrical about an axis (265)

My drawing shows a little particle in a cylindrical framework. "Symmetrical" includes the idea that the stress pattern has to be the same if you reflect the particle in an axis through it and the axis, so this rules out $T_{r\theta}$ and $T_{\theta r}$ shear stress components. You could have shear T_{zr}, but I think we are still in the plane strain and plane stress scenario. We have a slice of the bar and we are just expressing things x_1,x_2 in terms of r,θ so we still have a T matrix of the shape before and we have just altered the upper left quad of elements. We still have $T_{rz} = 0$ and $T_{\theta z} = 0$ by assumption. We might have some T_{zz} at some point, but not in a **plane stress** situation. So our only non-zero stress components are T_{rr} and $T_{\theta\theta}$. Symmetrical means that these do not vary with θ, so we then have $T_{rr}(r)$ and $T_{\theta\theta}(r)$.

So that was the first step. We now attempt to represent these two guys in their Airy Function form which in polars was 5.27.3 where the ∂_θ term now goes away. This Airy form is OK if it solves $\nabla^4\varphi = 0$ as we know. This is written now in cylindrical coordinates but nothing varies in z or θ, so $\nabla^4\varphi$ is pretty

simple as shown in 5.28.3. Since this is a 4th order ODE, there must be 4 independent solutions and they are all trivially listed off in 5.28.4, each with a constant A,B,C,D. Then the Airy formulas give T_{rr} and $T_{\theta\theta}$ as in 5.28.5.

Conclusion: with cylindrical symmetry and our plane stress framework, a static stress pattern must have this very restricted form! In any azisym problem, all we have to do is find the coefficients.

5.29 What displacement structure is implied by the previous section (265)

This is a fascinating section, another Lai tour de force. The final **u** result is 5.29.17 and 18 where you see the four constants appearing. He could have argued right here that we must have B = 0 since a linear θ term makes no sense in an azisym situation. It might be there for a cylindrical wedge problem where the angle θ is restricted. So the main result is u_r with just A and C.

The interest is how Lai *arrives* at these results. He first takes our restricted T_{rr} and $T_{\theta\theta}$ forms from the last section and inserts them into our polar-coordinates version of the plane-strain equations for E. We then end up with three partial differential equations

$\partial_r u_r = F(r)$	5.29.1	E_{rr}
$(1/r)\partial_\theta u_\theta + (1/r)u_r = G(r)$	5.29.2	$E_{\theta\theta}$
$(1/r)\partial_\theta u_r + \partial_r u_\theta - (1/r)u_\theta = 0$	5.29.3	$E_{r\theta}$

The problem is solving these three PDE's to find expressions for $u_r(r,\theta)$ and $u_\theta(r,\theta)$.

Comment: This is another of my big knowledge holes. This is a coupled *system* of first-order linear PDE's in 2 variables with two functions to solve for, with some given source functions of r sitting in there. In general, I would have no idea how to solve such a system of equations. I don't even know how to solve one first order PDE in any general way. This is the Courant Hilbert business.

Luckily, Lai is going to show how to solve the above system in complete detail. I did it all and it all makes sense, every single step. The first step is to "integrate" $\partial_r u_r(r,\theta) = F(r)$ to get 5.29.4 where we are stuck with some new unknown function f(θ) as the "integration constant" relative to r. Later we get another unknown such "integration function" g(r). So we have now to find not 2 but 4 functions. He is able to come up with a certain "separation" as in 5.29.8 and he solves it all. The result is shown in the two lines top of page 267. I won't detail all the steps here, but it is all done very honestly.

He then makes the claim that the terms with constants H,G and F are terms associated with rigid body motions. Before I could verify that by showing that these terms give an antisymmetric $\nabla\mathbf{u}$ object. I suppose you could somehow show that here in polar coordinates using the official form for $\nabla\mathbf{u}$ in polar coordinates. I will put off this task and accept it. If we throw out these terms which cause rigid body displacement patterns in our azisym problem (we don't have any particular one specified yet), we get the part that has significance. We never care about just rotating or translating the physical object. So the significant part of the result is shown in 5.29.17 and 18 and we still have our four original constants A,B,C,D which arose in the previous section. Notice that it is the "integration functions" f(θ) and g(r) that

we associate with the rigid body displacements. This always seems to be what happens, but I don't have a theorem saying why that is the case. Maybe there is a superposition theorem at work here.

In separate doc "Most general rigid 2D displacement in polars.doc" I have verified that Lai's claim is correct, that the F,G and H terms arise from 2D rigid displacements! The GH come from translations, while the F term comes from rotations. It took me a while to get this right.

5.30 Thick walled cylinder with pressures in and out. (267)

OK, finally we arrive at a real problem which is going to use all the above stuff. Knowing PVC properties, I might calculate how much water pressure a pipe can withstand!

We first set the T_{rr} inside and out as shown 5.30.1. We then set B = 0 for the θ reason stated, and this gives T_{rr} and $T_{\theta\theta}$ as shown in 5.28.5 (general azisym) which he copies into 5.30.2. It is then just a question of getting constants A and C in terms of the inside and outside pressures p_i and p_o.

$-p_i = A/a^2+2C \qquad -p_o = A/b^2+2C$ // the T_{rr} BC's

Therefore

$-p_i+ p_o = A/a^2 - A/b^2 = A(1/a^2-1/b^2) = A(b^2-a^2)/(a^2b^2) \quad \Rightarrow \quad A = (p_o-p_i)\, a^2b^2/(b^2-a^2)$

$-a^2p_i = A+2Ca^2 \qquad -b^2p_o = A+2Cb^2$

$-a^2p_i + b^2p_o = 2C\,(a^2-b^2) \quad \Rightarrow \quad C = (1/2)\,(b^2p_o-a^2p_i)/(a^2-b^2)$

We then get

$T_{rr} = A/r^2+2C = (p_o-p_i)\, r^{-2}a^2b^2/(b^2-a^2) + (b^2p_o-a^2p_i)/(a^2-b^2)$

$= (b^2-a^2)^{-1}\{\ (p_o-p_i)\, r^{-2}a^2b^2 \ - (b^2p_o-a^2p_i)\ \}$

$= (b^2-a^2)^{-1} \quad [\ p_i\ \{\ -r^{-2}a^2b^2 + a^2\}\ + p_o\{\ r^{-2}a^2b^2 - b^2\}\ ]$

$= -p_i\ \{\ (b^2-a^2)^{-1}(\ r^{-2}a^2b^2 - a^2)\}\ - p_o\{\ (b^2-a^2)^{-1}\ (-r^{-2}a^2b^2 + b^2)\ \}$

$= -p_i\ \{\ a^2(b^2-a^2)^{-1}(\ (b/r)^2 - 1)\}\ - p_o\{\ b^2(b^2-a^2)^{-1}\ (-(a/r)^2 + 1)\ \}$

$= -p_i\ \{\ ((b/a)^2-1)^{-1}(\ (b/r)^2 - 1)\}\ - p_o\{\ (1-(a/b)^2)^{-1}\ (-(a/r)^2 + 1)\ \}$ // 5.30.4 first

To get the $T_{\theta\theta}$ result, replace $r^{-2} \rightarrow - r^{-2}$ and you get then the second of 5.30.4.

Suppose as in the PVC pipe case we have $p_o \approx 0$. Then we get $T_{rr} < 0$ which means in the radial direction a particle feels compression (just as $T_{ii} = -p$ for hydrostatic test) , BUT in the tangential direction we have $T_{\theta\theta} > 0$ which means your PCV pipe is trying pulled apart in that direction (a "tensile stress"). It all makes perfect sense. Notice that $T_{\theta\theta}$ is maximum at b = r which is the outside surface

(which has to do the most stretching). So if the PCV pulls apart in the tension sense, that will happen first on the outside surface of the pipe. I would have to think about the maximal sheer stress situation. In our polar coordinate directions, there is no shear at all. I could guess max shear is the same number maybe.

We know our displacement pattern from 5.29.17 (B = 0 and our A,C above).

I agree with closing comment on how u would be different for plain-strain solution. Here we have been talking plain stress. We have been using formulas for the plain-stress situation.

Example 5.30.1. Pipe with small ID relative to OD, outside pressure only. In the limit b >> a the results depend only on a. Could you "crush' the pipe? The results shows $T_{\theta\theta} = -2p_o$ at r = a, the inside of the pipe, so I guess with enough p_o you could make $T_{\theta\theta}$ = yield strength.

5.31 Bending of a curved beam (268)

The picture shows the cross section of this "beam" which is assumed thin in the direction normal to paper. He wants to torque both ends and see what happens. The torque pattern shown should bend the beam down a bit. This beam starts out being part of a annular ring so polar coordinates are appropriate.

Now he makes the azisym claim for such a beam. Why would that be?? In our pure bending example page 247 we had only $T_{11}(\mathbf{x}_T)$ and said that T_{11} was the same for all cross sections. So I guess here we are going to make the ansatz that we have T_{rr} and $T_{\theta\theta}$ which are the same at any cross section of the curved beam. The T_{rr} is like the T_{11} and the $T_{\theta\theta}$ arises because it is curved.

The upshot: we are going to try to fit things with the 5.28.5 plane-stress model. This time the B term is going to survive!

The first question is: what do we get for total force on the right end? Well, you have to integrate $T_{\theta\theta}$ over that end. If you do this and make use of the "no tractions on either curved side" 5.31.2, you find that the total force at the right end is R = 0. This would apply for the left end by symmetry as well.

Next, and as usual, he will compute M at the right end using r x F with origin at as in the drawing. Then torque = $\tau = rF = rT_{rr}(r)drh$ and $\tau/h = rT_{rr}(r)dr$ so this is what you integrate to get M but it is -M which is fine.

Now 5.31.2 is two equations in 3 unknowns A,B,C. The integral for M comes out M(A,B,C) as in 5.31.8. If we regard M as a "given", we then have 3 in 3 and we solve for A,B,C as in 5.31.9. The fact that there exists a solution means this IS the solution within our context. The solution of course involves the radii a and b and the applied "couple" M. You could then go on somehow to compute E_{ij} and u_i and then find how much it bends down along the center line, but of course it is all polar coordinates now. Lai just stops at page 269.

5.32 Initial Stress in a Welded ring (270)

An application of the previous section. Assume you cut a tiny wedge of small angle α out of the ring, and then you take that as your "curved beam" of the previous section. We are then allowed to use the "curved beam" results. Since we also have a "stress distribution which is azisym", we can use results from the top of page 267. Luckily, we computed there the displacements u_r and u_θ for the general azisym solution. But in the current ring problem we know that in order to bend the ring into a closed shape, we need $u_\theta = \alpha r$, because this will cause perfect abutment. Setting those two u_θ forms equal determines B. But the middle equation of 5.31.9 lets us write B in terms of M and N for the curved beam and thus we know M. So if you do your "weld" (perfectly), each end of this circular beam has this moment acting on it. We of course

know everything about the ring. I assume this ring is a thin flat washer to be consistent with earlier assumptions. Closing comment confuses me. I know that $T_{r\theta}= 0$ everywhere, yes, but T_{rr} is not 0. At the faces of this "beam" we already know that the applied total force is 0 from doing the beam problem. But I don't see the connection between the first fact and the second.

5.33 Trig atoms for Airy Functions

Suppose we try a set of functions $\varphi_{nc} = f(r)\cos(n\theta)$ and $\varphi_{ns} = f(r)\sin(n\theta)$. Trying these in the $\nabla^4\varphi=0$ Airy requirement gives 5.33.1 as a 4th degree ODE for f(r). I expect 4 f(r) solutions and this leads to the general solutions for φ_{nc} and φ_{ns} shown in 5.33.4. So far so good.

What happens when n = 0? This is a non-trivial question. Recall that n=0 is always a special case in the Laplace world and brings in logs. In this case we have to go back to the PDE 5.27.6 and I don't know how to count the solutions of a PDE. Lai finds the 5 solutions shown in 5.33.9 in addition to the constant solution which no one cares about. I guess there are no more independent solutions, but I have no idea how one would show that! Again, my PDE "hole".

In the case n = 1 pairs of solutions become the same since $r^{n+1} = r^{-1}$ and $r^{-n} = r^{-1}$. We then seem to have only 6 distinct solutions from the set of 8 in 5.33.4. But Lai comes up with 4 new ones so we end up with a whopping 10 solutions for n = 1! There are 5 cosine ones and 5 sine ones.

OK, all mysterious to me, he is just stating facts and I can verify them, but I cannot show that there are not other solutions he may have missed. I don't know how to count solutions! Not sure how to find more on this if I wanted to.

So we have three cases to worry about: n=0, n=1 and other n (he has not said integer yet). For each case, we can of course stuff our general φ into the Airy formulas and obtain T_{rr} and $T_{\theta\theta}$ and $T_{r\theta}$ and he does this on page 272. He splits up the n=1 case into two sets of 5 solutions each.

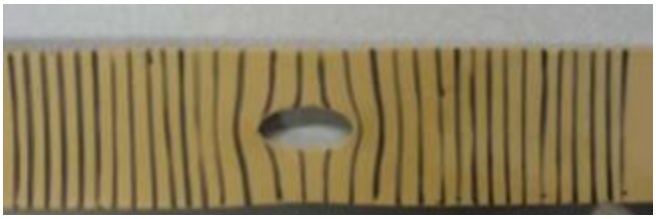
Example 5.33.1. This is an example "doing a fit". We have a hollow cylinder b>a and a set of BC's shown top page 273 (four of them). We "try" $\varphi = f(r)\cos(n\theta)$ with its 4 coefficients and we find that they are all set by the four BC's and we get a solution for this plane-stress problem. Lai then takes the $b/a\to\infty$ and we get simpler results which of course depend only on the inside radius a. This is of course a contrived toy problem, but it well illustrates the method.

Example 5.33.2. This is just an exercise in following instructions, all OK.

5.34 Hole in Plate under Tension (274)

This certainly must be a Hit Parade classic problem, and I have a cloth picture above showing what probably happens. Lai "calls in" a lot of chits on this problem that were staged elsewhere. First, if we want to get $T_{11} = \sigma$ into polars, we do the matrix thing of Problem 5.71 to get 5.34.2.

The key idea is an imaginary large circle as shown with radius b and we put a BC on this circle which is just the general St Venant as-if-no-hole stress field value. He then does a proper "BC superposition" of the problem into problem 1 and problem 2, and it turns out we have already solved both of those in earlier Exercises. These two solutions are shown p 275 and are then added to get 5.34.8. We can then plot the $T_{\theta\theta}$ around the hold boundary, and I plot it on the picture, it is very simple. At the top and bottom of the hole the background axial stress is tripled ! It is 0 at 30 degrees and in compression below that. So look at the nice web picture again,



Here you see that the tripled stress makes the material stretch a lot at the top and bottom of the hole. The sides of the hole are under compression in $T_{\theta\theta}$ but hard to see that in the picture, except cloth might ripple at that location. This tripling effect is called "stress concentration". So a hole makes a material "weaker" in a serious way, the failure stress is cut to 1/3 of what it would otherwise be.

5.35 Hole in Plate under Pure Shear (276)

Again we take our pure shear background matrix and convert it to polars by the same method. The same outer ring at b. Establish BC's in the same way as before (of course no surface traction on the hole boundary since nothing is touching it!) He "tries" a certain solution (from an exercise!) which has 4 coefficients to find, and applying the four BC's gets the result. Here they have made a mistake which is repeated twice which I will report in at some point. Their error is verified in a PDF I have found,.

$$(121) \qquad \sigma_{rr} = S\left(1 - 4\frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right)\sin(2\theta)$$

$$(122) \qquad \sigma_{\theta\theta} = S\left(-1 - 3\frac{a^4}{r^4}\right)\sin(2\theta)$$

$$(123) \qquad \sigma_{r\theta} = S\left(1 + 2\frac{a^2}{r^2} - 3\frac{a^4}{r^4}\right)\cos(2\theta)$$

http://en.wikiversity.org/wiki/Introduction_to_Elasticity/Plate_with_hole_in_shear

This problem is a bit less practical than the previous one I would say. But both are excellent.

5.36 Wedge Loaded at Apex (277)

Another winner. He makes an ansatz that $T_{\theta\theta} = 0$ on the wedge sides. No surface tractions on same gives $T_{r\theta} = 0$. Thus, thus 5.36.1 is four BC's.

You then do a force balance deal. His picture is very helpful and I agree with 5.36.2 which are done at some arbitrary radius r.

He then rolls out an n=1 proposal from his ample tool kit (n=1 general case on page 272) and we have two constants to find. The barred $B_5 = 0$ due to the vertical force balance equation, while the left right balance relates P to B_5. The final result is T_{rr} only, 5.36.7. The ansatz is justified.

In this toy problem, the wedge is infinite in the radial direction, so we never have to worry about what happens on the right surface. The solution shows $T_{rr} \sim 1/r$ and peaks at $\theta = 0$ as you might expect. I like this problem.

5.37 The Flamant Problem (277)

We just set $\theta = \pi/2$ in the previous problem and we have a 2D point force acting on 2D half space. This would be a knife edge acting on a half plane. The solution is a simple limit of the previous problem and is extremely simple. This type of wedge problem is associated with Flamont's work in 1892.

Alfred-Aimé Flamant (born in Noyales , Aisne , October 31, 1839 - disappeared during the war from 1914 to 1918) was a French engineer, former disciple of Barre de Saint-Venant [1] and chemistry professor Charles Leon Durand- Claye.

He was a French civil engineer and buddy of Saint-Venant and Joseph Valentin Boussinesq, another big name in this field, but more so in the fluid realm. This one was a "hydraulic engineer". Boussinesq did however do the 3D version of the Flamont Problem and it will show up on page 293 where in fact it is called the Boussinesq Problem.

And so ends this very nice section on more or less 2D plane stress type problems. It was very good. Notice that we have not really done any truly 3D problems.

A.4 3D Elastostatic Potential Theory (279)

5.38 Fundamental potential functions for elastostatic problems (279).

If we replace ρB by B and call this B the body force *per volume*, and if we replace (λ+μ) by the factor shown in 5.38.2 (derived on p 212), then it is true that the Navier equation of page 216 can be written for a static situation as: (see "essay" and see p 212 table for λ)

$$\mu/(1-2\nu)\,\nabla e + \mu\nabla^2\mathbf{u} + \mathbf{B} = 0 \qquad e = \nabla\bullet\mathbf{u} \qquad \lambda+\mu = \mu/(1-2\nu) \qquad \lambda = 2\mu\nu/(1-2\nu).$$

Now out of the blue Lai tells us to "try" this form for **u**

$$\mathbf{u} = \mathbf{\Psi} - 1/[4(1-\nu)]\,\nabla(\mathbf{x}\bullet\mathbf{\Psi} + \Phi)$$

where **Ψ** is a vector potential and Φ is a scalar potential. What happens when we insert this mess into the Navier equation above?

Derivation of 5.38.3

$$\mu/(1-2\nu)\,\nabla e + \mu\nabla^2\mathbf{u} + \mathbf{B} = 0 \qquad e = \nabla\bullet\mathbf{u}$$
$$\mathbf{u} = \mathbf{\Psi} - 1/[4(1-\nu)]\,\nabla(\mathbf{x}\bullet\mathbf{\Psi} + \Phi)$$

For the moment, let's define $a = \mu/(1-2\nu)$ and $b = -1/[4(1-\nu)]$ so these equations are

$a\nabla e + \mu\nabla^2\mathbf{u} + \mathbf{B} = 0 \qquad e = \nabla\bullet\mathbf{u}$

$\mathbf{u} = \mathbf{\Psi} + b\,\nabla(\mathbf{x}\bullet\mathbf{\Psi} + \Phi)$

This is so totally messy that I think it is best to stick to components, so

$a\,\partial_i e + \mu\nabla^2 u_i + B_i = 0 \qquad e = \nabla\bullet\mathbf{u} = \partial_j u_j$

$u_i = \Psi_i + b\,\partial_i[\, x_k\Psi_k + \Phi\,]$

or

$a\,\partial_i\partial_j u_j + \mu\nabla^2 u_i + B_i = 0$

$u_i = \Psi_i + b\,\partial_i[\, x_k\Psi_k + \Phi\,]$

or

$-B_i = a\,\partial_i\partial_j u_j + \mu\nabla^2 u_i$

$u_i = \Psi_i + b\,\partial_i[\, x_k\Psi_k + \Phi\,]$

So now we can just jam in u_i to get

$-B_i = a\,\partial_i\partial_j\{\,\Psi_j + b\,\partial_j[\, x_k\Psi_k + \Phi\,]\} + \mu\nabla^2\{\,\Psi_i + b\,\partial_i[\, x_k\Psi_k + \Phi\,]\}$

$= a\,\partial_i\partial_j\Psi_j + ab\partial_i\partial^2{}_j[\, x_k\Psi_k + \Phi\,] + \mu\nabla^2\,\Psi_i + \mu b\nabla^2\partial_i[\, x_k\Psi_k + \Phi\,]$

$= a\,\partial_i\partial_j\Psi_j + ab\partial_i\nabla^2[\, x_k\Psi_k + \Phi\,] + \mu\nabla^2\,\Psi_i + \mu b\nabla^2\partial_i[\, x_k\Psi_k + \Phi\,]$

$= a\,\partial_i\partial_j\Psi_j + ab\partial_i\nabla^2(x_k\Psi_k) + \mu\nabla^2\,\Psi_i + \mu b\nabla^2\partial_i(x_k\Psi_k) + ab\partial_i\nabla^2\Phi + \mu b\nabla^2\partial_i\,\Phi$

$= a\,\partial_i\partial_j\Psi_j + ab\partial_i\nabla^2(x_k\Psi_k) + \mu\nabla^2\,\Psi_i + \mu b\partial_i\,\nabla^2\,(x_k\Psi_k) + ab\partial_i\nabla^2\Phi + \mu b\partial_i\,\nabla^2\,\Phi$

$= a\,\partial_i\partial_j\Psi_j + (a+\mu)b\partial_i\nabla^2(x_k\Psi_k) + \mu\nabla^2\,\Psi_i + (a+\mu)b\partial_i\nabla^2\Phi$

Now consider

$\nabla^2(x_k\Psi_k) = \partial_i\partial_i(x_k\Psi_k) = \partial_i\,(x_k\partial_i\Psi_k + \delta_{ik}\Psi_k) = \partial_i\,(x_k\partial_i\Psi_k + \Psi_i)$

$= [x_k\nabla^2\Psi_k + \delta_{ik}\partial_i\Psi_k + \partial_i\,\Psi_i] = [x_k\nabla^2\Psi_k + \partial_k\Psi_k + \partial_i\,\Psi_i]$

$= [x_j\nabla^2\Psi_j + 2\partial_j\Psi_j]$

Then continuing above we get

$= a\,\partial_i\partial_j\Psi_j + (a+\mu)b\partial_i[x_j\nabla^2\Psi_j + 2\partial_j\Psi_j] + \mu\nabla^2\,\Psi_i + (a+\mu)b\partial_i\nabla^2\Phi$

$= a\,\partial_i\partial_j\Psi_j + (a+\mu)b\{\,x_j\partial_i\,\nabla^2\Psi_j + \delta_{ij}\nabla^2\Psi_j + 2\partial_i\partial_j\Psi_j\,\} + \mu\nabla^2\,\Psi_i + (a+\mu)b\partial_i\nabla^2\Phi$

$= a\,\partial_i\partial_j\Psi_j + (a+\mu)b\{\,x_j\partial_i\,\nabla^2\Psi_j + \nabla^2\Psi_i + 2\partial_i\partial_j\Psi_j\,\} + \mu\nabla^2\,\Psi_i + (a+\mu)b\partial_i\nabla^2\Phi$

$= [2(a+\mu)b + a]\,\partial_i\partial_j\Psi_j + (a+\mu)b\,\{\,x_j\partial_i\,\nabla^2\Psi_j + \nabla^2\Psi_i + \partial_i\nabla^2\Phi\,\} + \mu\nabla^2\,\Psi_i$

Now what is the leading constant? Maple says it vanishes:

```
> a := u/(1-2*v);
```

$$a := \frac{u}{1-2\nu}$$

```
> b := -1/(4*(1-v));
```

$$b := -\frac{1}{4-4\nu}$$

```
> c := 2*(a+u)*b+a: simplify(%);
```

$$0$$

So we then have

$= (a+\mu)b\ \{\ x_j \partial_i\ \nabla^2\Psi_j\ + \nabla^2\Psi_i + \partial_i\nabla^2\Phi\ \} + \mu\nabla^2\ \Psi_i$

$= (a+\mu)b\ \{\ x_j \partial_i\ \nabla^2\Psi_j\ + \nabla^2\Psi_i + \mu/[(a+\mu)b]\ \nabla^2\ \Psi_i\ + \partial_i\nabla^2\Phi\ \}$

$= (a+\mu)b\ \{\ x_j \partial_i\ \nabla^2\Psi_j\ +\ \{\ 1 + \mu/[(a+\mu)b]\}\ \nabla^2\ \Psi_i\ + \partial_i\nabla^2\Phi\ \}$

So what is this {..} coefficient?

```
d := 1 + u/(b*(a+u)): simplify(%);
```

$$-1+4\nu$$

So we then have

$= (a+\mu)b\ \{\ x_n \partial_i\ \nabla^2\Psi_n\ +\ (-1+4\nu)\ \nabla^2\ \Psi_i\ + \partial_i\nabla^2\Phi\ \}$

Finally, this leading coefficient is

```
e := (a+u)*b: simplify(%);
```

$$\frac{1}{2}\,\frac{u}{-1+2\nu}$$

so our final result is then

$= -\mu/[2(1-2\nu)]\ *\ \{\ x_n \partial_i\ \nabla^2\Psi_n\ +\ (-1+4\nu)\ \nabla^2\ \Psi_i\ + \partial_i\nabla^2\Phi\ \}\ =\ -B_i$

or

$-\mu/[2(1-2\nu)]\ *\ \{\ x_n \partial_i\ \nabla^2\Psi_n\ -\ (1-4\nu)\ \nabla^2\ \Psi_i\ + \partial_i\nabla^2\Phi\ \}\ + B_i\ = 0$

which finally agrees with 5.38.3.

Derive 5.38.4.

Now suppose we assume that

$\nabla^2\ \Psi_i = -B_i/\mu$ and $\nabla^2\Phi = +\ x_n B_n/\mu$

Insert these in the above equation to get

$$-\mu/[2(1-2\nu)] * \{ x_n \partial_i \nabla^2 \Psi_n - (1-4\nu) \nabla^2 \Psi_i + \partial_i \nabla^2 \Phi \} + B_i = 0$$
$$-\mu/[2(1-2\nu)] * \{ x_n \partial_i [-B_n/\mu] - (1-4\nu) [-B_i/\mu] + \partial_i (x_n B_n/\mu) \} + B_i = 0$$
$$-\mu/[2(1-2\nu)] * \{ x_n \partial_i [-B_n] - (1-4\nu) [-B_i] + \partial_i (x_n B_n) \} + \mu B_i = 0$$
$$-\mu/[2(1-2\nu)] * \{ -x_n \partial_i B_n + (1-4\nu) B_i + \partial_i (x_n B_n) \} + \mu B_i = 0$$
$$\mu/[2(1-2\nu)] * \{ -x_n \partial_i B_n + (1-4\nu) B_i + \partial_i (x_n B_n) \} - \mu B_i = 0$$
$$\{ -x_n \partial_i B_n + (1-4\nu) B_i + \partial_i (x_n B_n) \} - [2(1-2\nu)] B_i = 0$$
$$\{ -x_n \partial_i B_n + (1-4\nu) B_i + x_n \partial_i B_n + \delta_{in} B_n \} - [2(1-2\nu)] B_i = 0$$
$$\{ -x_n \partial_i B_n + (1-4\nu) B_i + x_n \partial_i B_n + B_i \} - \mu[2(1-2\nu)] B_i = 0$$
$$\{ (1-4\nu) B_i + B_i \} - [2(1-2\nu)] B_i = 0$$
$$\{ (2-4\nu) B_i \} - [2(1-2\nu)] B_i = 0$$
$$2 (1-2\nu) B_i - [2(1-2\nu)] B_i = 0 \qquad \text{QED.}$$

So I have shown that *if* the potentials satisfy the two Poisson equations shown, then they also solve the static Navier equation !!! I have not shown whether or not there might be some potentials which satisfy the Navier which do not satisfy the two Poissons.

Derive page 280 top alternative form

Start with what we know

$$\mathbf{u} = \mathbf{\Psi} - 1/[4(1-\nu)] \nabla(\mathbf{x} \bullet \mathbf{\Psi} + \Phi)$$
$$= \{ 4(1-\nu) \mathbf{\Psi} - \nabla(\mathbf{x} \bullet \mathbf{\Psi} + \Phi) \} / [4(1-\nu)]$$

Multiply by 2μ

$$2\mu\mathbf{u} = \mu\{ 4(1-\nu) \mathbf{\Psi} - \nabla(\mathbf{x} \bullet \mathbf{\Psi} + \Phi) \} / [2(1-\nu)]$$

Now replace

$$\mathbf{\Psi} = [-2 \boldsymbol{\psi} (1-\nu)/\mu]$$
$$\Phi = [-2 \varphi (1-\nu)/\mu]$$

and get

$$2\mu\mathbf{u} = \mu\{ 4(1-\nu) [-2 \boldsymbol{\psi} (1-\nu)/\mu] - \nabla(\mathbf{x} \bullet [-2 \boldsymbol{\psi} (1-\nu)/\mu] + [-2 \varphi (1-\nu)/\mu]) \} / [2(1-\nu)]$$
$$= \mu\{ 2 [-2 \boldsymbol{\psi} (1-\nu)/\mu] - \nabla(\mathbf{x} \bullet [- \boldsymbol{\psi}/\mu] + [-\varphi/\mu]) \}$$
$$= \{ 2 [-2 \boldsymbol{\psi} (1-\nu)] - \nabla(\mathbf{x} \bullet [- \boldsymbol{\psi}] + [-\varphi]) \}$$
$$= -4 (1-\nu) \boldsymbol{\psi} + \nabla(\mathbf{x} \bullet \boldsymbol{\psi} + \varphi)$$

Then for no body forces, we get these simpler looking equations

$$2\mu\mathbf{u} = -4 (1-\nu) \boldsymbol{\psi} + \nabla(\mathbf{x} \bullet \boldsymbol{\psi} + \varphi)$$
$$\nabla^2 \boldsymbol{\psi} = 0 \qquad \nabla^2 \varphi = 0$$

Solve the bottom two Laplace equations (in Cartesians), then the first line tells you the displacement **u** ! Not obvious how BC's fit in.

TEN "EXAMPLES": these follow below

Example 5.38.1 -- this is Lai's derivation of 5.38.4, it is a lot faster than mine above but fine.

Example 5.38.2 (Cartesian) -- what if only the scalar potential φ is non-vanishing? Then $2\mu\mathbf{u} = \nabla\varphi$ so φ is like an electrostatic potential and $-2\mu\mathbf{u}$ is like the **E** field. He computes E_{ij} as usual (Cartesian) to get a trivial result, then he shows that e = 0 and therefore T = E ! (from Hooke)

Example 5.38.3 (281) (cylindrical, azisym) -- conversely, what if only $\psi_z \equiv \psi(r,z)$ is non-vanishing? In this case we find that $2\mu\mathbf{u} = -4(1-\nu)\psi\hat{\mathbf{z}} + \nabla(z\psi)$ and we use the cylindrical version of ∇

$$\nabla(z\psi(r,z)) = \partial_r(z\psi)\,\hat{\mathbf{r}} + \partial_z(z\psi)\,\hat{\mathbf{z}} = z\partial_r\psi\hat{\mathbf{r}} + [\,z\partial_z\psi + \psi]\,\hat{\mathbf{z}}$$

$$\Rightarrow\ 2\mu\mathbf{u} = -4(1-\nu)\psi\hat{\mathbf{z}} + z\partial_r\psi\hat{\mathbf{r}} + [\,z\partial_z\psi + \psi]\,\hat{\mathbf{z}}$$

$$\Rightarrow\ 2\mu u_r = z\partial_r\psi$$
$$2\mu u_z = -4(1-\nu)\psi + z\partial_z\psi + \psi = (-3+4\nu)\psi + z\partial_z\psi$$
$$2\mu u_\theta = 0$$

He next quotes a result from e, but I think you need to find E first and that requires the cyl E equations. In "essay" I quote all these forms which are

$E_{rr} = \partial_r u_r$	$= (1/2\mu)\, z\partial_r^2\psi$
$E_{\theta\theta} = (1/r)(\partial_\theta u_\theta + u_r)$	$= (1/2\mu)\,(1/r)\, z\partial_r\psi$
$E_{r\theta} = [\,(1/r)(\partial_\theta u_r - u_\theta) + \partial_r u_\theta]/2 = E_{\theta r}$	$= 0$
$E_{rz} = (\partial_z u_r + \partial_r u_z)/2 = E_{zr}$	$= (1/4\mu)\{\,\partial_z(z\partial_r\psi) + \partial_r((-3+4\nu)\psi + z\partial_z\psi\,\}$
$E_{\theta z} = (\partial_z u_\theta + (1/r)\partial_\theta u_z)/2 = E_{z\theta}$	$= 0$
$E_{zz} = \partial_z u_z$	$= (1/2\mu)\,\{\ (-3+4\nu)\,\partial^2_z\psi + \partial_z(z\partial_z\psi)\,\}$

Now it seems to me that $e = E_{ii}$ (covariant) so we should then have
(note that $e = E_{ii} = E_{rr} + E_{\theta\theta} + E_{zz}$)

$$2\mu e = z\partial_r^2\psi + (1/r)\, z\partial_r\psi + (-3+4\nu)\,\partial_z\psi + \partial_z(z\partial_z\psi)$$

$$= z\partial_r^2\psi + (1/r)\, z\partial_r\psi + (-3+4\nu)\,\partial_z\psi + \partial_z\psi + z\partial^2_z\psi$$

$$= z\partial_r^2\psi + (1/r)\, z\partial_r\psi + (-2+4\nu)\,\partial_z\psi + z\partial^2_z\psi$$

but this seems to disagree with 5.38.17. But wait, we also know that $\nabla^2\boldsymbol{\psi} = 0$ and we have then to go look that thing up to find, in the case $\boldsymbol{\psi} = \psi\hat{\mathbf{z}}$, that

$$\nabla^2\boldsymbol{\psi} = \hat{\mathbf{r}}\,[\,0+0+0] + \hat{\boldsymbol{\theta}}\,[\,0+0+0\,] + \hat{\mathbf{z}}\,[\,\nabla^2\psi] = (\nabla^2\psi)\,\hat{\mathbf{z}}$$

and therefore with the scalar Laplacian we have $\nabla^2\psi = 0$ which says

$$(1/r)\partial_r(r\partial_r\psi) + \partial_z{}^2\psi = 0$$

$$\partial_r{}^2\psi + (1/r)\,\partial_r\psi + \partial_z{}^2\psi = 0$$

Therefore our result above can be written

$$2\mu e = (-2+4\nu)\,\partial_z\psi = -2(1-2\nu)\,\partial_z\psi \qquad \text{// agrees with 5.38.17}$$

Well I *now* see that Lai computes all this stuff as I have above on page 282. His final step is to use Hooke's Law (covariant) to find T_{ij} now that we are given e and E_{ij}: $T_{ij} = \lambda e\delta_{ij} + 2\mu E_{ij}$. From above we also have $\lambda = 2\mu\nu/(1-2\nu)$, so this is just "turn the crank". I would do this in Maple if I wanted to do it. Since my $E_{r\theta}=0$ and $E_{\theta z}= 0$, it is clear that the corresponding T's vanish so I check those on page 281.

Example 5.38.4 (cylindrical, azisym) -- This time we assume that (I use "f" since I don't have two kinds of lower case φ symbols)

$$\varphi = (1-2\nu)f(r,z) \qquad \boldsymbol{\psi} = \psi_z\hat{\mathbf{z}} = (\partial_z f)\,\hat{\mathbf{z}}$$

Can I superpose two of the examples above? The u in terms of potentials is linear. And e = E_{kk} is linear. And $T_{ij} = \lambda E_{kk}\delta_{ij} + 2\mu E_{ij}$ is also linear, so I would say yes. Let's try this approach. But it turns out I don't have the φ = (1-2ν)f(r,z) as an example, I only had it in Cartesians.

OK, I will just "go with" his solution. We have two of four potential components as shown above in terms of a scalar function f. Getting **u** is just a plug-in. Getting E_{ij} uses my cylindrical forms noted above and from that get e. Then getting T_{ij} uses the covariant Hooke's. So I will learn nothing new by doing the algebra here, so let's just accept the Lai results here.

Example 5.38.5 (sphericals, radial only) -- Lai replaces the usual (r,θ,φ) by (R,β,θ). One reason is that the symbol φ is already in use as the scalar potential. Another reason is that we just used θ as the azimuth in our cylindrical examples, so might as well use it as azimuth here. Since he uses r as ρ of cylindricals, I guess he will use R as r for sphericals. In any event, here are the assumed potentials : (2 of 4 ≠ 0)

$$\psi = \psi(r)\hat{\mathbf{r}} \qquad \varphi = \varphi(r)$$

so we get

$$2\mu\mathbf{u} = -4\,(1-\nu)\,\psi(r)\hat{\mathbf{r}} + \nabla(\mathbf{x}\bullet[\,\psi(r)\hat{\mathbf{r}}] + \varphi(r))$$

$$2\mu\mathbf{u} = -4\,(1-\nu)\,\psi(r)\hat{\mathbf{r}} + \nabla(r\psi(r)) + \nabla\varphi(r)$$

$2\mu\mathbf{u} = -4(1-\nu)\psi(r)\hat{\mathbf{r}} + \partial_r(r\psi(r))\,\hat{\mathbf{r}} + \partial_r\varphi(r)\,\hat{\mathbf{r}}$

$\Rightarrow \qquad 2\mu u_r = -4(1-\nu)\psi + \partial_r(r\psi(r)) + \partial_r\varphi(r) \qquad$ others = 0

$2\mu u_r = -4(1-\nu)\psi + \psi + r\partial_r\psi + \partial_r\varphi(r)$

$2\mu u_r = (-3+4\nu)\psi + r\partial_r\psi + \partial_r\varphi \qquad$ // agrees 5.38.35

Now I have to use the spherical E_{ij} equations and I don't want to mess with that right now. As usual, that would give E_{ij} and then e and then we can get T_{ij}. This Example is assigned as **Problem 5.80**, meaning that unlike most examples, Lai does not do the computation for this one, he just gives the results.

Example 5.38.6 (spherical and cylindricals, azisym) -- Potentials are assumed to be

$\psi = 0 \qquad \varphi = \varphi(r,z) \quad = \hat{\varphi}(R,\beta)$

OK, so finally he is doing the very first Example in the other two systems. He uses $\hat{\varphi}$ to distinguish the functional form for sphericals, fine. So in this case right off the bat we get

$2\mu\mathbf{u} = \nabla\varphi$

and we can then use the cyl and sphere gradient forms to write this out in the two cases. $u_\theta = 0$ in both sphere and cyl systems. Then we really need to examine all the E_{ij} equations in both systems and do all the stuff, then get e, then get T_{ij} from the two covariant Hooke's. This is **Problem 5.81**. As in the Cartesian first example, we get e = 0 in both systems here. Since e = 0 we will have $T_{ij} = 2\mu E_{ij}$ from the two covariant Hooke's laws. I guess the fact that e = 0 cannot depend on what you use for a coordinate system. That is to say, if e(x,y,z) = 0, then e(r,θ,z) = 0 as well, etc.

Example 5.38.7 (spherical, azisym) -- this is the spherical version of Example 5.38.3 where we assume there is only $\psi_z \equiv \psi(R,\beta)$. Notice that we have ψ_z here even though z is not a coordinate! This first step is same as the previous example,

$2\mu\mathbf{u} = -4(1-\nu)\psi\hat{\mathbf{z}} + \nabla(z\psi)$

but now we have to use the spherical gradient and use these facts

$z = R\cos\beta$

$\hat{\mathbf{z}} = \cos\beta\,\hat{\mathbf{R}} - \sin\beta\,\hat{\boldsymbol{\beta}}$

so then

$2\mu\mathbf{u} = -4(1-\nu)\psi[\cos\beta\,\hat{\mathbf{R}} - \sin\beta\,\hat{\boldsymbol{\beta}}] + \nabla(R\cos\beta\,\psi(R,\beta))$

Now

$$\nabla f(R,\beta) = \partial_R f\, \hat{\mathbf{R}} + (1/R)\partial_\beta f\ \ \hat{\boldsymbol{\beta}}$$

so we get

$$2\mu\mathbf{u} = -4(1-\nu)\,\psi[\cos\beta\,\hat{\mathbf{R}} - \sin\beta\,\hat{\boldsymbol{\beta}}] + \partial_R(R\cos\beta\,\psi)\,\hat{\mathbf{R}} + (1/R)\,\partial_\beta(R\cos\beta\,\psi)\,\hat{\boldsymbol{\beta}}$$

$$2\mu\mathbf{u} = -4(1-\nu)\,\psi[\cos\beta\,\hat{\mathbf{R}} - \sin\beta\,\hat{\boldsymbol{\beta}}] + \cos\beta\,\partial_R(R\psi)\,\hat{\mathbf{R}} + \partial_\beta(\cos\beta\,\psi)\,\hat{\boldsymbol{\beta}}$$

$$2\mu\mathbf{u} = -4(1-\nu)\,\psi[\cos\beta\,\hat{\mathbf{R}} - \sin\beta\,\hat{\boldsymbol{\beta}}] + \cos\beta\,[\psi + R\partial_R\psi]\,\hat{\mathbf{R}} + [-\sin\beta\,\psi + \cos\beta\,\partial_\beta\psi]\,\hat{\boldsymbol{\beta}}$$

$$2\mu\mathbf{u} = [(-3+4\nu)\,\psi + R\partial_R\psi]\cos\beta\,\hat{\mathbf{R}} + [(3-4\nu)\sin\beta\,\psi + \cos\beta\,\partial_\beta\psi]\,\hat{\boldsymbol{\beta}} \qquad \text{// agrees 5.38.47}$$

Then we have to use the spherical E_{ij} equations to get E and then e, and finally the covariant Hooke's Law to get T_{ij}. This one is assigned as **Problem 5.82**.

Example 5.38.8A (cylindrical, uniform T_{zz}= S field -- what are the potentials?) The claim is that the correct potentials are these

$$\boldsymbol{\psi} = (Bz)\hat{\mathbf{z}} \quad = \qquad \varphi = A(z^2 - r^2/2) \qquad A,B \text{ as in } 5.38.57.$$

This is a superposition of Examples 3 (only $\psi_z \equiv \psi(r,z)$) and 6 (only $\varphi = \varphi(r,z)$) above. Those examples state the T_{ij} in cylindricals, so we just compute them and add them.

Example 5.38.8B (spherical, uniform T_{zz}= S field -- what are the potentials?)

We get the potentials by trivial direct conversion of the cylindrical ones as shown. But how would you get the spherical coordinate stresses? Lai does it superposing two earlier examples (6 and 7), but what is the systematical relation of the two sets of T_{ij} ? The two coordinate triads must be related by a rotation, so we should be able to use our generic form with the matrices. I think I know these facts

$$(\hat{\boldsymbol{\beta}}\ \ \hat{\boldsymbol{\theta}}\ \ \hat{\mathbf{R}}) = R_z(\theta)\,R_y(\beta) \qquad \text{// spherical}$$

$$(\hat{\mathbf{r}}\ \ \hat{\boldsymbol{\theta}}\ \ \hat{\mathbf{z}}) = R_z(\theta) \qquad \text{// cylindrical}$$

Then it would seem that

$$(\hat{\mathbf{r}}\ \ \hat{\boldsymbol{\theta}}\ \ \hat{\mathbf{z}})\,R_y(\beta) = (\hat{\boldsymbol{\beta}}\ \ \hat{\boldsymbol{\theta}}\ \ \hat{\mathbf{R}})$$

and then maybe we could say

$$T' = M\,T\,M^T \qquad M = R_y(\beta)$$

I will save this for some other time, but I think the method can be made to work.

Example 5.39.9 (Cartesian/spherical, a very specific example)

$$\varphi_1 = [\, 2z^2-(x^2+y^2)] \;=\; [\, 2z^2-(R^2-z^2)] \;= [3z^2-R^2] \;= [3R^2\cos^2\beta - R^2] = R^2[3\cos^2\beta - 1]$$
$$\varphi_2 = R^{-5}\,\varphi_1 \;= R^{-3}[3\cos^2\beta - 1]$$

Why are these each Laplace solutions? Convert to usual coordinates

$$\varphi_1 = r^2[3\cos^2\theta - 1]$$
$$\varphi_1 = r^{-3}[3\cos^2\theta - 1]$$

The atomic forms in sphericals are given by

$$\begin{array}{lcccl} & \text{expo} & \text{osc} & \text{osc} & \\ (1) & [\, r^n,\, r^{-n-1}] & [\, P_n^m(z),\, Q_n^m(z)] & [\, \sin(m\varphi), \cos(m\varphi)] & z = \cos\theta \end{array}$$

If we take m=0 and n=2 we get (ignoring the Q's

$$(1) \qquad [\, r^2,\, r^{-3}]\; [\, P_2(z),\, Q_2(z)] \qquad\qquad z = \cos\theta$$

and it happens that

$$P_2(z) = (1/2)\,(3z^2 - 1)$$

so his two potentials are two known atomic form Laplace solutions in sphericals.

Now recall from above that

$$\varphi_2 = R^{-5}\,\varphi_1 \;= R^{-3}[3\cos^2\beta - 1] \;= \hat{\varphi}_2(R,\beta)$$

which fits into Example 6, so he can just read off the stresses from that Example, computing the derivatives as needed. Fine.

Example 5.39.10 (cylindricals) A single potential $\varphi(r,\varphi,z)$ has this form

$$\varphi(r,\varphi,z) = z\,\ln(\sqrt{r^2+z^2} + z) - \sqrt{r^2+z^2}$$

In sphericals this says

$$\hat{\varphi}(R,\beta,\theta) = R\cos\beta\,\ln(R + R\cos\beta) - R$$

$$= R\cos\beta\,\{\, \ln R + \ln(1 + \cos\beta)\,\} - R$$

$$= z\,\{\, \ln r + \ln(1+z)\,\} - r \qquad\qquad \text{// in usual sphericals with } z = \cos\theta$$

Can I construct this from spherical atoms?

expo osc osc

(1) $[\, r^{n}, r^{-n-1}] \, [\, P_n^m(z), Q_n^m(z)] \, [\, \sin(m\varphi),\cos(m\varphi)]$ $z = \cos\theta$

The only way to get a ln is from a Q_n, but all the Q_n have logs of the form ln[(1+z)/(1-z)] so how could you ever obtain z ln(1+z) by itself? I see no way. What about from the alternative atoms

osc expo osc

(2) $(1/\sqrt{r}\,)[\, r^{i\tau}, r^{-i\tau}] \, [\, P_{i\tau-1/2}^m(z), Q_{i\tau-1/2}^m(z)] \, [\, \sin(m\varphi),\cos(m\varphi)]$ $n = i\tau - 1/2$

Try $\tau = 0$ to get

(2) $(1/\sqrt{r}\,)[1, 1] \, [\, P_{-1/2}(z), Q_{-1/2}(z)]$

but these things are K functions so that seems unlikely to fly. I certainly see nothing in the cylindrical atoms. But I see Lai's Laplace proof on page 289! Look again at cylindricals

expo osc osc

(1) $[\, e^{+kz}, e^{-kz}\,] \, [\, J_m(k\rho), N_m(k\rho)] \, [\, \sin(m\varphi),\cos(m\varphi)]$

We certainly can only have m = 0 so this gives

(1) $[\, e^{+kz}, e^{-kz}\,] \, [\, J_0(k\rho), N_0(k\rho)]$

So then the question is this: can you somehow write

$$z \ln\left(\sqrt{r^2+z^2} + z\right) - \sqrt{r^2+z^2} \;=\; \int_0^\infty dk\, [A_k e^{+kz} + B_k e^{-kz}]\, [\, C_k J_0(kr) + D_k N_0(kr)]\, dk$$

The answer must be that yes, this must be possible. So it is a weird combination of atoms that makes the functions of interest. Consider

6.611

1. $$\int_0^\infty e^{-\alpha x} J_\nu(\beta x)\, dx = \frac{\beta^{-\nu}\left[\sqrt{\alpha^2+\beta^2} - \alpha\right]^\nu}{\sqrt{\alpha^2+\beta^2}} \qquad [\mathrm{Re}\,\nu > -1, \quad \mathrm{Re}\,(\alpha \pm i\beta) > 0]$$

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which says

$$\int_0^\infty dk\, e^{-zk} J_0(rk) = (z^2+r^2)^{-1/2} = 1/\sqrt{r^2+z^2} \qquad \text{// which is 1/R, known solution}$$

Another one is this

$$8. \qquad \int_0^\infty e^{-\alpha x}\, Y_0(\beta x)\, dx = \frac{-2}{\pi\sqrt{\alpha^2+\beta^2}} \ln \frac{\alpha+\sqrt{\alpha^2+\beta^2}}{\beta}$$

which says

$$\int_0^\infty dk\, e^{-zk} Y_0(rk) \sim 1/\sqrt{r^2+z^2} \ln \{[z+\sqrt{r^2+z^2}]/r\}$$

and this is what we are looking for

$$z \ln(\sqrt{r^2+z^2} + z) - \sqrt{r^2+z^2}$$

so we are in the right ball park. I am asking a general question here which is: Given a known Laplace solution, how to you write it as a sum of atoms? I can probably answer that even with the second kind functions, but let's let it ride.

I accept Lai's algebra on page 289 so it must be that

$$\varphi(r,\varphi,z) = z \ln(\sqrt{r^2+z^2} + z) - \sqrt{r^2+z^2}$$

is in fact a Laplace solution.

Conclusion: I accept all the conclusions of this Example. I don't think I have seen this particular potential before.

More on cylindrical atoms. Consider this atomic form

$$(2) \qquad [\sin(\kappa z), \cos(\kappa z)] \; [I_m(\kappa\rho), K_m(\kappa\rho)] \; [\sin(m\varphi),\cos(m\varphi)]$$

and our interest in this case would be with m = 0 so

$$(2) \qquad [\sin(\kappa z), \cos(\kappa z)] \; [I_0(\kappa\rho), K_0(\kappa\rho)]$$

and then consider this integral

$$13. \qquad \int_0^\infty K_0(\beta x) \sin \alpha x\, dx = \frac{1}{\sqrt{\alpha^2+\beta^2}} \ln\left(\frac{\alpha}{\beta}+\sqrt{\frac{\alpha^2}{\beta^2}+1}\right)$$

But this seems just to replicate the result found above, $\alpha = z$ and $\beta = r$, but it does show another way to think of these atomic integrals.

In general, suppose we have

$$f(r,z) = \int_0^\infty dk\, k\, J_0(kr)[A(k)\, e^{+kz}]$$

How might we find A(k) ? Use the Hankel transform with $\nu = 0$?

$f(r; z) = \int_0^\infty dk\, k\, J_0(kr)\, [A(k)\, e^{+kz}]$ // expansion

$[A(k)\, e^{+kz}] = \int_0^\infty dr\, r\, J_0(kr)\, f(r; z)$ // projection

But this only works if the integral $\int_0^\infty dr\, r\, J_0(kr)\, f(r; z)$ happens to come out in the form $A(k)\, e^{+kz}$.

Consider this example taken from section 5.40 below

$f(r; z) = \ln(\sqrt{r^2+z^2} + z)$

We could at least try it and look for this integral

$\int_0^\infty dr\, r\, J_0(kr) \ln(\sqrt{r^2+z^2} + z)$

How about this with x = r, a = z, b = k

6.776 $$\int_0^\infty x \ln\left(1 + \frac{a^2}{x^2}\right) J_0(bx)\, dx = \frac{2}{b}\left[\frac{1}{b} - a\, K_1(ab)\right]$$

$\int_0^\infty dr\, r\, J_0(kr) \ln(1 + z^2/r^2) = (2/k)\,(1/k - z\, K_1(zk))$

Well, not the right form, and we don't get the form $[A(k)\, e^{+kz}]$, so no progress on this question. Remember that the general form could be something like this

$\int_0^\infty dk\, [A_k\, e^{+kz} + B_k e^{-kz}]\, [\, C_k\, J_0(kr) + D_k N_0(kr)]\, dk = f(r,z)$

and we don't really know how to invert such a thing! Better methods would be needed.

5.39 Delta Force at center of infinite 3D medium (290). [Kelvin Problem]

I am a little confused about this boundary condition. Lai wants to have $\mathbf{F} = F_z\hat{\mathbf{z}}$ but only "at the origin", so what does this mean? It must mean $\mathbf{F} = \delta^{(3)}(\mathbf{r})\, F_z\hat{\mathbf{z}}$ so that $\int dV\, \mathbf{F} = F_z\hat{\mathbf{z}}$ if you integrate over any volume which includes the origin, no matter how small. In practice, the force is spread out over some little sphere which includes the origin and then F_z is the total force on that sphere. In this case, it would seem that as you make that sphere smaller, the stress approaches infinity since it acts on a smaller area. I think we would have to avoid this limit in order to stay in the "infinitesimal" elastic solids model.

So here is what Lai does in this little section. First, assume a certain potential which is quite simple

$\boldsymbol{\psi} = A(1/R)\hat{\mathbf{z}}$ $\varphi = 0$

We know from the **Example 3** how to quickly write down the stress tensor elements this implies. The T_{ij} matrix (this is cylindricals by the way) is shown in 5.39.9. We take an assumed sphere (a math surface) and compute the total force on this sphere caused by the stress $\mathbf{t} = T\mathbf{n}$ (implied by our assumed potential), ie, we integrate this over the entire surface of this sphere. This integral gives a certain total force acting on the sphere. Since the sphere is stationary in a static problem, there must be some other force acting somehow on the sphere which keeps it from accelerating. Since the total force we get by integration does not depend on R_0, which in itself seems rather amazing, that "other force" must be located at the center of the sphere. Thus, we are able to deduce the value of F_z in my $\mathbf{F} = \delta^{(3)}(\mathbf{r})\, F_z \hat{\mathbf{z}}$ formula above. By inverting the logic, we conclude that applying such a localized force on an infinite isotropic medium, we can deduce the value of the constant A and thus we really have everything given in terms of F_z. Since we know the potential, we trivially know the displacement $\mathbf{u}$. All these results are stated at the bottom of page 292.

Now, things really are singular and Lai quietly spaces over this fact. For example, the u_z displacement (5.39.19) goes to ∞ at the origin, and is extremely large "near" the origin, so we have a conflict with our infinitesimal theory in some region of the origin.

This problem is for some reason associated with **Kelvin**, no reference is given.

Apart from the singularity problem, it is pretty hard to imagine how you would apply this delta z force at the origin. Perhaps you cut a tube-like hole in your solid and reach into it with a probe and try to apply that force with the probe. For a 2D medium we can at least visualize how you might apply a delta force at the center of an infinite medium, coming in from the 3rd dimension with your probe. I note in passing that Lai has never said anything about 2D elastic solids.

In some sense, you might think of this problem as analogous to the electric field of a point charge located at the origin. This is described by $\rho = \delta^{(3)}(\mathbf{r})\, q$. Or perhaps to get the vector sense, as the electric field of a dipole $\mathbf{p} = \delta^{(3)}(\mathbf{r})\, \mathbf{P}$. We never have trouble thinking about such a dipole despite the delta. It means that if you do a local volume integral, you get dipole strength $\mathbf{P}$.

5.40 Delta Force at center of *half*-infinite 3D medium (293). [Boussinesq Problem]

In this problem at least we can imagine more easily how the force would be applied -- from just above the "center point" (which could be any point on the z=0 plane of course).

First, Lai proposes a potential structure which matches the one assumed in **Example 4,** but now he picks a particular auxiliary function $\varphi = C \ln(R+z)$. A condition of that Example is that $\nabla^2\varphi = 0$ and Lai shows that this is the case, although (as in the last section) I was unable to write this as an integral of cylindrical atoms. I don't know how to find an integral representation for a given function of two variables! We can then use the expressions of that Example 4 to compute everything, and this is relegated to **Problem 5.86** , this is just plug-in and differentiate work.

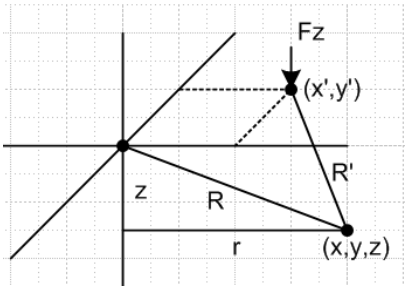
Now we take our half-space and consider a slice right at the top as shown page 294. From the potential we know the normal force on the top z=0 plane to be the integral of T_{zz} there, but T_{zz} is singular at the origin, so we do our integral instead at z = h (we lower our slice = disk). We find that there is a total normal force on the plane z = h in the amount $F_z = -2C\pi$ where C is the constant in our potential. We next imagine very small h so this plane is just below the surface. Since the surface is not accelerating, something must be pushing down with an equal and opposite force. But we see that T_{zz} at the surface is 0 everywhere except at the origin (since $T_{zz} = C3z^3/R^5$). Therefore this balancing force must be of the form $\mathbf{F} = \delta^{(3)}(\mathbf{r})\, F_z \hat{\mathbf{z}}$ as in our last problem. We then can relate F_z to C. Then we can interpret the potential

and all derived quantities such as displacement and stress tensor elements as resulting from a delta force applied to the top center point of the half space. The T_{ij} and u_i are shown bottom of page 294. As expected, if you go onto the positive (down) z axis where R = z and r=0, you find that $u_z \sim 1/z$ and diverges at the surface, so our infinitesimal assumption is violated at the origin, no comment from Lai.

In Example 5.40.1 we convert the T_{ij} and the u_i from cylindrical to Cartesian coordinates. The first is trivial as shown, while the second uses the famous matrix trick and you have to find the matrix M which is just a z rotation by θ, but you have to get the sign of θ right. This is **Problem 5.87**.

5.41 Distributed normal stress on *half*-infinite 3D medium (296).

Consider a modification of the above section where the F_z delta force is placed at some other location (x',y') on the surface relative to the origin (previously F_z was located at the origin):



We know that

$R^2 = x^2 + y^2 + z^2$ and $R'^2 = (x-x')^2 + (y-y')^2 + z^2$

Clearly it is distance R' that is relevant for all our expressions, so we have

$u_z(x,y,z) = F_z/(4\pi\mu R') * (z^2/R'^2 + 2(1-\nu))$

Now put differential force $F_z = q(x',y')dx'dy'$ at the point shown to get

$du_z(x,y,z) = q(x',y')dx'dy'/(4\pi\mu R') * (2(1-\nu) + z^2/R'^2)$ // agrees with 5.41.1

Then if we have some continuous force distribution on the surface (q is really a normal stress). We can then integrate the above to find the $u_z(x,y,z)$ from some real-world applied normal stress distribution.

Example 5.41.1 considers the case where the stress distribution is a disk of uniform stress (radius r_o and normal stress q_0) which I would call an indenter. Lai takes the special observation point right on the central axis, and in this case the two integrals are easily done with result u(0,0,z) = 5.41.4. As we approach the surface from inside the half-space, this approaches a finite value = 5.41.5. Now as long as $q_0 r_0$ is adequately small , we will have $u_z << 1$ in some sense, and now we are OK again with our infinitesimal approximation.

Comment 1: I think if we were able to compute u_z etc. for an arbitrary observation point (the integrals might be difficult), this would NOT be the indenter problem treated in a later section. The reason is that if

we apply a uniform stress on the disk, probably we are not going to get a uniform displacement on the disk, so the disk is not an indenter. Conversely, probably the actual indenter does not apply a uniform stress on the disk of surface it touches.

Comment 2: I can imagine some fancier Jackson version of this book doing all these problems with correct delta functions and perhaps some kind of Green's Functions and all that good stuff. But this is an introductory book and there is no need for all that extra baggage at this point.

5.42 Thick spherical shell with inner and outer pressures (297).

This is an application of the results of **Example 5** (p 284) with $\psi = BR$ and $\varphi = A/R$, so we use the results of that exercise to state the T_{ij} and the u_i in spherical coordinates. Now just for fun we try to impose the BC's that $T_{RR} = p_i$ on the inside of a hollow sphere and $T_{RR} = p_o$ on the outside. It turns out that we CAN do this, and when we do this, we obtain values for A and B as shown in 5.42.8. The implication is that therefore this potential describes the isotropic material in this thick spherical shell : it is a solution of the static Navier equation, and it meets our two BC's. The other two stress components are equal in sphericals because they describe how this shell wants to pull apart in the two surface tangent directions. In the solution, there is only $u_R(R)$ displacement as symmetry would suggest. A pressurized tennis ball comes to mind as an application. I would expect in this case more displacement toward the inner surface. I like this problem.

5.43 Spherical Hole in 3D axial stressed material (298).

This is the 3D analog of a problem we solved earlier. Spherical coordinates are used in this solution since these match the BC surface (ie, the hole boundary). A field of constant axial stress $T_{zz} = S$ was found in **Example 8** to correspond to the potential structure shown top p 299 and from that Example 8 we know the spherical stress T_{ij} components as in 5.43.2 with A and B as shown.

Now the game is add to this potential some "adder" potential (which Lai calls a disturbed field) such that the total surface traction will be 0 on the entire surface of the hole -- that is of course the BC (and of course the adder potential must not affect distant stresses). As usual, Lai throws out a trial potential which has the same general structure as the axial field one, 5.43.4. We compute the stress T_{ij} from this and then add it to the axial one (in sphericals though) and the total is then as in 5.43.8 and 9. The surface traction on the hole boundary is $\mathbf{t} = T\hat{\mathbf{R}}$ where R is component 1, and in the usual manner, this means that we must have $T_{i1} = 0$ for all three i values, that is, $T_{iR} = 0$ at R=a. Now there were three constants C_i in our trial adder field form, and these appear in the total stress T_{ij} where I wrote "total". It turns out that you CAN match the three BC's shown and this sets the three constants, and the problem has a simple solution.

So far we have no expression for $T_{\beta\beta}$ but you could compute it in that Example 8. The claim is that the max tensile stress will be at the top and bottom of the hole (in picture), which really means all along the equator of the hole, and $T_{\beta\beta}$ is trying to rip the material apart. My pencil calculation shows that the "multiplier" of S ranges from 1.9 to 2.2 roughly, and is not as large as the 3.0 encountered in the 2D version of this problem. Still, we have the amplification effect. The 3D-ness softens the effect.

The "adder" potentials have (azisym) spherical harmonics n = 0,1, and 2.

5.44 The cylindrical indenter problem (300).

First, Lai states the required BC's:

(1) $u_z = w_0$ for $r \le a$. Note that the z axis points down and $w_0 \ge 0$. This is the indented region.
(2) $T_{iz} = 0$ at the surface $z = 0$ for $r > a$ $(i = 1,2,3 = r,\theta,z)$ (no surface traction outside indenter)

Even though the curve u_z might have the shape shown in Lai's picture, the surface $z = 0$ is still a perfect full flat plane, and outside the indenter, r>a, those $T_{iz} = 0$ because nothing is touching the surface of the half-space. Of course in our infinitesimal model, we should have $u_z << a$ anyway.

Now, we are going to use a combination of two past Examples. The first is **Example 4**, p 282 where we study a potential structure like that shown in 5.44.3 (he has changed from script φ to F). But now F is going to be the imaginary part of a certain new φ in 5.44.4, and we then get to use certain math results that were found in **Example 10**, p 288. From Example 4 we crib u_z and various T_{ij}, and then from Example 10 we crib various computed derivatives. These derivatives are evaluated at $z = 0$ (meaning on the planar surface of the half-space). We see certain "discontinuous functions" which remind me of the Weber's discontinuous J_0 integrals which appeared in Sneddon.

So right off the bat now we have $u_z(r; z=0)$ as in 5.44.8: a constant inside the indenter area, and an arc sin outside. At this point, we have a constant A, but it is related to w_0 as in 5.44.11 so we then know all about $u_z(r; z=0)$ in terms of w_0, a, ν, μ.

Next he looks at T_{zz} as in 5.44.12 for $r \le a$. This blows up $1/\sqrt{}$ at the rim since the indenter has a sharp edge, similar to the charge density on a charged metal disk, and has some finite value at indenter center. Earlier I noted that probably $T_{zz} \neq$ constant under the head of an indenter like this one and we now see that to be correct.

If you then integrate this T_{zz} over the footprint you get P, so now we can relate w_0 to P as in 5.44.14.

You might try to use an indenter with a couple of tests to measure μ and ν for a material. Most tests on the web go into the plastic region and are not elastic region tests. Perhaps in this way you can only measure $(1-\nu)/\mu$ as in 5.44.14.

5.45 The general and spherical indenter problems (302)

Lai considers some arbitrary azisym shape of an indenter and assumes the touch goes out to r=a. It could be a cylinder, a sphere, an ellipsoid, or some very strange azisym shape. Perhaps it can even be a non-convex shape. I have a feeling this arbitrary problem is treated just as an integration of a tiny delta indenter of the cylindrical variety, but Lai does not present it that way.

The assumed potential as in integral of the cylindrical indenter potential support function φ against some unknown scalar function f(t), where recall $z^* = z + it$. Somewhere I have read about complex potentials like this, perhaps in Goldstein, maybe Sneddon, maybe Ahlfors.

We turn the same math crank as before and u_z comes out as 5.45.12, integrals over f(t). We can set this to $w_0 + w(r)$ as in 5.45.13. Lai claims this integral equation can be solved for f(t) as a function of w(r) and the result is given in 5.45.17. We are referred to an Appendix for this calculation. So fine, you insert whatever indenter shape you want to use, and out pops f(t), and then you know everything.

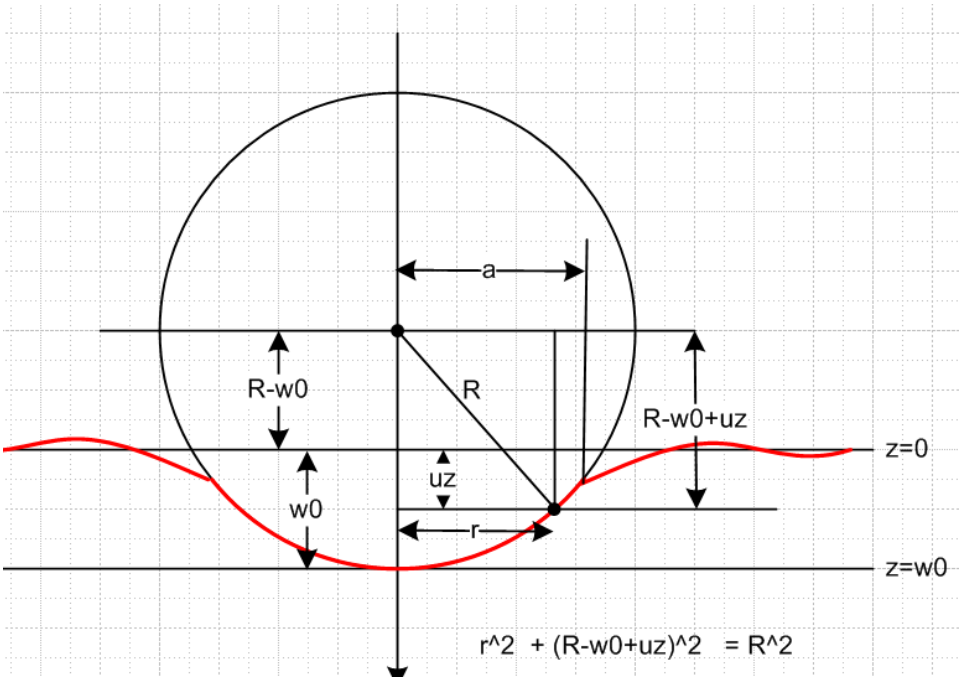
Meanwhile, bottom of page 303 we compute T_{zz} for $r \le a$ and come up with a way to get P in terms of an integral of f(t).

In Example 5.45.1 we apply this method to the same cylindrical indenter of Section 5.44. We get a certain value for constant B (which shows up in the solution to the integral equation), and everything comes out as in the last section, an excellent test of our method.

Then starting bottom page 304 he considers the spherical indenter case. Since we are in the infinitesimal case, we know the indentation will be very small, so he assumes a $\leq\leq$ R of the sphere and that makes the math easier. The w(r) curve is now a quadratic fit, and out pops everything. The f(t) is shown in 5.45.30 and T_{zz} as in 31. The smooth edge requires B = 0 and I agree, since T_{zz} does not blow up now at the smooth ball touch edge, so f(t) is 33. From this we can compute whatever we want. He first computes P as in 35, then a is related to P as in 36. He compute u_z and I plotted it in Maple, nothing interesting, it does not overshoot as his drawing suggests (see sphere indenter.mws).

It turns out that w_0 is very simply related to a (since quadratic). The final T_{zz} vanishes at the touch edge (unlike the cylinder case which was infinite there).

Here is the picture Lai should have drawn for his spherical indenter (his drawing makes it seem that R = w_0 which is not in general true):



The triangle shows that

$$r^2 + (R-w_0+u_z)^2 = R^2$$

$$R-w_0+u_z = \sqrt{R^2-r^2}$$

$$u_z = w_0 + \sqrt{R^2-r^2} - R \qquad \text{// and this is 5.45.24 page 304}$$

Appendix 5A.1 Solving the integral equation used in Section 5.45 on general indenters.

I skip this for now. It might be Sneddon like, it might not. I know I can understand it if I need to.

Part A Problems (pp 309-319, problems 5.1 through 5.90)

He has been saving them up, we have 90 problems here, grouped by subsection:

Part A, opening section
Part A.1
Part A.2
Part A.3
Part A.4

PART B: The *Anisotropic* Linear Elastic Solid (319)

5.46 Hooke's Law for general anisotropic case (aka constitutive equations) (319)

All the above concerned isotropic solids. We are now back with those 81 elasticity coefficients that we mentioned long ago. This is a short section, pp 319-333, some 14 pages.

So I am now going to just start over counting how many C_{abcd} are unique.

Counting. This time I will start with this assumed fact (spring energy is quadratic, 5.2.20 p 206)

$$U = (1/2)\, C_{abcd}E_{ab}E_{cd} \quad \Rightarrow$$

$$C_{abcd} = (\partial^2/\partial E_{ab}\partial E_{cd})(U)$$

From this single fact (last line above) we learn that

(a)	$C_{abcd} = C_{bacd}$	swap elements of left pair
(b)	$C_{abcd} = C_{abdc}$	swap elements of right pair
(c)	$C_{abcd} = C_{dcab}$	swap the left and right pairs

One idea is to imagine that, just for purposes of symmetry (not value),

$$C_{abcd} = X_{ab}Y_{cd} \qquad 81 = 9 * 9 \text{ elements}$$

Condition (a) says that X is symmetric.
Condition (b) says that Y is symmetric.
Condition (c) says that X = Y.

Ignoring (c) for the moment, we could write this, knowing that X and Y are symmetric:

$$C_{abcd} = X_{ab}Y_{cd} \qquad 6 * 6 = 36 \text{ unique elements}$$

We could list these off by making the requirement $a \leq b$ and $c \leq d$.

Now consider that, if X = Y, then the following two elements are the same: (condition (c))

$C_{1123} = X_{11}X_{23}$
$C_{2311} = X_{23}X_{11}$

The question then is: how many ab, cd are there such that ac $\leftrightarrow$ cd gives a new condition, and we maintain the requirement $a \leq b$ and $c \leq d$. Obviously if ab = cd, we do not have a condition. So off hand, you might say there were 6*5 = 30 conditions. But 12,34 is the same as 34,12. So if we were to list off our 30 cases of ab,cd where $ab \neq cd$, then we count every one twice, because ab,cd $\approx$ cd,ab. So there are only 15 unique conditions due to Condition (c) and then we have 36-15 = 21 which is the right answer.

I will now write them all out just for fun. For each pair (left and right), I will restrict to $a \leq b$ so there will only be 6 values and same for $c \leq d$. I will draw all the 36, then I will mark with * the ones which are duplicates due to condition c.

1111 = 11 11
1112 = 11 12
1113 = 11 13
1122 = 11 22
1123 = 11 23
1133 = 11 33

1211 = 12 11*
1212 = 12 12
1213 = 12 13
1222 = 12 22
1223 = 12 23
1233 = 12 33

1311 = 13 11*
1312 = 13 12*
1313 = 13 13
1322 = 13 22
1323 = 13 23
1333 = 13 33

2211 = 22 11*
2212 = 22 12*
2213 = 22 13*
2222 = 22 22
2223 = 22 23
2233 = 22 33

2311 = 23 11*
2312 = 23 12*

2313 = 23 13*
2322 = 23 22*
2323 = 23 23
2333 = 23 33

3311 = 33 11*
3312 = 33 12*
3313 = 33 13*
3322 = 33 22*
3323 = 33 23*
3333 = 33 33

I count 15 asterisks, so this says there are only 36-15 = 21 unique C coefficients.

Derive (5.46.1) p 319.

This thing is set up to show the 6x6 = 36 elements as described above. This 6x6 matrix is symmetric exactly due to Condition (c) above -- we can swap the left and right pairs! Due to this symmetry, there are 6 diagonal elements and 30 off-diagonal elements. Thus, there are 15+6 = 21 unique C coefficients and they all appear in this matrix. This of course agrees with the 21 shown above.

So then the only question left is why there are those factors of 2 in the right side vector. Go back to

$$T_{ab} = C_{abcd}E_{cd} = \Sigma_{c=1}^{3}\Sigma_{d=1}^{3}C_{abcd}E_{cd} = \Sigma_{c=d}C_{abcd}E_{cd} + \Sigma_{c\neq d}C_{abcd}E_{cd}$$

$$= \Sigma_{c=d}C_{abcd}E_{cd} + 2\Sigma_{c<d}C_{abcd}E_{cd}$$

This 2 arises because $\Sigma_{c\neq d} = 2\Sigma_{c<d}$ because $\Sigma_{c\neq d}$ denotes 6 elements and $\Sigma_{c<d}$ only 3 elements, and of those 6 elements, they are equal in pairs since $E_{cd} = E_{dc}$. So

$$T_{ab} = \Sigma_{c=d}C_{abcd}E_{cd} + \Sigma_{c<d}C_{abcd}(2E_{cd})$$

where we are now doing n=6 matrix math. Thus you get a 2 in front of the E's which have c≤d.

Given this, then (5.46.2) is just a rewrite where one could express the C_{ij} in terms of the C_{ijkl} if one wanted. For example, $C_{11} = C_{1111}$ whereas $C_{14} = 2C_{1123}$. But I think we should really ignore 5.46.1 now and take 5.46.2 as our starting point. But we need 5.46.1 to understand why C_{ij} is symmetric!

C_{ij} is called the **stiffness matrix**, while C_{abcd} is the **elasticity tensor** of page 204.

page 320: Now consider in general this equation where A = vector, M = symmetric matrix, U = scalar.

$$U = A^{T}MA$$

This is not quite the same situation as I treat in tensor doc section 5 c and d. There I had $M = A^T D A$ where D was a diagonal matrix and A and M are also matrices, but things seem similar. Since M is real symmetric, we know we can diagonalize it like so

$$D = S^{-1}MS = S^T MS \quad \Rightarrow \quad M = SDS^{-1} = SDS^T$$

where S is some real orthogonal matrix $S^{-1} = S^T$. Therefore we can write

$$U = A^T MA = A^T SDS^T A$$

But $A^T S = (S^T A)^T$ so we have with $B = S^T A$,

$$U = (S^T A)^T D(S^T A) = B^T DB = \Sigma_i D_{ii} B_i^2.$$

Now in the above, A was an arbitrary real vector, so B (being a rotation of A) is also arbitrary. Thus, if we want to have $U > 0$ for all B_i , we need to have all $D_{ii} > 0$. Now consider

$$M = SDS^T \quad \Rightarrow M_{ii} = \Sigma_j S_{ij} D_{jj} S^T_{ji} = \Sigma_j D_{jj} S_{ij}^2.$$

From this last equation we see that M_{ii} is a sum of positive terms and therefore $M_{ii} > 0$ as the text claims. Meanwhile,

$$\det(M) = \det(SDS^T) = \det(D) = \Pi_i D_{ii} > 0$$

so $\det(M) > 0$ as well, the second claimed fact in the text. A matrix of this type is called a positive definite matrix. It is just a similarity transform of a diagonal matrix having positive diagonal elements (the eigenvalues).

Finally, since $\det(M) \neq 0$, we know that M^{-1} exists and we have

$$M^{-1} = [SDS^{-1}]^{-1} = SD^{-1}S^{-1} = SD^{-1}S^T$$

$$(M^{-1})_{ij} = \Sigma_k S_{ik}(D^{-1})_{kk}(S^T)_{kj} = \Sigma_k (1/D_{kk}) S_{ik}S_{jk}$$

We see at once that M^{-1} is real symmetric. Furthermore

$$(M^{-1})_{ii} = \Sigma_k (1/D_{kk}) S_{ik}^2 > 0$$

$$\det(M^{-1}) = \det(SD^{-1}S^{-1}) = \det(D^{-1}) = \Pi_i (1/D_{ii}) > 0$$

so M^{-1} is therefore also positive definite and symmetric.

So, if C is the **stiffness matrix**, then $S = C^{-1}$ is the **compliance matrix**.

Why are all submatrices of M also positive definite? Well, by "submatrix" Lai means submatrices which are on the diagonal, as you would see in a block-diagonal situation (coming soon). But such a submatrix is obviously also symmetric to start with. If we consider $U = A^{\mathbf{T}}MA$ as above for vectors **A** which have zeros in certain positions corresponding to the non-blocks, call these A_1 then we have $U = A_1^{\mathbf{T}}MA_1$ and in this thing, only the block M elements will appear and these elements make the submatrix M_s so then we have $U = A_1^{\mathbf{T}}M_sA_1$ where implied sums are only over the block matrix. Then we just repeat everything above applied to this smaller world and we reach the same conclusions.

Example 5.46.1. (320) Here Lai shows in his own simple way all the facts I have just shown above.

Summary: In the most general anisotropic Hooke's law, there are 21 unique coefficients C_{ij} which are the unique elements of a symmetric matrix C which is symmetric positive definite and whose inverse exists and is the same. Recall that in isotropic there were 2 unique coefficients!

5.47 Planes of material symmetry (321)

Suppose the solid has a reflection symmetry axis, how does this affect C ? Consider

$T_{ab}(x,y,z) = C_{abcd}E(x,y,z)$ for all points x,y,x

This equation is written in x-space. Now let x'-space be the space in which just y is reflected.

x',y' z' = x,-y, z

We can write $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ as $x'_i = F_{ij}x_j$ where F = diag(1,-1,1) and F is a "rotation" in that it is real-orthogonal with detF = -1. We would write in this case in developmental notation,

$T'_{ab} = F_{aa'}F_{bb'}T_{a'b'}$ or $T' = FTF^{\mathbf{T}}$

$E'_{ab} = F_{aa'}F_{bb'}E_{a'b'}$ or $E' = FEF^{\mathbf{T}}$

$C'_{abcd} = F_{aa'}F_{bb'}F_{cc'}F_{dd'}C_{a'b'c'd'}$

Now, suppose the material is identical in nature if you reflect it in this way, perhaps due to its crystalline structure of what have you. Then I would say there cannot possibly be any difference between C_{abcd} and C'_{abcd} . The two situations $T_{ab}(x,y,z) = C_{abcd}E(x,y,z)$ and $T'_{ab}(x',y',z') = C'_{abcd}E'_{cd}(x',y',z')$ could not be distinguished from each other. Think of it as two experiments or measurements. In this case, I think

$C'_{abcd} = C_{abcd}$ // agrees with Lai 5.47.6 p 321

// I am now going to do a bunch of stuff out of order, before it appears in the text:

A. Monoclinic case.

This in turn says that

$$C_{abcd} = F_{aa'}F_{bb'}F_{cc'}F_{dd'}C_{a'b'c'd'}$$

which is now a condition on C.

Now consider this set of facts

n	$\delta_{n,2}$	$-2\,\delta_{n,2}$	$(1 - 2\,\delta_{n,2})$
1	0	0	1
2	1	-2	-1
3	0	0	1

One can then write F_{ab} in this manner

$$F_{ab} = \delta_{a,b}\,(1 - 2\,\delta_{a,2})$$

and then

$$C_{abcd} = (1 - 2\,\delta_{a,2})\,(1 - 2\,\delta_{b,2})\,(1 - 2\,\delta_{c,2})\,(1 - 2\,\delta_{d,2})\,C_{abcd}\ .$$

This says that $C_{abcd} = -\,C_{abcd}$ so that $C_{abcd} = 0$ whenever C_{abcd} has an odd number of 2 indices! This is because each such "2 index" creates a (-1) in the above equation (other indices create a (+1)).

Theorem: If $\hat{\mathbf{u}}_n$ is a symmetry axis, $C_{abcd} = 0$ if C_{abcd} has an odd number of n indices.

It is fascinating to see what this means for the C_{abcd} matrix as written in 5.46.1. The first only is "block":

n=1

1	1	1	1	0	0
1	1	1	1	0	0
1	1	1	1	0	0
1	1	1	1	0	0
0	0	0	0	1	1
0	0	0	0	1	1

// agrees with 5.48.8 p 324

Notice that there are 4+6 + 2 + 1 = 13 unique constants in this case!

n=2

1	1	1	0	1	0
1	1	1	0	1	0
1	1	1	0	1	0
0	0	0	1	0	1
1	1	1	0	1	0
0	0	0	1	0	1

So you can see that you only get the block diagonal form in the x symmetry case, just from the way one chose to lay out the coefficients. I won't bother with n=3. You could get block form by shuffling columns and rows, but why not just take x if you have this one symmetry. This case is called **monoclinic**.

B. Orthotropic Case.

Now suppose instead *both* the x and y planes show symmetry. This means that both these equations are true:

$$C_{abcd} = (1 - 2\,\delta_{a,1})\,(1 - 2\,\delta_{b,1})\,(1 - 2\,\delta_{c,1})\,(1 - 2\,\delta_{d,1})\,C_{abcd}$$
$$C_{abcd} = (1 - 2\,\delta_{a,2})\,(1 - 2\,\delta_{b,2})\,(1 - 2\,\delta_{c,2})\,(1 - 2\,\delta_{d,2})\,C_{abcd}$$

Then coefficients vanish if they have an odd number of 1's *or* an odd number of 2's. We need some kind of case table:

Odd number of 1's and even number of 2's:	1223	1333	1113				3odd
Odd number of 2's and even number of 1's:	2213	2333	2223				3 odd
Even number of 1's and even number of 2's:	3333	2233	2222,	1122	1133,	1111	3 even
Odd number of 1's and odd number of 2's:	1233	1222	1112				3 even

Notice that the classification on the left forces the #3's even or odd as shown on the right. So,

# 1's	# 2's	#3's	C_{abcd}
odd	even	odd	0
even	odd	odd	0
even	even	even	1
odd	odd	even	0

This table lists all possible cases, the other four never occur. Notice that C = 0 if you have an odd number of 1's, 2's or 3's. So **having sym in any two axes implies it in the third**. Now what does C look like

1	1	1	0	0	0	
1	1	1	0	0	0	
1	1	1	0	0	0	
0	0	0	1	0	0	
0	0	0	0	1	0	
0	0	0	0	0	1	// agrees with 5.49.3 p 325

We get the block form, and there are now 3+3+3 = 9 coefficients. This is the **orthotropic** case.

C. Transverse isotropic case.

Now suppose you have a case of what I might call **cylindrical symmetry**. Yes, you have the 2D anisotropic situation. The axisym axis is taken to be the z axis. Recall from above

$$C'_{abcd} = F_{aa'}F_{bb'}\,F_{cc'}F_{dd'}\,C_{a'b'c'd'}$$

where this time F_{ab} rather than being a reflection in the x,y,or z plane, is a reflection in a plane that is rotated by angle β from the x=0 plan, and we have this symmetry for any β. This probably means the material is isotropic in the x-y direction.

How do you reflect in a plane at angle β? If **x** is a point in space, then let $\mathbf{x}_t$ = (x,y). Then we have

$$[R_z(\beta)\,\mathbf{x}_t] \rightarrow -[R_z(\beta)\,\mathbf{x}_t] \qquad \text{// This is 2D parity.}$$
$$z \rightarrow z \qquad \text{// no change in z}$$

Therefore I would say that $F_{ab} = R_z(\beta)_{ab}$ where this is the 3D matrix. So the condition on C then is

$$C_{abcd}(\beta) = R_z(\beta)_{aa'}\, R_z(\beta)_{bb'}\, R_z(\beta)_{cc'}\, R_z(\beta)_{dd'}\, C_{a'b'c'd'}$$

We know this C matrix will be a subset of the one shown above, since the one above applies for just the two reflections (x and y) which occur for β = 0 and β = π/2. But how on earth do we make sense out of the general case above? I tried fiddling in some separate notes on transverse isotropic but could not answer even such a simple question.

Given that we must have the general form 5.49.3, and given that there exists some solution, then the method used by Lai on pp 325-327 is just fine. He comes up with four simple conditions which reduce the number of independent elements from 9 to 5 and the result is shown page 327. I accept it.

Now lets return to notes on the book.

Example 5.47.1 (Thermal Expansion Tensor) (p321) Before Lai attacks the rank 4 tensor case of C_{abcd} he wants to so the rank 2 case with α_{ab} as a thermal coefficient tensor in the equation $e_{ab} = \alpha_{ab}(\Delta\theta)$. If some 3D material sees an increase in temperatur, it will expand in some tensor manner where e_{ab} is I guess something like E_{ab}, some form of strain (maybe one that was mentioned earlier).

(a) What happens if the material has x symmetry? We do just as I did in the above 3D case and write out that $\alpha_{ab} = F_{aa'}F_{bb'}\alpha_{ab}$ as in (i). We end up with a 4 elements being zero if $\alpha_{ab} = \sigma'_{ab}$ and we get (iv).

(b) What happens if we have x and z reflection symmetry. He shows that adding just the z symmetry kills off two more matrix elements and we end up with α being diagonal as in (viii).

(c) What happens in the 3 transverse isotropic case. He shows that we still have our diagonal matrix for α, but two elements are the same, $\alpha_{11} = \alpha_{22}$ as in (xii). So only 2 parameters.

(d) What happens if you have transverse isotropy re both the 3 and 1 axis? Then also $\alpha_{22} = \alpha_{33}$ and you then get that $\alpha = \alpha_{11}\,1$ where 1 is the identity matrix, only one parameter. Lai notes that this is a common situation. I presume such a material is really isotropic in all three directions, but he does not say this.

Recall that in the general case, C_{abcd} has 21 independent elements.

5.48 Monoclinic symmetry (323)

I did this above and obtained the Lai results. There are 13 independent C_{abcd} elements.
Symmetric under reflection in one axis.

5.49 Orthotropic symmetry (324)

I did this above and obtained the Lai results. There are 9 independent C_{abcd} elements.
Symmetric under reflection in two axes (same as in all three axes).

5.50 Transverse isotropic symmetry (325)

I did this above and obtained the Lai results. There are 5 independent C_{abcd} elements.
Isotropic in the x-y plane.

5.51 Full isotropic symmetry (327)

If we assume transverse iso in both the 3 and 1 direction, we get the four new conditions (5.51.1). Somehow this only counts as 3 new conditions, and we then reduce to 5-3 = 2 independent parameters, and the C matrix takes the form 5.51.3 which has only C_{11} and C_{12}. He claims then to give the connection between these and our famous μ and λ, but he does not derive this result. This is a little exercise which he fails to assign. Recall that $T_{ij} = \lambda tr(E)\delta_{ij} + 2\mu E_{ij}$ so you have to write this out in its 6x6 sense. I just did this on scratch paper and it was easy, so writing it as $T_a = C_{ab}E_b$ in the 6x6 sense I found this

λ+2μ	λ	λ	0	0	0
λ	λ+2μ	λ	0	0	0
λ	λ	λ+2μ	0	0	0
0	0	0	2μ	0	0
0	0	0	0	2μ	0
0	0	0	0	0	2μ

which agrees with p 328 top.

5.52 Engineering constants: isotropic (328)

We have our 2 independent constants in (5.51.3) which shows matrix C, symmetric and pos def. We already saw that $S = C^{-1}$ will also be symmetric and pos def, and I know it will have the same block form. So 5.52.1 shows this "compliance matrix" S where the two constants are taken to be E_Y and ν, but the combination G is just shown as G which is really 1/μ. By "engineering constants", Lai must mean that this is the way practicing engineers like to see their compliance matrix parameterized. The various pos def facts have implications shown at the bottom of page 328: we learn that certain things must be positive and therefore $G > 0$, $E_Y > 0$, and $-1 < \nu < 1/2$. These facts from very general principles!

5.53 Engineering constants: transverse isotropic (329)

There are now those 5 constants. C was shown in 5.50.16 and the inverse S is shown here in 5.51.1 of course having the same block form, but the 5 constants are now: E_1, E_3, G_{12}, G_{13}, ν_{13}. Lai gives the C_{ij} in terms of these new parameters. The reason these constants are defined in this way is that they then have simple *interpretations*. By considering simple uniaxial stress cases, you learn that E_1 and E_3 are the Young moduli in the transverse and axial directions. We find that G_{12} is our shear modulus (formerly called μ) in the 12 plane. Recall from page 203 that ν was the ratio of strains in two perp directions (right from the git go), so ν_{ij} is always going to be such a strain ratio for two perp directions. This is shown for ν_{13} in p 329 (i). As before, S is pos def so various things must be positive as shown.

5.54 Engineering constants: orthotropic (330)

Same idea but now we have 9 constants. Once again S is stated, it has the same old form, but now there are lots of different parameters hiding in this S matrix. Again he expresses the C_{ij} in terms of these engineering parameters. The 9 parameters are: $E_{1,2,3}$; $G_{23,31,12}$; $\nu_{21,31,32}$ and they have the interpretations already described above. As before, the various positive items are shown.

5.55 Engineering constants: monoclinic (332)

We are up to 13 constants now, and we have the different block form for S which goes with C. There are 3 new parameters with η_{ij} names, and one new one called μ_{56}. There new guys are called coupling coefficients and I don't understand their interpretations. You can see that they appear in the "new" non-vanishing matrix elements compared to orthotropic. Well, the coupling guys cause things like shear under uniaxial stress which normally does not happen.

Comments: So that wraps up Lai's section on the anisotropic elastic solid. He does not say why he never examines the "positive things" for the C matrix, just for the S matrix. He does not give engineering constant names to the S elements in the general case where there are 21 constants (and where we have the full 6x6 filled out block form, though symmetric.).

You can just imagine the complexity of a problem in the general case, but at least we have a matrix that is down to 6x6.

There is no doubt a group of people (physicists) who try to compute these constants from the actual atomic or molecular or cellular structure of a material. You would worry about chemical bond strengths under stress maybe. Sounds very complicated.

I found some stuff on "composites" and another PDF, so yes, people do worry about this.

Part B Problems (p 333, Problem 5.91 through 5.103)

PART C: The *Isotropic* Large Deformation of Elastic Solids

Note that "large deformation" just means elastic theory outside of the "infinitesimal" region where we were allowed to use the E strain tensor. It does not mean we are doing any kind of plastic theory. An application of large deformations would be pulling a rubber band.

5.56 Change of Frame (334)

The key equation here is this: // this is a Galilean Transformation of reference frame

$$\mathbf{x}^*(t) = Q(t)\,(\mathbf{x}(t)-\mathbf{x}_0) + \mathbf{c}(t)$$

where Q is a "rotation". Here, **x**(t) is a point in some blob object at time t, and **x***(t) is that same point observed by a moving camera platform. The point $\mathbf{x}_0$ is some selected origin for the rotation Q. I have written this up a bit in the Goldstein folder as Chasle theorem.doc. As the camera platform translates, the translation offset **c(t)** changes, and as the camera platform rotates, Q(t) changes. Of course $QQ^T = 1$.

Now think of the above equation as being some underlying transformation F in the sense of tensor doc. It is not linear due to the translation part **c(t)**. The "R matrix" is just Q. The metric tensor is Cartesian so there is only one kind of vector possible.

Lai now shows that certain vector-like objects transform as vectors under Q, and some do not! And the same is true for scalars and rank-2 tensors. When something transforms as a true tensor, Lai refers to it as being **objective** or **indifferent**. You can have vectors which are objective or non-objective.

I well know that rules for the transformation of tensors of rank 0,1,2 and Lai shows these just above the line on page 335.

Example 5.56.1. Lai shows that dx is a true (objective) vector, and |d**x**| is an objective scalar. Of course these are the canonical objects of tensor doc theory.

Example 5.56.2. The velocity vector **v** and the gradient tensor $\nabla\mathbf{v}$ do NOT transform as tensors of rank 1 and rank 2 and are therefore non-objective. In tensor doc I said that **v** = d**x**/dt always transformed as a vector since dt was a scalar and d**x** was a vector. How do we reconcile this? Well, I hinted at saying that underlying F must be time independent, and I just updated tensor doc to clarify this point. So **v** = d**x**/dt only transforms as a vector if underlying F is time-independent, and of course that is not the case here.

Here are the transformations showing the extra pieces

$$\mathbf{v}^* = Q\,\mathbf{v} + \dot{Q}(\mathbf{x}-\mathbf{x}_0) + \dot{\mathbf{c}}(t) \quad = Q\,\mathbf{v} + \text{two extra pieces} \quad \text{so non-objective}$$

$$(\nabla^*\mathbf{v}^*) = Q(\nabla\mathbf{v})Q^T + \dot{Q}Q^T \quad = Q(\nabla\mathbf{v})Q^T + \text{one extra piece} \quad \text{so non-objective}$$

Example 5.56.3. Recall that d**x** = F d**X** defines the "deformation tensor F". If we assume that the camera platform (frame) is non-rotated at time t_0 (meaning $Q(t_0) = 1$), then in the camera frame we will have d**x*** = F* d**X*** with fact F* = QF. Since this does not say $F^* = QFQ^T$, I conclude that F is not an objective tensor.

Example 5.56.4. Here Lai shows that the right Cauchy-Green $C = F^TF$ transforms as C* = C and since this does not say $C^* = QCQ^T$, we conclude that C is a non-objective tensor. On the other hand, Lai shows that the left Cauchy-Green $B = FF^T$ is in fact an objective tensor and $B^* = QBQ^T$

In passing, he notes (and I prove) that the inverse of an objective tensor (if it exists) is objective. Therefore, he concludes that

$e = (1 - B^{-1})/2$ is objective Eulerian strain tensor

$E = (C-I)/2$ is not objective Lagrangian strain tensor

Back on page 118 this last object was called E^* and E was reserved as the name for the *infinitesimal* version of the Lagrangian strain tensor. But now I guess he will use symbol E for the full Lagrangian strain tensor.

His final comment is that the time derivative of a rank-2 tensor transforms this obvious way,

$\dot{T}^* = Q\dot{T}Q^T + \dot{Q}TQ^T + QT\dot{Q}^T = Q\dot{T}Q^T$ + two extra terms so non-objective

He writes non-objective as a single word nonobjective.

5.57 Constitutive Equation for Elastic Medium under large deformation (338)

Earlier we had T = CE as our Hooke's law business for the infinitesimal E, where C was the rank-4 elasticity tensor. What is going to happen here we wonder? Which stress tensor should we use, and which strain tensor?

Example 5.57.1. Lai first shows that the first P-K stress tensor T_0 transforms as $T_0^* = QT_0$ which means that T_0 is nonobjective. He claims that the second P-K goes as $\tilde{T}^* = \tilde{T}$ which is also nonobjective. Showing this is left to a problem 5.110, but I will do that problem right here:

$\tilde{T} = J F^{-1} T F^{-1,T}$

$\tilde{T}^* = J F^{*-1} T^* F^{*-1,T}$ since $J^* = \det(F^*) = \det(QF) = \det(F) = J$

$J (QF)^{-1} T^* (QF)^{-1,T} = J F^{-1} Q^{-1} T (F^{-1} Q^{-1})^T = J F^{-1} Q^{-1} T^* Q F^{-1,T}$

$= J F^{-1} (Q^{-1} T^* Q) F^{-1,T} = J F^{-1} T F^{-1,T} = \tilde{T}$ => $\tilde{T}^* = \tilde{T}$

Example 5.57.2. If we try a Hooke's law of the form T = kC (right C-G tensor C), we then have a problem because k is the same constant in * and unstar world ("the same material"), but we need T to transform as objective, and C transforms as $C^* = C$ so we fail to get T transforming right.

Example 5.57.3. If we try a Hooke's law of the form $\tilde{T} = kC$, then things are perhaps more reasonable since both $\tilde{T}$ and C transform in the same trivial way. This applies also to $\tilde{T} = f(C)$ with my power series arguments. This equation transforms into $\tilde{T}^* = f(C^*)$. Lai says that this is the Hooke's Law that is used for the general anisotropic elastic solid. Lai has required $f = f^*$ here, but for some reason in this case

that does not imply material is isotropic. Something is missing here. It is not clear that problem 5.111 explains things. I suspect this form won't be used.

Example 5.57.4. Now try instead a Hooke's Law of the form T = f(B). Since both T and B are objective, this is a good possibility. Lai tries it out, but he then needs to satisfy $QTQ^{\mathbf{T}} = f(QBQ^{\mathbf{T}})$ as on page 340. He then requires the appendix below to find the general form that f can take. Since he is assuming right off the bat that f* = f, he is assuming isotropic (think camera orientation Q). A special case of T = f(B) would be that T = kB and this would then be a true "Hooke's Law" for this case since linear. Here k is a constant, not a rank-4 tensor like C, not clear why.

Question: Why is it that we never worried about tensors being true tensors with respect to these camera moves in the infinitesimal theory, but we suddenly worry about it in the large deformation theory? The infinitesimal theory said that $T_{ij} = C_{ijk\ell}\, E_{k\ell}$ where $E = (\nabla \mathbf{u})^{\mathbf{S}}$. What is the transformation nature of the tensor E_{ij} under global rotations (which is what the camera business is). I think under global rotations, E_{ij} is in fact a true tensor, and no doubt so is T_{ij}, but what about C? It would have to be a true rank-4 tensor to make things work right. And I suppose it in fact is a true tensor, and that is why this equation is covariant from the get-go, and that is why the subject never comes up! Note that the components of C will then have to be different in different frames. I guess we always worked in the rest frame.

5.58 Constitutive Equation for Isotropic Elastic Medium (340)

We are in the context now of T = f(B) and are still talking "large deformation" (since we are not using the infinitesimal E tensor). From the previous section, T = f(B) is what tensor doc calls "covariant" under Galilean transformations and is therefore a viable physical law. In the primed (starred) frame this law would read T* = f*(B*) in Lai notation. If we then assume that f = f* (function is the same under rotations), then we are **assuming isotropic**!!! Then we have T* = f(B*) which means $QTQ^{\mathbf{T}} = f(QBQ^{\mathbf{T}})$ where Q is that arbitrary rotation, see 5.57.14. When Q = 1 this says T = f(B). We can then apply the results of the appendix (notes below) to conclude that equation T = f(B) must have this general form (which we write in two different alternate ways),

$T = a_0 I + a_1 B + a_2 B^2$	where a_o are functions of the eigenvalues of symmetric B.
$T = \varphi_0 I + \varphi_1 B + \varphi_2 B^{-1}$	where φ_o are functions of the eigenvalues of symmetric B.

Recall that in general T = f(B) is covariant only for an isotropic medium so we are in the isotropic context here. A linear Hooke's law would have only the $a_1 B$ term I guess. In the above, the a_i and φ_i are in general functions of space (see comment at the end of the appendix notes below).

The above alternate forms for T in terms of B are the "most general isotropic large deformation" constitutive equations.

Example 5.58.1. This is Lai's nice proof of the C-H theorem, but he omits a simple step for some reason. He shows that C-H is true in the basis in which symmetric matrix B is diagonal, but fails to take it from there to an arbitrary non-diagonal B. But all OK.

Lai comments on incompressible solids. Lai replaces φ_0 by -p in 5.58.8. At first I thought he was claiming that p was a constant, but the torsion example talks about p(r). So NOW I think all he has done here is *rename* $\varphi_0(x)$ to be -p(x). Maybe it is a constant in Cartesian coordinates, I don't know. The web has some derivations of this using Lagrange multipliers, probably not very simple. From page 130 I showed that incompressible means no volume change which in turn means 1 = detF = detC = detB, so in our current situation we would be interested in det(B) = 1, which means $I_3 = 1$ (appendix notes), as Lai says on page 341 2nd line of text up from the bottom.

Recall from the appendix notes below that you can write $\varphi_i = \varphi_i(\Lambda_i)$ or $\varphi_i = \varphi_i(I_i)$. The key fact about incompressible is that $I_3 = 1$ so then you really have $\varphi_i = \varphi_i(I_1,I_2)$ only. I would assume this form also applies to p.

In a very brief section, Lai makes an ansatz that you can develop both φ_1 and φ_2 from a potential function he calls A, a bit like the earlier planar Airy function. Then of course you can rewrite 5.58.8 as 5.58.10. When this was done in the past, we found that the potential had to satisfy some equation such as the Laplace or biharmonic equation, but nothing is said about that here. Another of Lai's frequent comment omissions.

Finally, Lai rolls out a version of 5.58.8 where the φ_i pair are replaced by simple expressions involving a certain μ and β with certain ranges given for these parameters. I presume that these are constants of some sort, so this would seem to be a very special case. This μ,β form is called the **Mooney-Rivlin "theory for rubber",** and "**hyperelastic isotropic solids**". So they just wanted to put in a plug for this subject, and they give a very rare reference (ie, Lai has hardly any references in his book) . It turns out that this reference is to a certain book which was originally published as a strange volume III/3 in some Encyclopedia of Physics. This book is



The Non-Linear Field Theories of Mechanics (Hardcover)
by C. Truesdell, W. Noll

and here is the place in the contents to which we are referred

http://www.amazon.com/The-Non-Linear-Field-Theories-Mechanics/dp/0387550984#reader_0387550984

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You see that page 349 which Lai quotes does in fact treat this theory of rubber. So here is a little "advanced topic" one could pursue, but I won't.

This book of **Truesdell and Noll** is a famous one in the world of continuum mechanics. It also appears on page 512 in the References. This book has three editions 1965,1992, 2004. This third edition appears partially in Google books so you can get a feel for things (first 90 or 600 pages). The contents is similar to that of Lai, but of course is going to be more advanced. Remember that Lai is "Introduction to CM". Truesdell also has a book on field theories. Amazon sells the T&N book for 271, one used copy at 70.

5.59 Simple Extension of an incompressible isotropic elastic solid (340)

The example here is just stretching a rectangular solid by an axial load, as we did before. The two transverse directions we know will shrink as the object is stretched. If nothing else, we know this must

happen due to the incompressibility of the solid. I think Lai means to write things like $x_1 = \lambda_1 X_1$ so that each point is "stretched" in a linear (and large) fashion from its original material position. I would guess that $\lambda_1 > 1$ and $\lambda_2 = \lambda_3 < 1$. If we then look at page 122 concerning the interpretation of the B tensor, we find that off-diagonal elements are associated with rotation of a particle which we don't have in our simple stretching example here. The diagonal elements are $B_{11} = (ds_1/dS_1)^2 = (\lambda_1 X_1/X_1)^2 = \lambda_1^2$ in our example, so I agree with Lai's statement of B and B^{-1}. Notice that one can regard $\Lambda_i \equiv \lambda_i^2$ as the eigenvalues of B, and then the incompressible condition says $\Lambda_1 \Lambda_2 \Lambda_3 = 1$ which in turn means that the product $\lambda_1\lambda_2\lambda_3 = 1$. In our example, the two transverse dimensions are 2 and 3 and are equivalent, we this becomes $\lambda_1\lambda_2^2 = 1$ which agrees with 5.59.1.

Now we apply our general form $T = -p\, 1 + \varphi_1 B + \varphi_2 B^{-1}$ and we know B, so T is diagonal with

$$T_{11} = -p + \lambda_1^2\varphi_1 + \lambda_1^{-2}\varphi_2 \qquad T_{22} = T_{33} = -p + \lambda_2^2\varphi_1 + \lambda_2^{-2}\varphi_2$$

Lai says these T_{ii} are "constants" so the $\partial_j T_{ij} = 0$ static equation will be OK. Perhaps the φ_i are also constants, I am not sure (I think yes). Since there are no normal side surface tractions, we must have $T_{22} = T_{33} = 0$ so we can use this to say $p = \lambda_2^2\varphi_1 + \lambda_2^{-2}\varphi_2$ which then is the same as $p = \lambda_1^{-1}\varphi_1 + \lambda_1\varphi_2$. So insert to get

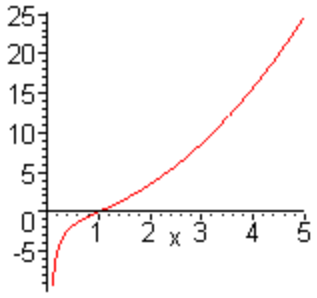
$$T_{11} = -\lambda_1^{-1}\varphi_1 - \lambda_1\varphi_2 + \lambda_1^2\varphi_1 + \lambda_1^{-2}\varphi_2 = \varphi_1(\lambda_1^2 - \lambda_1^{-1}) + \varphi_2(\lambda_1^{-2} - \lambda_1)$$

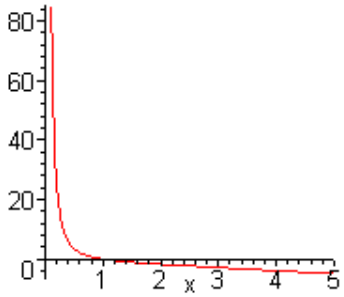
At this point the example ends abruptly. We have no idea what $\varphi_{1,2}$ are, but λ_1 is the amount of stretch in the axial direction (fractional) which can be large. You can see that if $\lambda_1 \to 1$ (no stretch). $T_{11} \to 0$. I think the φ_i will be constants in this problem and will be functions of μ and λ for isotropic material, but Lai is just silent about this. A quick wiki peek suggests that the φ_i are functions of the material in some experimental way only. These constants are called C_i in wiki, and I quote

Rubber [edit]

Elastic response of rubber-like materials are often modeled based on the Mooney—Rivlin model. The constants C_1, C_2 are determined by the fitting predicted stress from the above equations to experimental data. The recommended tests are uniaxial tension, equibiaxial compression, equibiaxial tension, uniaxial compression, and for shear, planar tension and planar compression. The two parameter Mooney–Rivlin model is usually valid for strains less than 100%.

So you could then take our equation $T_{11} = \varphi_1(\lambda_1^2 - \lambda_1^{-1}) + \varphi_2(\lambda_1^{-2} - \lambda_1)$ and maybe measure T_{11} at various stretches λ_1 and thereby deduce the values of constants φ_1 and φ_2 for a particular material! The two functions of λ_1 have these shapes,





Since the shapes are different, you should be able to obtain constants φ_1 and φ_2 from a few data points. I would actually do this for a rubber band I think. The form here seems highly non-intuitive. You would want to have interpretations for these constants.

5.60 Simple Shear of an incompressible isotropic elastic block (343)

You take a block into the plane of paper and shear it as shown by applying some kind of stress (coming soon). The first line on page 343 shows $\mathbf{x} = f(\mathbf{X})$ from which it is trivial to compute F and then B and B^{-1} as shown. Since the volume does not change, this is an incompressible situation and we find detB = 1 = I_3. The three I_i are computed as shown. Given this B, we compute T from the incompressible form in terms of p, φ_1 and φ_2. P is defined as shown. If the two faces parallel to plane of paper have no surface traction, then $T_{33} = 0$ and so P = 0 and then T has the 2x2 block form penciled in top page 344. From this Lai computes the surface tractions on the right face in the page 343 picture. We find that there is both a normal force and a shear force as shown p 344. Lai comments that both stress components are needed to get the block into the shape shown. Perhaps I could solve this if one of those components were missing. All results are functions of K and φ_1 and φ_2. Lai says nothing about these last two. But 5.60.13 gives a simple relation which holds for any values of the φ_i. He refers to the φ_i as "coefficients of the material". Perhaps that means they can be related to λ and μ or whatever pair you choose. [Again, I now think these φ_i are constants, but are probably not simply related to λ and μ which are parameters of the infinitesimal theory with its linear Hooke's law in terms of C_{abcd}.]

5.61 Bending an incompressible isotropic rectangular bar (344)

This is a very fancy example in many ways.

• First, we have a mapping from rectangle to piece of a thick annular ring as shown. The area of the annular ring piece is

$$\text{area} = 2 \int_{r_1}^{r_2} dr \int_{0}^{cb} d\theta\, r = 2cb \int_{r_1}^{r_2} dr\, r = 2cb\,(1/2)\,(r_2^2 - r_1^2) = cb\,(4a/c) = 4ab$$

where I used my pencil results on page 345. Thus, the area is unchanged by the deformation, as appropriate for an incompressible deformation. Lai does not mention this fact, but does show later that $I_3 = 1$ which implies this same thing.

• Second, we have *just* the kind of situation that was treated in the elaborate Section 3.29 where your before deformation is in one coordinate system, and your after deformation is in another! In this case before is Cartesian and after is Cylindrical. In that section I referred to this kind of deformation as if it were the flow of a fluid, and that is sort of what we are talking about here. I wrote a whole doc on this section 3.29. The F tensor has one foot in each coordinate system, such as $[F^{(x,X)}]_{ij}$ for the two Cartesian systems, but you can then compute $[F^{(x',X')}]_{mn}$ where x' and X' are some other curvilinear systems. In this case, it turns out that the B matrix is entirely in terms of x' coordinate labels (both rows and columns) since we have $[B^{(x',x')}]_{mn} = [F^{(x',X')} F^{(x',X')T}]_{mn}$. In any event, everything needed for our current example was computed back there,

$$[B^{(x',x')}]_{mn} = \Sigma_i[\{ [h^{(x')}{}_n / h^{(X')}{}_i] [h^{(x')}{}_m / h^{(X')}{}_i] (\partial x'^n/\partial X'^i)(\partial x'^m/\partial X'^i)$$

Page 136 gives all the B components like B_{rr} and all we have to do is compute derivatives. It also gives B^{-1}, but we could just invert our B. So a huge amount of math technology is going into the statement of the simple matrix B in this example!!! I could easily verify the matrix but won't bother.

• Once you have matrix B and B^{-1} (all with r,θ,z row and column labels), you can compute the I_i and of course these are in terms of these r,θ,z labeled components. $I_3 = 1$ as expected.

• Now, to make things more complicated, he is going to use the ansatz 5.58.9 which gives φ_1 and φ_2 in terms of some Airy-like function called $A(I_1, I_2)$, and derivatives are as shown wrt these I_i. So we install these A derivatives for the φ_i and obtain the entire T_{ij} matrix as shown bottom page 345.

• Next, we have to look back at the Cauchy equation of motion (which was more general than all our infinitesimal work), and we see that these equations require certain relations between the T_{ij}. These equations allow Lai to conclude that the hydrostatic p object can depend only on variable r.

• Next, the Cauchy equation is restated in terms of $\partial_r T_{rr}$ and $\partial_r A$ as in 5.61.9 which gets trivially integrated to get 5.61.10. He concludes that A = A(r) as well.

• There are no normal forces T_{rr} on either the inside or outside of the cylinder, so these are BC's. He then shows that $A(r_1) = A(r_2)$. This in turn imposes a condition that $r_1 r_2 = 1/c^2$ which I did not have in my original pencil work. I think this condition is imposed by the Cauchy equation! So you cannot just pick your value of a. Once you pick r_1 and r_2, that forces α (and c) and that in turn forces a as he shows mid page 346.

• The final step is to look at surface forces implied by our deformation. Lai shows that there is no net normal force on the "end caps" of the annular section. Not surprisingly, he finds some end couples on these ends (called M as usual). He can express M in terms of A(r).

And so ends this fancy example. A(r) is never determined, and I presume it will somehow be determined by the medium. Lai is totally quiet about how this will work. So, sadly, we can't say we have really "done" this problem of bending a rubber rectangular solid into a part of a cylinder because we don't know

how to find A(r). Maybe there is some Sneddon like integral equation involved here. Maybe he will return to this subject later in the book in some context?

5.62 Torsion and tension of an incompressible isotropic cylinder (344)

• Here we are going to twist a rubber cylindrical rod and see what happens. The "flow equations" are 5.62.1 where you see that the amount of twist per z distance is proportional to selected constant K. The rod is allowed to have an increased diameter at any point $r = \lambda_1 R$, and is allowed to axially stretch $z = \lambda_3 Z$. Before twisting, the volume is $\pi R^2 Z$ if we regard momentarily R and Z as rod dimensions. After twisting, the volume is $\pi r^2 z = \pi \lambda_1^2 R^2 \lambda_3 Z = \lambda_1^2 \lambda_3 (\pi R^2 Z)$. If we want the material to be incompressible, then we must have $\lambda_1^2 \lambda_3 = 1$, as shown in 5.62.1.

• In this problem, unlike the last, we will use cylindrical coordinates for both before and after. That implies a certain set of B and B^{-1} equations from Section 3.29, and I accept that Lai has done the various derivatives correctly to get 5.62.2. Since we are talking about a specific final position of the bar as determined by λ_1 and λ_2 and K, we regard these three quantities as constants! Thus, we see that the B_{ij} are at most functions of variable r. That is, $B_{ij}(r)$. As usual, we compute the invariants I_i from these matrix elements, and so this gives $I_i(r)$ as shown.

• The matrix B as shown will have some eigenvalues $\Lambda_i(r)$. They are functions of r since the B_{ij} are functions of r. Just for fun, I had Maple compute the eigenvalues:

$$B := \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_1^2 + r^2 K^2 & r K \lambda_3 \\ 0 & r K \lambda_3 & \lambda_3^2 \end{bmatrix}$$

```
eigenvalues(B)[1];
```

$$\lambda_1^2$$

```
eigenvalues(B)[2];
```

$$\frac{1}{2}\lambda_1^2 + \frac{1}{2}r^2 K^2 + \frac{1}{2}\lambda_3^2 + \frac{1}{2}\sqrt{\lambda_1^4 + 2\lambda_1^2 r^2 K^2 - 2\lambda_1^2 \lambda_3^2 + r^4 K^4 + 2 r^2 K^2 \lambda_3^2 + \lambda_3^4}$$

```
eigenvalues(B)[3];
```

$$\frac{1}{2}\lambda_1^2 + \frac{1}{2}r^2 K^2 + \frac{1}{2}\lambda_3^2 - \frac{1}{2}\sqrt{\lambda_1^4 + 2\lambda_1^2 r^2 K^2 - 2\lambda_1^2 \lambda_3^2 + r^4 K^4 + 2 r^2 K^2 \lambda_3^2 + \lambda_3^4}$$

You see that $\Lambda_1 = \lambda_1^2$ and is not a function of r, but the other two are functions of r. These Λ_i are never used by Lai in this example.

• In my appendix notes, I show that if $B_{ij} = B_{ij}(r)$ as in this case, then $\varphi_i = \varphi_i(r)$ as well and in general $\Lambda_i(r)$ as well. So I am happy with Lai's claims made just below 5.62.3.

• Next, since we know B, we can write out all the T_{ij} components as shown page 347.

• Next, we look at the same cylindrical Cauchy static equations as in a previous example and conclude p(r) again. On page 348 we have a long evaluation of an integral for N, the normal force one will find on the endcaps of the rubber cylinder that is being twisted. The final form is at page bottom and is given as an integral of φ_1 and φ_2 which in this problem we know are functions of r (unknown functions). Since r and R are simply related, you can regard things as $\varphi_i(R)$.

• Next the twisting moment M is computed as another integral top page 349. At this point there is nothing else you can do unless you come up with some candidate functions for $\varphi_i(R)$.

• What happens if K is small? In this case, Lai can do the integrals in the small K limit as shown 24 and 25, still of course in terms of the unknown $\varphi_i(R)$ which in this limit are now constants since the I_i are shown to be constants. So we now have a 2-constant fit for N and M where λ_3 is the axial stretch. You could take a rubber cylinder then and do some experiments to find these two constants.

• Lai wraps up with a few comments. The ratio M/K is interesting. For a given twist rate K, this tells how much twisting torque you have to apply. Large M/K means large "torsion stiffness" against twisting. An expression for M/K is given for small K and is seen not to depend on the constants φ_i. It relates M/K to N and λ_3 and R_0 and nothing else -- **Rivlin's universal relation.**

Appendix 5C.1. The most general form of F in T = F(B) for isotropic solid.

I am making these replacements in this appendix: T→B and S→T, so the notation then better fits what is being talked about in this Section C! Also, I replace $\lambda_i \rightarrow \Lambda_i$ because Lai keeps using λ_i to mean other things in all his examples.

We seek the most general form of matrix-valued function F such that the following is true for ALL matrices Q which are real orthogonal

$$QTQ^T = F(QBQ^T)$$

where B is some given symmetric matrix, and where T = F(B), which is the above with Q = 1. I guess my first approach would be to assume some power series for F, so that $F(B) = \Sigma_{n=0}^{\infty} d_n B^n$. Now, assume that (recall that $B = FF^T$ is symmetric)

$$B\mathbf{n}_i = \Lambda_i \mathbf{n}_i \qquad \text{// eigenvalues and eigenvectors of B}$$

The Λ_i are determined by the secular equation $\det(B-\Lambda I) = 0$ which has this form

$$\Lambda^3 + a\Lambda^2 + b\Lambda + c = 0 \qquad \Rightarrow \qquad \Lambda^3 = -[a\Lambda^2 + b\Lambda + c]$$

The Cayley-Hamilton theorem (see matrix notes section 7 page 19) says that for *any* square matrix B, B satisfies its own secular equation, so we can write

$B^3 + aB^2 + bB + c = 0 \qquad \Rightarrow \qquad B^3 = -\,[aB^2 + bB + c]$

[Note from page 40: $a = -I_1$, $b = I_2$, $c = -I_3$, and we also know that $I_3 = \Lambda_1\Lambda_2\Lambda_3 = \det(B)$.]
[Note: since the I_i are "invariants" (p 40), they are invariant under similarity transformations on B. Thus, we can just assume B has its diagonalized form B_d as shown below such that $B_d = \mathrm{diag}(\Lambda_1,\Lambda_2,\Lambda_3)$, and then we see that the I_i are functions only of the Λ_i as listed top page 41.]

This means we can replace B^3 as shown in our infinite sum, and similarly we can reduce B^4 and higher, so we conclude that it is possible to find constants a_n such that (ie, a quadratic)

$$T = F(B) = \Sigma_{n=0}^{2}\, a_n B^n$$

It seems pretty clear at this point that, whatever F is, we must have $a_n = a_n(B_{ij})$. That is to say, if you write $F(B) = \Sigma_{n=0}^{2}\, a_n B^n$, then the a_n can be functions only of the matrix elements of B.

Now assume that B is symmetric so it can be brought to diagonal form B_d by some RO matrix P,

$$B = PB_dP^{-1} \qquad\qquad B_d = P^{-1}BP$$

Then sandwich P^{-1}...P around the previous equation to get

$$P^{-1}F(B)P = \Sigma_{n=0}^{2}\, a_n\, P^{-1}B^nP = \Sigma_{n=0}^{2}\, a_n\, P^{-1}BPP^{-1}B \;.....P = \Sigma_{n=0}^{2}\, a_n\, B_d^n$$

The thing on the right is a diagonal matrix, so therefore $P^{-1}F(B)P$ is also diagonal. Call this

$$T_d \equiv P^{-1}TP = P^{-1}F(B)P\ .$$

We have just shown that if P diagonalizes B, then it also diagonalizes T, and we then have

$$T_d = \Sigma_{n=0}^{2}\, a_n\, B_d^n = F(B_d) \qquad\qquad B_d = \mathrm{diag}(\Lambda_1, \Lambda_2, \Lambda_3) \qquad\qquad \text{(xi)}$$

Now $\Sigma_{n=0}^{2}\, a_n\, B_d^n$ involves only the a_n and the Λ_i. Similarly, $F(B_d)$ involves only the Λ_i since that is all that B_d is made of. Therefore, the a_n are functions only of the numbers Λ_i. This conclusion would be true even if we did not use C-H to limit the sum to a quadratic.

[Note: We can reverse solve the three equations top page 41 for $\Lambda_i(I_n)$. I tried this in Maple and the result is a bit messy, involving roots of a cubic, but the point is that the solution exists and you can then think of $\Lambda_i = \Lambda_i(I_1,I_2,I_3)$.]

Question: what happens if two eigenvalues collide so that $\Lambda_2 = \Lambda_3$? The secular equation then has the form

$$(\Lambda-\Lambda_1)(\Lambda-\Lambda_2)^2 = 0 \qquad \Rightarrow \qquad (\Lambda-\Lambda_1)(\Lambda-\Lambda_2) = 0$$

C-H is still true and we have then

$$(B-\Lambda_1)(B-\Lambda_2)^2 = 0 \qquad => \qquad (B-\Lambda_1)(B-\Lambda_2) = 0$$

but now this has the form

$$B^2 + xB + yI = 0$$

and this lets you now replace B^2 in the sum with lower terms so you then get

$$T_d = \Sigma_{n=0}^{1} a_n B_d^n = F(B_d) \qquad \text{// linear, no longer quadratic}$$

And if all three roots are the same, you can replace the B term as well so that

$$T_d = \Sigma_{n=0}^{0} a_n B_d^n = F(B_d) = a_0 I \qquad \text{// constant in B}$$

These are the conclusions discussed on page 351.

Now go back to (and include in this case the chance that $a_2 = 0$ or $a_2 = a_1 = 0$ as just noted),

$$T = \Sigma_{n=0}^{2} a_n(\Lambda_i) B^n = F(B)$$

We can apply $Q..Q^{-1}$ to the left equation to get

$$QTQ^{-1} = \Sigma_{n=0}^{2} a_n(\Lambda_i) QB^n Q^{-1} = \Sigma_{n=0}^{2} a_n(\Lambda_i) (QBQ^{-1})^n = F(QBQ^{-1})$$

and this is true for ANY real orthogonal Q, as required at the very start. So the question was to find the most general form that F can have to make this be true for any Q, and the answer is this

$$F(B) = \Sigma_{n=0}^{2} a_n(\Lambda_i) B^n$$

Now one more detail. If B is invertible, then we can write C-H as

$$B^3 + aB^2 + bB + c = 0$$

$$B^2 + aB + b + cB^{-1} = 0$$

and this allows replacement of the B^2 term with a B^{-1} term and we can then write as shown bottom of page 349.

$$F(B) = a_0 I + a_1 B + a_2 B^2 = \varphi_0 I + \varphi_1 B + \varphi_2 B^{-1} \qquad \text{// see 5.58.5 and 5.58.2 p 340,1}$$

But just having B be symmetric does not guarantee invertibility (but I think F is always invertible so B is as well, flow is non-singular). We already shows that $a_n = a_n(\Lambda_i)$ and we can also write this as $a_n(I_i)$

with a different functional form. (see note above). Since the a,b,c in the secular equation are really the I_i we conclude that $\varphi_n = \varphi_n(I_i) = \varphi_n(\Lambda_i)$ as well.

Comments: In the above, we might then assume that B and T are fields so really $B_{ij}(x)$ and $T_{ij}(x)$. If you track the derivation, this must means that *everything* is a function of space, including the Λ_i and the a_i. So you could say for example that $a_i = a_i(\Lambda_n(x)) = \hat{a}_i(x)$. And of course the I_i are then $I_i(x)$ since the I_i are functions of the $B_{ij}(x)$.

Suppose we knew that the $B_{ij} = B_{ij}(r)$ only in some problem where r is the cylindrical radius. Then we would have $\Lambda_i(r)$ and $I_i(r)$ and $a_i(r)$ and $\varphi_i(r)$ as well.

Part C Problems (p 351, Problem 5.104 through 5.113)

And so ends this monstrous Chapter 5 on Elastic Solid Continuum Mechanics, about 150 pages. My notes here are about 90 pages!