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Introduction. There is a notion of "rigid body motion" of a fluid which surprised me at first. This is motion in which the fluid maintains its "shape", nothing moves internally. What this means is that there is

no "deformation" going on, no "relative internal motion" in the fluid. Technically, I think it means the rate of deformation tensor is 0. Here are some examples: (1) water in a square container which is accelerating to the right at constant a. The surface is slanted, a camera in a moving frame would see a static water mass. (2) Probably a centrifuge pattern is the same idea, but not quoted. (3) any obviously static situation. In all these cases of "rigid body motion", if you look at a particle's cube, you find only normal forces. The idea is that you *can have* shear forces on such a cube, but that would cause internal motion of the fluid, and that is excluded from this class of situations. So we end up with this definition: a fluid is unable to sustain shearing forces without continuously deforming.

Just think of a large math cube of water with a shear force on some wall of the cube. Water at that wall then has to be accelerating tangent to the wall in the direction of that shear force and there is nothing (no springs) to hold it back. Somehow that motion of the water there results in a non-zero rate-of-deformation tensor D.

Second idea is that you can separate off the incompressible fluids as a special sub-class.

6.1 Fluids. (353) Consider the equation $\mathbf{t} = T\mathbf{n} = \lambda\mathbf{n}$ acting on an internal cube of fluid. The part $\mathbf{t} = \lambda\mathbf{n}$ says that any surface traction on any face of the cube must be normal only, no shear. The part $\mathbf{t} = T\mathbf{n}$ is the usual relationship between a surface traction on a surface patch with normal $\mathbf{n}$ and the stress tensor T. So in a fluid, $T\mathbf{n} = \lambda\mathbf{n}$ for any internal direction $\mathbf{n}$ you choose for your cube wall alignment, and λ is therefore independent of direction. This then is an EV equation with only one eigenvalue λ, and that means all three eigenvalues are λ since this is a 3D equation. Since T is symmetric, it can be diagonalized and the diagonal matrix will then be $T' = \lambda I$. Then we have $T = RT'R^{-1} = \lambda I$ as well. In other words, in a fluid with no shear forces, the stress tensor T must be a multiple of the identity, so $T = \lambda I$ and this is normally written $T = -pI$ where p is the hydrostatic pressure. The sign is such that if $p > 0$, the surroundings of a particle cube will exert a positive normal force on each face of the cube.

6.2. Incompressible subclass of fluids (354). The equation of continuity for mass flow (I have separate notes on this in Stak Ch 6 support) says $\operatorname{div} \mathbf{v} = 0$ if $d\rho/dt = 0$, this last meaning incompressible. This is like $\operatorname{div} \mathbf{E} = 0$ in electrostatics and allows $\mathbf{v}$ to be represented by a potential, but that is not mentioned at this point. Remember that $\mathbf{v} = d\mathbf{x}/dt$, the velocity of a particle of continuous matter.

6.3 Hydrostatics (355). The Cauchy Equation of Motion of course applies to fluids as it does to any continuous material and says $\partial_j T_{ij} + \rho B_i = \rho a_i$. In the situation described above where $T_{ij} = -p\,\delta_{ij}$, we get $\partial_j T_{ij} = -\delta_{ij}\partial_j p = -\partial_i p$ so then we have $-\partial_i p + \rho B_i = \rho a_i$, so this is a massive simplification. In a gravity field, $B_3 = g$ if the 3 direction goes down, since F = mg and B is force per unit mass. Then the equation says $-\partial_i p + \rho g \delta_{i,3} = \rho a_i$. If a body of fluid is not accelerating in the rigid body sense of above, this says $-\partial_i p + \rho g \delta_{i,3} = 0$ which is three equations. Two say that $p = p(x_3)$ (functional dependence), and the third says

$$\partial_3 p(x_3) = \rho g \qquad \Rightarrow \qquad p(x_3) = \rho g h + \text{constant}$$

which is the famous high school formula for pressure in a fluid -- it depends only on height or depth. Lai did not mention that this is only valid when ρ = constant, so it applies for an incompressible fluid. In a compressible fluid that equation $\partial_3 p(x_3) = \rho(x_3) g(x_3)$ is still valid, but then you need some equation of state (a constitutive equation) for the fluid to make some connection between p and ρ so you can solve things. For air, that would be the usual gas law, and maybe g drops off with height in the usual way.

Example 6.3.1. A cylinder of less dense fluid floats in a more dense fluid tethered to the bottom. We do force balance and solve for tension T in the anchor line and the result is basically a derivation of **Archimedes Principle**: there is a buoyancy force up on the barrel equal to the weight of the liquid displaced. So we have really derived this Principle right here.

Example 6.3.2. (355) This is the accelerating cube of water with a free surface that ends up tilted at some angle. The basic equation here is $-\partial_i p + \rho g \delta_{i,3} = \rho a \delta_{i,1}$ since we are accelerating in the x_1 direction. If you write this out as three separate equations you get (i) as a system of PDE's of first order. You can just guess the solution (ii) and verify that all three equations are happy. Then place origin on the surface as shown as O. The rest is just algebra and you find that the tilt angle has tangent a/g, and the surface has a planar shape as shown (set $p = p_0$ to find this), and $p = p(x_3)$ depends only on depth even in this situation with a tilted surface, something that might not have been obvious. It really is obvious though if you just realize that the cause of the pressure is the weight of the mass above a point in the fluid. So this is then just a check.

6.4 Newtonian Fluids (357).

The D matrix $D = (\nabla \mathbf{v})^{\mathbf{S}}$ tells us two things. The diagonal elements tell us the rate at which each direction of the cube is expanding (fractionally), while the off-diagonal elements tell us the rate at which the cube is shearing in each direction pair. Page 97 showed in fact that $2D_{12}$ = rate of decrease of angle θ_{12} if it starts at a right angle. Both of these "motions" of the cube are NOT rigid body motion. On the other hand, the spin tensor $W = (\nabla \mathbf{v})^{\mathbf{A}}$ corresponds to a rigid body rotation of the cube with no deformation.

Now in a Newtonian fluid you first assume things are isotropic. You then assume that T_{ij} has the pressure term contribution $-p(x)\delta_{i,j}$ and has a second contribution Lai calls T'_{ij} which is going to be linear in the D_{ij} tensor. Thus

$$T'_{ij} = Q_{ijab} D_{ab}$$

and then this is exactly like $T_{ij} = C_{ijab} E_{ab}$ in terms of tensor properties and then you end up with the μ and λ form when you do the simple tensor analysis.

My big question right now is why the spin tensor W is ignored? Why is there no contribution of this form

$$T'_{ij} = S_{ijab} W_{ab} \qquad S_{ijab} = \kappa \, \delta_{ij} \varepsilon_{ab}$$

where I show what seems to be a reasonable tensor form for S which is symmetric on the first two indices and antisymmetric on the second two. You might think that a little cube would in fact "rotate" rigidly as part of its reaction to shear stress, so I don't see why this is thrown out. Well, I think this may come back into play later when we ponder flow with "vorticity". Or maybe there really is some tensor isotropy reason for no such term.

So let's ignore W for now, and then we know that

$$T'_{ij} = Q_{ijab} D_{ab} = \lambda \Delta \delta_{ij} + 2\mu D_{ij} \qquad \Delta = \mathrm{tr}(D)$$

Note: Whereas δ_{ij} is an isotropic tensor in 3D, ε_{ab} is an isotropic tensor only in 2D, but here we are talking 3D! So the above makes no sense, and I think we can just rule out a W contribution because there is no isotropic tensor S_{ijab} which is symmetric on the first two and antisymmetric on the second two. Needless to say, Lai does not mention this. Say again: we need a form like that shown above where ε_{ab} is some isotropic 3D tensor, but there exists no such tensor, so the form cannot be implemented.

6.5 Interpretation of λ and μ (358).

If you do the "river" situation where v_1 varies in the x_2 direction across the river, you have $v_1(x_2)$ and Lai easily shows that $T_{12} = \mu\, \partial_2 v_1$. A large μ means a given change of velocity across the river requires a large shear stress, so you have a large **viscosity** in this case, so μ is viscosity.

Comment: Earlier I wondered about a shear force on the wall of a math water cube and why it creates some D tensor action. Here you see one answer. That shear force causes a relative motion of layers parallel to the wall, which means $\partial_2 v_1 \neq 0$ near the wall, and that results in T_{12} as just shown, and that in turn requires that there be some D_{12} as per the above equation. In other words, tiny particle cubes just inside the wall boundary of the big cube are being "sheared" by this velocity differential, and the means we have D going. This shearing could only be stopped if you had viscosity $\mu = 0$, and perhaps that situation arises in a superfluid, I don't know. [it arises later in inviscid fluids!]

Lai next shows that

(average of normal T'_{ii} elements) $= (\lambda+2\mu/3)\,\Delta$.

Recall that $\Delta = \mathrm{tr}(D)$ tells you the rate at which your particle's volume is expanding. Δ is the "rate of dilatation". The average normal stress on the LHS is a sort of pressure-like quantity that is perhaps trying to compress the particle to a smaller volume. If $(\lambda+2\mu/3)$ is very large, then it takes a huge average normal stress to achieve a given rate of compression Δ. Think of compressing cold tar which is very viscous in this bulk sense. So $(\lambda+2\mu/3)$ is called the **bulk viscosity**.

If fluid is incompressible, $\Delta = 0$ always. It would be possible to have a compressible fluid but it happens that $(\lambda+2\mu/3) = 0$ (Stokes assumption). In either of these two cases (incompressible or Stokes), the average of the normal stresses will by -p, as shown in 6.5.6.

6.6 Incompressible Newtonian Flow (359).

First, Lai shows that when $\Delta = 0$, you get mean normal stress = p, that pressure. He discusses how the value of p sort of decouples from things and is **indeterminate**. Whatever happens of interest will happen at any p you want, I think. But some BC will set p.

Now we are going to think of that flow velocity vector **v** as our variable of interest, sort of like the displacement **u** in elastic solids. Constitutive means F = kx which in this case is T = k v, so to speak, and we have in particular that

$$T_{ij} = -p\delta_{ij} + 2\mu D_{ij} = -p\delta_{ij} + \mu(\partial_j v_i + \partial_i v_j). \qquad \text{// constitutive equations}$$

This is a model for how this particular continuous material behaves (ie, incompressible fluid).

Example 6.6.1 gives a simple math detail: it computes $\partial_j T_{ij}$ for the above T, and one term goes away because div **v** = 0 for incompressible. Notice that λ is gone because Δ = 0 for incompressible. The result is

$$\partial_j T_{ij} = -\partial_i p + \mu\, \partial_j^2 v_i \;=\; -\partial_i p + \mu \nabla^2 v_i$$

which as a vector is

$$(\text{div}T) \;= -\nabla p \;+\; \mu \nabla^2 \mathbf{v}$$

6.7 Navier-Stokes for incompressible fluids (360).

We then stick this $\partial_j T_{ij}$ into the usual Cauchy equation of motion, that is one detail. The other detail is that we recall from Chapter 2 about acceleration, with its convective term,

$$\mathbf{a} = D_t \mathbf{v} \;= \partial_t \mathbf{v} + (\nabla \mathbf{v})\mathbf{v}$$

Sticking in these two pieces gives 6.7.6 which is the world-famous (incompressible) **Navier-Stokes equation**

$$\rho\, [\, \partial_t \mathbf{v} + (\nabla \mathbf{v})\mathbf{v}\,] \;= \rho \mathbf{B} \;-\nabla p \;+\; \mu \nabla^2 \mathbf{v} \qquad \text{and} \qquad \text{div } \mathbf{v} = 0$$

So this thing is three PDE's in the three variables v_i, and variables are the x_i and t. It has two terms similar to the heat equation (∇^2 and ∂_t) so it has elements of diffusion. There does not seem to be a wave possibility since no ∂_t^2 action, but the non-linear term $(\nabla \mathbf{v})\mathbf{v}$ could somehow alter things.

But p seems also an unknown. It is in N=3 space dimensions at least. ρ = constant since we are incompressible. μ the only other constant. This is N-S for incompressible fluids, and if you use the more general result with Δ back in there you get the general N-S.

We are now ready for some examples:

Unidirectional Flow Examples.

Example 6.7.1 (361). Uni flow in x_1 means only v_1 exists. We first learn that, in this case, $\partial_1 v_1 = 0$ which says $v_1(x_2,x_3)$ only, so v_1 cannot vary along the direction of the flow. This is nothing but continuity -- the water has to flow somewhere, it goes straight ahead, so it cannot speed up or slow down along a streamline of this nature. Next, we learn that diagonal D elements are also 0 in this case, very obvious. We then conclude that normal stress T_{ii} is -p on all three cube faces, so uni flow does not change this fact! I think this would be true even if we have $v_1(x_2,x_3,t)$ so flow is accelerating in uni fashion.

Example 6.7.2. (362) It took me a while to grasp the picture. Imagine some down-tilted river channel (with depth) that you view from the side. Gravity is in $-\hat{z}$, but all water is flowing down the river which is defined as the x_1 direction of a little 123 system. The equal arrows are perhaps misleading because the

velocity is not the same at all depths in this channel (at least I don't think it is). But all flow at any point in the river is in the down-river x_1 direction. Perhaps the top line is a free surface where river meets air.

The thing shown in this section is that the quantity $p/(\rho g) + z$ is a constant in any plane perp to the flow. So in such a plane, this relates p to z. This quantity is called the **piezometric head**, but I am not really ready to understand this term yet.

Example 6.7.3. Here we continue the previous example a bit. At surface point B we know $p = p_a$ so we then learn that $p_A = p_a + (\rho g h)\cos\theta = p_a + \rho g(z_B - z_A)$. But this result seems quite obvious, that the pressure is determined by the depth down from the surface in the gravity direction. But I guess the point here is that this "obvious fact" is true even though the water is moving in some arbitrary but unidirectional manner (and could even be accelerating).

6.8 Navier-Stokes for incompressible fluids in cylindricals and sphericals (364).

I verified every detail of cylindrical, and the first of the sphericals. Here is what you have to do:

(1) look up $-(1/\rho)\nabla p$ in each system
(2) look up $(\nabla \mathbf{v})\mathbf{v}$ in each system
(3) look up $(\mu/\rho)\ \nabla^2\mathbf{v}$ in each system
(4) look up div **v** in each system (yes, div **v** = 0 in any system if incompressible)

Luckily for me, I have done all this work in tensor doc. I know how to compute these things in ANY curvilinear system in fact. We are in the framework of vector **v** expanded on the unit vectors as usual.

These are amazingly complex equations! The $(\nabla \mathbf{v})\mathbf{v}$ part creates 4 to 5 terms all by itself, and then the $\nabla^2\mathbf{v}$ makes its usual huge mess of 7 to 9 terms. Obviously we are going to have to look for "simple situations" (motifs) where most terms go away and we can understand something. And this is all just for incompressible flow! Ouch!

6.9 Boundary Conditions (365).

It is experimentally found that for all fluids, the fluid is at rest with respect to any hard boundary (the **non-slip** condition). This could be a river bottom or river wall, for example. You might think that some tangential velocity might occur at a boundary, but no. Perhaps from $T_{12} = \mu\ \partial_2 v_1$ you would have a discontinuity in $\partial_2 v_1$ if tangential velocity were allowed (since I guess it is 0 just inside the boundary), and then T_{12} has a kink, but perhaps you cannot have that in the physical world if you look at the micro scale model.

6.10 Streamline vs Pathline, Steady vs Unsteady, Laminar vs Turbulent (365).

In **steady flow**, the velocity field is constant in time, so $\mathbf{v}(\mathbf{x})$ and not $\mathbf{v}(\mathbf{x},t)$. A **streamline** (at time t) in any flow is a path tangent to the velocity field at all points along the streamline. In a steady flow, the streamlines are all frozen in place. Particles just flow along the streamlines in this case. The streamlines of course can be curved, and the flowing particles will have acceleration which varies with time and position as they are held to the streamline as they flow. So steady flow does not imply that everything

does not depend on time, since we just showed that acceleration of a particle (and of course the velocity) can vary in time as that particle flows along the streamline.

In **unsteady flow**, the streamlines move in time and we have some general field **v**(**x**,t). I think that at any instant in time, a particle follows a streamline. But if the streamline is itself moving in some way, the particle follows a path which doesn't match any streamline at any time, it is just some path that you can deduce by integration. This thing is called a **pathline**.

So in steady flow, a pathline is a streamline since the streamlines are frozen.

To trace out a pathline on a surface, you put in one **reflector** and do a long time lapse on it.
To find streamlines, you put in a plane of reflector powder on the surface and do a short time lapse.

In **laminar flow**, layers don't mix even if going at different velocities. This was tested by Reynolds using little threads of dye in a fluid. This is true only for low "Reynolds Number" situations (yet to come), and in general laminar applies when things are relatively slow and relaxed and not violent. The other side of this coin is **turbulent flow**. Somehow that nonlinear term in the N-S equation is associated with such flow.

6.11 Plane Couette Flow (368). " cou-et' "

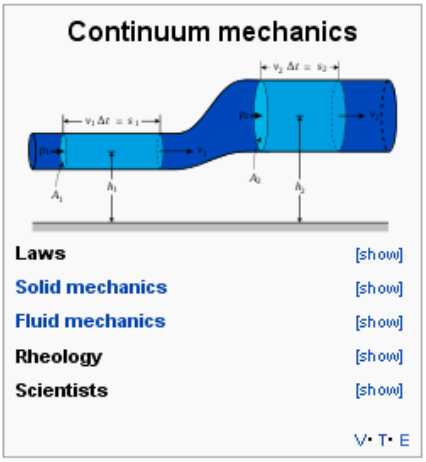
The Cartesian NS equations are top page 361. The plan is to identify simple flow situations in which most of the terms are zero and you describe the flow for some simple prototype situation. This section is the first of quite a few simple flow situations Lai is going to present. As usual, you can make an ansatz concerning the solution, and then show that it is a viable one -- that is, it solves the NS and continuity.

But first, let us read a little wiki. The first entry is on "rheology".

Rheology

From Wikipedia, the free encyclopedia

Rheology (/riːˈɒlədʒi/) is the study of the flow of matter, primarily in the liquid state, but also as 'soft solids' or solids under conditions in which they respond with plastic flow rather than deforming elastically in response to an applied force.[1] It applies to substances which have a complex molecular structure, such as muds, sludges, suspensions, polymers and other glass formers (e.g. silicates), as well as many foods and additives, bodily fluids (e.g. blood) and other biological materials.



Newtonian fluids can be characterized by a single coefficient of viscosity for a specific temperature. Although this viscosity will change with temperature, it does not change with the flow rate or strain rate. Only a small group of fluids exhibit such constant viscosity, and they are known as Newtonian fluids. But for a large class of fluids, the viscosity change with the strain rate (or relative velocity of flow) and are called non-Newtonian fluids. Rheology generally accounts for the behaviour of non-Newtonian fluids, by characterizing the minimum number of functions that are needed to relate stresses with rate of change of strains or strain rates. For example, ketchup can have its viscosity reduced by shaking (or other forms of mechanical agitation, where the relative movement of different layers in the material actually causes the reduction in viscosity) but water cannot. Ketchup is a shear thinning material, as an increase in relative velocity caused a reduction in viscosity, while some other non-Newtonian materials show the opposite behaviour: viscosity going up with relative deformation, which are called shear thickening or dilatant materials. Since Sir Isaac Newton originated the concept of viscosity, the study of liquids with strain rate dependent viscosity is also often called *Non-Newtonian fluid mechanics*.[1]

The term *rheology* was coined by Eugene C. Bingham, a professor at Lafayette College, in 1920, from a suggestion by a colleague, Markus Reiner.[2][3] The term was inspired by the aphorism of Simplicius (often misattributed to Heraclitus), *panta rei,* "everything flows"[4]

The experimental characterization of a material's rheological behaviour is known as *rheometry*, although the term *rheology* is frequently used synonymously with rheometry, particularly by experimentalists. Theoretical aspects of rheology are the relation of the flow/deformation behaviour of material and its internal structure (e.g., the orientation and elongation of polymer molecules), and the flow/deformation behaviour of materials that cannot be described by classical fluid mechanics or elasticity.

Notice certain terms here: muds, sludges, polymers. Then Newtonian fluids as a small class in which the viscosity μ is not a function of rate of deformation D (which wiki calls "strain rate", a nice term). Nice comment on ketchup. Newton's credit is given, and origin of the rheology word.

Now regarding Mr. Couette (1858-1943): French, work done circa 1890 at age ~30, only child:

> Couette is best known for his contributions to rheology and the theory of fluid flow. He designed a concentric cylinder viscometer that he used to accurately measure the viscosity of fluids. The laminar flow observed in the gap between the two cylinders is known as Couette flow. He studied the boundary conditions of a fluid and showed that the "no slip" condition was satisfied for the fluids and wall materials tested.

Feynman Vol II section 41 is on fluid flow and 41-6 is on Couette flow. I will read it soon.

So now Couette flow. We start with a horizontal "river" that is infinitely wide, d units deep with origin on the bottom which bottom is at rest. The top surface is a solid plate moving to the right (x_1) at velocity v_0 . What do the streamlines look like? Lai implies that this is meant to be a "steady flow" situation, so no

time dependence of the velocity field v. He first makes the ansatz that $\mathbf{v} = v_1\hat{\mathbf{x}}_1$ (= $v_1\mathbf{e}_1$) . Then continuity says $\partial_1 v_1 = 0$ since the other two components vanish. At this point we then have $v_1(x_2,x_3)$ [no time dependence]. But then we are really treating this as a 2D problem where things are constant in cross sections parallel to the plane of paper (slices along the river), and x_3 is this infinite direction out of paper, so we have then $v_1(x_2)$. Then the complicated NSE's become just

$0 = \rho B_1 - \partial_1 p + \mu \partial_2^2 v_1$
$0 = \rho B_2 - \partial_2 p$
$0 = \rho B_3 - \partial_3 p$

We will use these in the next section. But for Couette, we make the further ansatz that p is constant going down the river x_1, and we also set $\mathbf{B} = 0$, ignoring gravity. Then we have

$0 = \mu \partial_2^2 v_1$
$0 = - \partial_2 p$
$0 = - \partial_3 p$

This says p = constant, and the first says that $v_1 = Ax_2 + B$. Applying the BC's we get B = 0 and

$v_1 = (v_0/d)x_2$

The upshot is that you have a set of layers, and going down from the top, each layer is slower than the one above. The effect of the bottom being non-moving is felt going up by the friction of each layer against the next, called viscosity. The velocity field is as simple as it could possibly be, linear top to bottom.

Question: What is T and D for this Couette flow?

$D_{12} = 1/2 * \partial_2 v_1 = 1/2 * (v_0/d)$

$T_{12} = 2\mu D_{12} = \mu (v_0/d)$

The rate of deformation is constant at all x_2 levels in the flow and so is the shear force T_{12}. Therefore, both the upper and lower plates exert a shear force T_{12} on the fluid, same but opposite traction. The upper traction drives fluid to the right, lower tries to stop the flow.

At the moving plate there is stress T_{12} pushing against the plate. This means a force on the plate, and a force from the plate on the fluid, and Fv = power, so work is being done on the fluid, it must be heating up! Lai has not mentioned this yet (but he will).

Since the streamlines are all "to the right", if you were to put some kind of non-diffusing dye in one layer, it would stay in that layer. There is no mixing of layers, but I know that diffusion would cause mixing, so we have to say no mixing if we ignore diffusion?

So why doesn't NS "know about" diffusion? If you look at just the time and ∇^2 terms, it seems that diffusion is "in there" already, but the Couette case above shows no diffusion somehow. Lai does not talk about diffusion at all (1 page) so I will have to wait on this questions.

6.12 Plane Poiseuille Flow (368). // pwa-zoy'-ee

Functional form is same ansatz as above, but the geometry is different. In Couette it was the moving top plate that gave rise to the **v** profile. But now both top and bottom are non-moving plates. To get some flow, we have to have some $\partial_1 p$ as a pressure driving force going down river. The river at this point is still horizontal. Think of it as a wide (out of paper) waveguide or duct of height 2b, origin now at center of the height. We still have **B** = 0, so NS now says

$0 = \partial_1 p + \mu \partial_2^2 v_1$ (*)
$0 = \partial_2 p$
$0 = \partial_3 p$

So $p = p(x_1)$ only, and as before $v_1 = v_1(x_2)$ only. We now have two unknowns to solve for, p and v_1. So we now make the next ansatz which is that $p = -\alpha x_1 + p_o$ so that p decreases moving to the right, and then $\partial_1 p = -\alpha$ = constant. Then the first NS above says

$0 = -\alpha + \mu \partial_2^2 v_1 \quad \Rightarrow \quad \partial_2^2 v_1 = \alpha/\mu \quad \Rightarrow v_1 = A x_2^2 + B x_2 + C \quad \Rightarrow \partial_2^2 v_1 = 2A$

and of course at once $2A = \alpha/\mu$ so $A = \alpha/2\mu$. The top and bottom BC then say

$0 = Ab^2 + Bb + C$
$0 = Ab^2 - Bb + C$

The difference says B = 0, the sum says $Ab^2 = C$ so we have

$v_1 = A(x_2^2 - b^2) = (\alpha/2\mu)(x_2^2 - b^2)$. $\partial_2 v_1 = (\alpha/\mu)\, x_2$

Thus, we arrive at a **parabolic shape** as shown page 369. The free parameter is α, the driving force in effect. Increase that and the quadratic nose becomes larger and more pointed in mid duct.

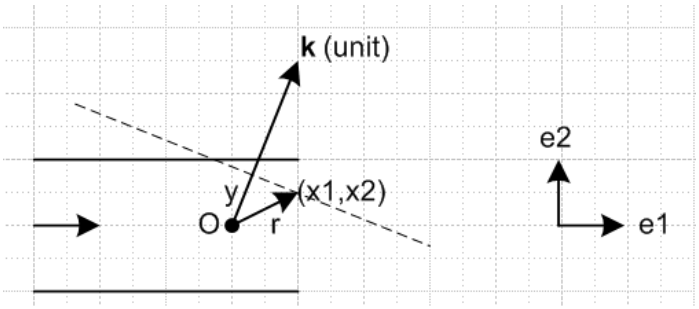
The v_1 at the max point is just $v_{1,max} = (\alpha/2\mu)b^2$.

Lai shows at this point that Q, the total flow rate per unit cross section dx_3 and average velocity is

$Q = (\alpha/\mu)(2/3)b^3$ $\bar{v} = (\alpha/\mu)(1/3)b^2 = 1/3$ of v_{max}.

So you get the idea for the first time that as viscosity μ increases, your throughput decreases linearly. Everything is inversely linear to μ here.

On page 370 we add in the effect of gravity (-**k** direction) AND we allow the duct to be tilted relative to the horizontal. No picture is drawn, but I made one below. Lai shows that we get the same three equations (*) above, but with the replacement $p \rightarrow p + \rho g y$ where y is the obvious vertical coordinate pointing down.



So call this $P = p+\rho gy$. Think of this as an adjusted pressure

$$P = p - [-\rho gy] \quad = p + \rho gy$$

where P is the solution to the previous problem with ansatz $P = -\alpha x_1 + P_0$ which decreases as you go along the duct, providing the driving force.

Now back to the three equations now in P. We assume that $\partial_1 P = -\alpha$ and go through the same steps as above and we will get exactly the same v_1 result! This seems a little amazing. The velocity profile in the duct stays the same regardless of how the duct is rotated, it can even be vertical.

Suppose it is vertical so $x_1 = y$. Then what is the actual pressure p doing?

$$p = P - \rho gy = (-\alpha y + P_0) - \rho gy \quad = P_0 \; - \alpha y - \rho gy$$

Due to the second term, the pressure increases as you go downward to larger negative y, as you would expect. But in addition to this, we get an extra $-\alpha y$ which drives the flow up with $v_{1,max} = (\alpha/2\mu)b^2$. If we set $\alpha = 0$, there is no flow, the velocity profile goes to 0, $p = P_0 - \rho gy$ as in any static situation.

Question: What is T and D for this Plane Poiseuille flow?

$$D_{12} = 1/2 * \partial_2 v_1 \; = \; 1/2 * \; (\alpha/\mu)\, x_2$$

$$T_{12} = 2\mu D_{12} = (1/\alpha)\, x_2$$

Now since origin at center, we have at the plates $T_{12} = \pm (b/\alpha)$. Each plate applies the same backwards dragging surface traction on the flow trying to slow the flow down. If you just injected such a flow into your duct without securing the duct, the duct would move to the right, dragged by the water.

6.13 Hagen-Poiseuille Flow (371).

Now we come to the much more practical situation of pushing a flow through a cylindrical pipe, whereas the plane case above perhaps would apply to a special machine where water is pumped between parallel plates for some reason. Everyone wants to know about water through a pipe! We have all seen it a million times, especially with a transparent plastic tube. Obviously we are going to do cylindricals here for our NS equation.

I verified all the equations in the text except this one:

Derivation of 6.13.10:

```
v := (a/(4*u))*(d^2/4 - r^2);
```

$$v := \frac{1}{4}\frac{a\left(\frac{1}{4}d^2 - r^2\right)}{u}$$

```
Int(v*2*Pi*r,r = 0..d/2);
```

$$\int_0^{\frac{1}{2}d} \frac{1}{2}\frac{a\left(\frac{1}{4}d^2 - r^2\right)\pi r}{u}\,dr$$

```
int(v*2*Pi*r,r = 0..d/2);
```

$$\frac{1}{128}\frac{a\,\pi\,d^4}{u}$$

The integral above is $\int dA\ v$ where $dA = 2\pi r dr$ for a ring shaped dA. Then

$\bar{v} = (1/A) \int dA\ v = (4/\pi d^2) *$ the above $= \alpha d^2/32\mu$ as claimed

Interesting that it turns out that $\bar{v} = 1/2\ v_{max}$ exactly for this flow.

The claim is now made that you can repeat the previous argument about the effect of gravity, and you get the exact same conclusion. The pressure ends up being

$p = P - \rho gy = (-\alpha y + P_0) - \rho gy = P_0 - \alpha y - \rho gy$

and the solution for velocity profile stays the same. I could verify this claim, but it seems so reasonable that I won't. So now we know about laminar flow through a tube! This is something I never knew before. Note that

$Q = (\pi/128)(\alpha/\mu)\ d^4 \qquad \alpha = -\partial_z p$

If you had a pump situation where you could only produce a fixed α, then doubling the tube diameter increases the flow by 16 times! I wonder if this describes the flow in my PVC sprinkler pipes? Maybe I now know all about sprinkler systems! An easy case would be PVC pipe of length L, driving pressure p_1 and perhaps final pressure $p_2 < p_1$ right at the head. Then $\alpha = (p_1 - p_2)/L$ is the pressure gradient in the pipe. For my PVC, d ≈ 2 cm. For water μ = 1 cgs unit says http://en.wikipedia.org/wiki/Viscosity . Suppose the pipe is 10 meters = L long and Δp = 50 psi just for example. The cgs unit of pressure is a dyne/cm^2 = 0.1 Pa [I think the cgs unit is called a **centipoise**]

1.0	psi	=	68947.57293	dyn/cm2

So I then have $\alpha = 50 * 69{,}000 / 1000$ cm $= 50*69 = 3450$ cgs units. Then

$$Q = (\pi/128)(\alpha/\mu)\, d^4 = (\pi/128)(3450/1)\, 16 \text{ cm}^3/\text{sec} = 1355 \text{ cm}^3/\text{sec}$$

1 gallon[US, liquid] = 3 785.411 784 cubic centimeter

so then $Q = 1355/3785 = 0.36$ gal/sec = 21 gallons/minute

Meanwhile, here are some measurements I did some years ago,

Water Flow Tests Done July 24, 2005 at 5 PM.						
cu ft	cu ft/min	What?	cu ft/hr	gal/min	gal/hr	HCF/hr
17166.7						
17166.3	0.4	shower	24.00	3.0	180	0.24
17165.8	0.5	hose thin	30.00	3.7	224	0.3
17165.2	0.6	hose full	36.00	4.5	269	0.36
17161.9	3.3	front	198.00	24.7	1481	1.98
17158.6	3.3	rear	198.00	24.7	1481	1.98

I found then that my entire front system with many heads at that time did 25 gal/min which is certainly in the same ball park. Now I don't really know if the flow is laminar or not.

6.14 Plane Couette Flow with two layers, gravity on, no tilt (372).

This might be oil on top of water in your duct river with a driving top plate. But the real purpose here is to study the boundary between two laminar layers of different fluids with labels 1 and 2. The new features here are: (1) v_1 is continuous at the boundary. This is a form of the "no slip" condition, but applied between two moving layers. (2) the stress tensor components T_{ij} are also continuous across the boundary, which can be interpreted in this case to say that the total force on a straddling cube must be 0 since it is not accelerating. We end up with four BC's (I numbered them on the right), and we have four unknowns $A_{1,2}$ and $B_{1,2}$ in our two v_1 equations, and the problem has a final solution on page 374, nothing too fancy happening here. The layers have thickness $b_{1,2}$ and $\mu_{1,2}$ and $\rho_{1,2}$ as well.

This problem did allow for some pressure on the two sides p_1 and p_2 and we start off with equations for these as well as the two v_1's, see 6.14.5. We have two more unknowns here $C_{1,2}$. By setting the pressures equal at the boundary, we get $C_1 = C_2$ and this is called p_0, the pressure on the boundary, whatever it might be. In each layer, pressure varies vertically as expected. In this problem, the flow duct is horizontal, we are not doing the tilt thing here, and gravity is on.

Again, at least in this solution, there is no mixing at all between the two layers. You do wonder what perturbations might do for this simple solution.

6.15 Couette Flow (374).

The geometry now changes from an upper and lower infinite plane to two infinite cylinders which rotate relative to each other with the flow in between. You could think of the inner stopped and the outer moving, to make it look like the plane problem, but might as well be general. If outside cylinder rotates CW and inside is at rest or going some slower CW rate, we expect the **v** pattern shown in the page 374 figure. Cylindrical coordinates, NS equation, no gravity, orientation is a non-issue.

From the usual symmetry arguments, we have only $v_\theta(r)$, the continuity check is trivial, and then we write the NS and write its most general solution. This looks like some radial equation I have encountered somewhere, but I can't place it. I verify that the solution works and since there are two independent solutions, that is it. This is already a homo equation, so no more. The BC's are that fluid velocity not slip on the inner and outer boundaries, and this gives constants A and B and then the solution. In the special case that $\Omega_1 = \Omega_2$ the solution is just $v_\theta = \Omega_1 r$ which is rigid body motion of the entire shell of fluid.

Since there is shear stress on the cylinders from the fluid, you could compute the torque acting on each cylinder to keep this thing going (again, the fluid heats up somehow). I did not check M but could. He uses the phrase "viscous stress" which seems appropriate.

6.16 Trying to drive a transverse wave into a fluid (375).

This is our first time-dependent example. We have $v_1(x_2,t)$ only, continuity OK, NS is shown, and solution to that is shown as well. We try a solution

$v = \exp[i(kx-\omega t)]$ $\quad\quad$ $\partial_t v = -i\omega \exp[i(kx-\omega t)]$

$\partial_x = ik \exp[i(kx-\omega t)]$ $\quad\quad$ $\partial^2_x = -k^2 \exp[i(kx-\omega t)]$

so get $\quad i\omega\rho \exp[i(kx-\omega t)] = \mu k^2 \exp[i(kx-\omega t)]$

$i\omega\rho = \mu k^2$ $\quad\quad$ $k = (i\omega\rho/\mu)^{1/2} = (\omega\rho/\mu)^{1/2} [1 + i]/\sqrt{2} = \beta + i\beta$

Solution is then

$v = \exp[i(kx-\omega t)] = \exp[i((\alpha + i\beta)x-\omega t)] = \exp[i(\alpha x-\omega t)] \exp[-\beta x]$

$\mathrm{Re}[v] = \cos(\beta x_2-\omega t) \exp[-\beta x_2]$ $\quad\quad$ $\beta = (\omega\rho/2\mu)^{1/2}$ $\quad\quad$ $\beta = \alpha$

and this is the solution shown of this "heat equation" PDE. We always expect damping with distance. At $x_2 = 0$ this is $\cos(\omega t)$ so this is the metal plate drive (assuming no slip!). The solution indicates a wave with $\mathbf{v} = v_1 \hat{\mathbf{x}}_1$ (so it is transverse) and it damps out fast going into the fluid. It is travelling in the x_2 direction as we can see. Note that $v = \beta/\omega$ and $\beta\lambda = 2\pi \Rightarrow \lambda = 2\pi/\beta$ and $\beta = 2\pi/\lambda$ so we have

$\exp[-\beta x_2] = \exp[-2\pi x_2/\lambda]$ $\quad\quad$ in one wave it is at $e^{-2\pi} = 1/535$

so it is pretty much totally gone within a single wavelength of propagation. Not a very good wave carrier in this transverse mode.

6.17 Dissipation functions for Newtonian fluids (376).

We had the general formula from Chapter 4 as in 6.17.2, and now we just throw in our fluid T_{ij} and we find that

[note added 9.14.12: $P_s = tr(T^T \nabla v)$; $= tr(TD)$ if T symmetric; now $T_{ij} = -p\delta_{ij} + \lambda\Delta\delta_{ij} + 2\mu D_{ij}$ so $tr(TD) = -ptr(D) + \lambda\Delta tr(D) + 2\mu tr(DD) = -p\Delta + \lambda\Delta^2 + 2\mu tr(D^2)$; for incompressible, get $P_s = 2\mu tr(D^2)$.]

$P_s = 2\mu\ tr(D^2)$ $= \Phi_{inc}$ // incompressible $tr(D) = \Delta = div\mathbf{v} = 0$

$P_s = 2\mu\ tr(D^2) + \lambda\ [tr(D)]^2 - p\ tr(D)$ $= \Phi - p\ tr(D)$ // compressible

Clearly - p tr(D) is the work of the pressure compressing a particle, whereas the other terms involve μ and λ. So

$\Phi_{inc} = 2\mu\ tr(D^2)$

$\Phi = 2\mu\ tr(D^2) + \lambda\ [tr(D)]^2$

Since D is symmetric, it must be diagonalizable into D' where

$D'_{ij} = \delta_{ij}\lambda_i$ and $[D'^2]_{ij} = \delta_{ij}\lambda_i^2$

so

$tr(D) = \Sigma_i\lambda_i$ $tr(D^2) = \Sigma_i\lambda_i^2$

so

$\Phi = 2\mu\ tr(D^2) + \lambda\ [tr(D)]^2 = 2\mu\ \Sigma_i\lambda_i^2 + \lambda\ (\Sigma_i\lambda_i)^2$

But that is not really the form I want. So have to just brute force it:

First, Maple tells me $tr(D^2)$ as

```
d := matrix(3,3);
```

$$d := \text{array}(1 .. 3, 1 .. 3, [\ ])$$

```
sum(d[i,i],i=1..3);
```

$$d_{1,1} + d_{2,2} + d_{3,3}$$

```
d2 := evalm(d &* d);
```

$$d2 := \begin{bmatrix} d_{1,1}^2 + d_{1,2}d_{2,1} + d_{1,3}d_{3,1} & d_{1,1}d_{1,2} + d_{1,2}d_{2,2} + d_{1,3}d_{3,2} & d_{1,1}d_{1,3} + d_{1,2}d_{2,3} + d_{1,3}d_{3,3} \\ d_{2,1}d_{1,1} + d_{2,2}d_{2,1} + d_{2,3}d_{3,1} & d_{1,2}d_{2,1} + d_{2,2}^2 + d_{2,3}d_{3,2} & d_{2,1}d_{1,3} + d_{2,2}d_{2,3} + d_{2,3}d_{3,3} \\ d_{3,1}d_{1,1} + d_{3,2}d_{2,1} + d_{3,3}d_{3,1} & d_{3,1}d_{1,2} + d_{3,2}d_{2,2} + d_{3,3}d_{3,2} & d_{1,3}d_{3,1} + d_{2,3}d_{3,2} + d_{3,3}^2 \end{bmatrix}$$

```
sum(d2[i,i],i=1..3);
```

$$d_{1,1}^2 + 2\,d_{1,2}d_{2,1} + 2\,d_{1,3}d_{3,1} + d_{2,2}^2 + 2\,d_{2,3}d_{3,2} + d_{3,3}^2$$

which agrees with 6.17.6.

Now I will use my $\Phi = 2\mu\ tr(D^2) + \lambda\ [tr(D)]^2$ to verify 6.17.10 in Maple. Here it is

f = his version of Φ in 6.17.10
g = my version of Φ $\Phi = 2\mu\, tr(D^2) + \lambda\, [tr(D)]^2$

Here I show that f-g = 0:

```
f1 := (a + 2*b/3)*(D11 + D22 + D33)^2;
```

$$f1 := \left(a + \frac{2}{3}b\right)(D11 + D22 + D33)^2$$

```
f2 := (2*b/3)* ( (D11-D22)^2 + (D11-D33)^2 + (D22-D33)^2);
```

$$f2 := \frac{2}{3}b\,((D11 - D22)^2 + (D11 - D33)^2 + (D22 - D33)^2)$$

```
f3 := 4*b*(D12^2 + D13^2 + D23^2);
```

$$f3 := 4\,b\,(D12^2 + D13^2 + D23^2)$$

```
g1 := 2*b*(D11^2+D22^2+D33^2 + 2*D12^2+2*D13^2+2*D23^2);
```

$$g1 := 2\,b\,(D11^2 + D22^2 + D33^2 + 2\,D12^2 + 2\,D13^2 + 2\,D23^2)$$

```
g2 := a*(D11+D22+D33)^2;
```

$$g2 := a\,(D11 + D22 + D33)^2$$

```
f := f1+f2+f3;
```

$$f := \left(a + \frac{2}{3}b\right)(D11 + D22 + D33)^2 + \frac{2}{3}b\,((D11 - D22)^2 + (D11 - D33)^2 + (D22 - D33)^2) + 4\,b\,(D12^2 + D13^2 + D23^2)$$

```
g := g1+g2;
```

$$g := 2\,b\,(D11^2 + D22^2 + D33^2 + 2\,D12^2 + 2\,D13^2 + 2\,D23^2) + a\,(D11 + D22 + D33)^2$$

```
expand(f-g);
```

$$0$$

.

Example 6.17.1. Assume we have shear flow as in plane Couette flow where k = (v_0/d) sec^{-1}. Suppose v_0 = 1 m/sec and d = 1 m so that k = 1 sec^{-1}. River is 1 m deep. Now

$D_{12} = (1/2)\,\partial_2 v_1 = (1/2)$ k sec^{-1}.

Then we get

$\Phi_{inc} = 2\mu * 2\, D_{12}^2 = 4\mu\,(k/2)^2 = \mu k^2$

Now we are told for the first time that μ = 1 mPa-sec for water = 10^{-3} Pa-sec which is the SI unit. Then

Φ_{inc} = 10^{-3} Pa-sec * 1 sec^{-2} = 10^{-3} Pa/sec = 10^{-3} Nt/m^2/sec

= 10^{-3} Nt-m /sec/m^3 = 10^{-3} watts/m^3 = 1 mW/m^3

So if I examine a huge 3 foot cubed box of water doing this, the entire thing is only heating up at a rate f 1 mW. That seems awfully low. I guess the idea is that μ is small for water. I think the lesson here is that you won't heat water very fast by stirring it.

6.18 Energy Equation for Newtonian fluids (378).

We derived this in an earlier section p 184. The equation says

rate of increase in stored internal energy

= stress power applied - heat loss rate at boundaries + internal heat sources

If the heat flow is assumed linear conduction only (Fourier), then the heat loss term becomes $\nabla^2\Theta$ and I guess this represents heat coming in from the boundaries somehow. This is a "heat equation" type term I saw many times in Stak.

Next, for incompressible we throw in the stress power as Φ_{inc}. The claim is that this term can usually be neglected (my example above supports this notion) based on the argument that heat coming in from boundaries typically swamps the minute viscous heat generation. Finally, $u = c\Theta$ says internal energy is linear in temperature with heat capacity as constant, and then we end up with 6.18.4 which is the famous heat equation, *except* in place of ∂_t we have D_t. This did not appear in Stak. Why not? Well, in Stak we did heat flow in something like a solid material where there was no velocity field **v** so then $\nabla\mathbf{v} = 0$. But here we have continuum particles flowing around, a whole different situation. One could spend a lot of time pondering this I think, but let's read on for now.

Example 6.18.1. Here we assume the velocity flow pattern of plane Couette, something specific and simple. And we assume that Θ is steady state. The second D_t term is $(\nabla v)v$ or $\mathbf{v} \bullet \nabla\Theta$. In our example, **v** is to the right, and $\nabla\Theta$ is up (by ansatz $\Theta(x_2)$) and then $\mathbf{v} \bullet \nabla\Theta = 0$. Since steady $\partial_t\Theta = 0$ as well, so yes, the full $D_t\Theta = 0$ so LHS = 0. The energy equation is then obtained (not ignoring the small term) and we get the solution as shown p 379. I now see what that line is. If we want to ignore the Φ_{inc} term, then we can think of κ as being very large (see 6.18.3). In this case we ignore two of the solution terms and then we get a simple linear Θ and that is what Lai plotted in the figure! Notice that heat is flowing down through the layers here, even though the layers don't mix. This is just how it works in a solid, the layers don't mix there either. There is major heat action at the plate boundaries, so that is the justification for ignoring the small viscous heat generation term. The total heat coming in from the top matches the total heat leaving at the bottom. A good example, Lai skips the words.

6.19 The Vorticity Vector (379).

Now the fun begins. I went off at this point and wrote "the spin tensor and related stuff" in Lai folder to clarify many things. I had to break a few log jams to get this section flowing!

Lai's first item here is the following theorem:

Theorem 1: if you choose your axes as the principle axes of D, and if you make **n** be any of the three unit vectors in that system, you find that

$D_t\mathbf{n} = W\mathbf{n}$ // a separate always-true fact is $W\mathbf{n} = \boldsymbol{\omega} \times \mathbf{n}$

[Note added 9.14.12: The above then says $D_t\mathbf{n}_i = \boldsymbol{\omega} \times \mathbf{n}_i$ for each normal vector in the principle axis system. Think of this principle axis system as rotating Frame S'. Compare to $(d\mathbf{e}'_n/dt)_S = \boldsymbol{\omega} \times \mathbf{e}'_n$ which is

1.25 of frames doc. The basis vectors do in fact rotate together in frames doc according to **ω**. But here we only get this Shearing effect when Frame S' is that special principle axis system.]

I derive this fact in those spin tensor notes and so does Lai p 379-380. If **n** are not the principle axes, then the equation is much more complicated,

$$D_t\mathbf{n} = W\mathbf{n} + D\mathbf{n} - [D_t(ds)/ds]\,\mathbf{n} \qquad (*)$$

Implication #1: The three unit normalized eigenvectors $\mathbf{n}_i$ of D all rotate according to the same **ω** (W) rotation. So the axes all rotate together (locked-hands George Shearing) causing the cube to rotate. Meanwhile, we have

$$[D_t(ds)/ds] = \mathbf{n}_i \bullet D\mathbf{n}_i = \lambda_i$$

so each edge of the cube is changing length. This is what Lai says in A on page 380. What the cube does *not* do is shear!! Remember that the three principal axes as EV's are orthogonal. In the spin tensor notes Appendix A I show an example of how shear "goes away" in the principal axis system.

So if you look specifically at a sugar cube whose edges are principle axes, it rotates and scales edges, but it does not shear.

Note: in the principle axis system, the last two terms in (*) above cancel.

Theorem #2. On page 17 of my Chap 1,2 Lai notes, I show that the dual vector **ω** which is associated with the antisymmetric tensor $W = (\nabla\mathbf{v})^A$ is this

$$\boldsymbol{\omega} = (1/2)\,\text{curl}\,\mathbf{v}\ .$$

Proof:

$$W_{jk} \equiv (\nabla\mathbf{v})^A_{jk} = (1/2)\,[\,\partial_k v_j - \partial_j v_k]$$

Then

$$2\,\omega_i = -\,\varepsilon_{ijk}W_{jk} = -\,\varepsilon_{ijk}\,(1/2)\,[\,\partial_k v_j - \partial_j v_k] = \varepsilon_{ijk}\partial_j v_k = [\text{curl v}\,]_i \qquad // = \zeta_i$$

(6.19.10)

$$\Rightarrow\ 2\boldsymbol{\omega} = \text{curl}\,\mathbf{v}.$$

Now consider our general situation :

$$D_t(d\mathbf{x}) = (\nabla\mathbf{v})d\mathbf{x} = Dd\mathbf{x} + Wd\mathbf{x} = Dd\mathbf{x} + \boldsymbol{\omega} \times d\mathbf{x}$$

[Note added 9.14.12: Think of d**x** as "a little dumbbell". The last term means, as I now understand from frames doc Fig 1.3, that due to this term, the dumbbell d**x** as a vector rotates in a "conical motion", so the dumbbell d**x** is "spinning" around the ω axis on a cone. This spinning action is the W dx term, and that is why W is called the spin tensor, very simple.]

If the velocity field has no curl, then W = 0 and we simply have

$$D_t(d\mathbf{x}) = Dd\mathbf{x}$$

In this case, the little sugar cube does not rotate, it shears and stretches and nothing else. *If in addition* we use the principal axes, then when W = 0 we also have

$$D_t\mathbf{n} = 0$$

which says the principle axes are frozen in time and there is no shearing. In this case, in these axes, since D' is diagonal, a sugar cube does not rotate, does not shear, but just stretches its axes in some manner determined by the eigenvalues of D'. Surely this must be a great simplification of things -- and this then is irrotational flow.

Definitions:

vorticity vector = $\boldsymbol{\xi} = 2\boldsymbol{\omega}$ = curl v $\boldsymbol{\omega}$ associated with $(\nabla\mathbf{v})^{\mathbf{A}} = W$

vorticity tensor = 2W

Lai then writes the $\boldsymbol{\xi}$ vector in Cartesians, cylindricals, and sphericals.

Example 6.19.1. In my spin tensor doc appendix, I show how simple shear motion when viewed in the principle axis system consists of a combination of rotation about the origin and edge stretching, where ω describes the rotation part. I showed that changing frames of reference (and changing accordingly the reference cube) removes the shearing action. Here we find the $\boldsymbol{\xi} = 2\boldsymbol{\omega} = -k\,\hat{\mathbf{x}}_3$ which agrees exactly with my picture.

Example PL 1: Plane Couette Flow. $v_1 = k\,x_2$ with $k = v_0/d$, so this is the same as the above example.

Example PL 2: Plane Poiseuille Flow. Has $v_1 = (\alpha/2\mu)(b^2 - x_2^2) = v_1(x_2)$, so this is similar to the previous cases in nature and we have

$$\boldsymbol{\xi} = \text{curl}\,\mathbf{v} = -\partial v_1/\partial_{\mathbf{x}2} = +(\alpha/\mu)x_2$$

You can see this exists everywhere but is zero in midstream.

Example 6.19.2 Couette Flow (space between two rotating cylinders), we find some curl pointing in the z direction.

Fact: whenever you see the velocity vectors piled up with different lengths, you know there is some curl **v** there because a small closed line integral does not vanish, very simple. So "rivers" are not irrotational flow (I had a misconception).

6.20 Irrotational Flow (381).

This section shows what I already know, that when curl $\mathbf{v} = 0$, you can represent $\mathbf{v} = -\nabla\varphi$ where φ is a potential solving the Laplace equation. (he uses the same minus sign as in electrostatics). Notice that this satisfies continuity div $\mathbf{v} = 0$ for incompressible situation since ρ = constant, and in fact that is there the Laplace equation comes from.

The question arises: does this mean for any Laplace φ there is some irrotational flow situation?

6.21 Irrotational Flow for inviscid impressible (382).

The discussion starts with **inviscid** fluids which for the first time in my life I understand to be those which have viscosity $\mu = 0$ (inviscid means non-viscous just as incompressible means non-compressible). But inviscid also assumes that bulk viscosity $k = 0$ and that in turn means that $\lambda = 0$, and the upshot is that you end up with only $T_{ij} = -p\delta_{ij}$. Now the full NS equation is this (see 6.7 of meta notes)

$$\rho [\partial_t \mathbf{v} + (\nabla \mathbf{v})\mathbf{v}] = \rho \mathbf{B} - \nabla p + \mu \nabla^2 \mathbf{v} + (\mu/3 + k) \nabla (\text{div } \mathbf{v})$$

and if you assume inviscid, then this reduces to

$$\rho [\partial_t \mathbf{v} + (\nabla \mathbf{v})\mathbf{v}] = \rho \mathbf{B} - \nabla p$$

which is 6.21.2. It is still non-linear due to the material derivative second term. This "inviscid NS" is called **Euler's equations of motion**, yet another Euler name attachment. Notice that once we assume inviscid, Euler is the result, and we do not require that div $\mathbf{v} = 0$. Thus, this Euler thing describes an inviscid fluid which can be either compressible or incompressible.

Note that Euler did fluids in 1757

Now we assume incompressible, meaning ρ = constant, and only then can you write Euler as in 6.21.6. If you then use the potential for v_i in the left term, this directly becomes 6.21.9 which is a pretty amazing equation which implies that

$$-\partial_t \varphi + v^2/2 + p/\rho + \Omega = f(t), \text{ and not } f(t,\mathbf{x}) \qquad \mathbf{v} = -\nabla\varphi$$

$$v^2/2 + p/\rho + \Omega = \text{constant} \qquad \text{// for steady flow} \qquad \Omega = gy \text{ for gravity.}$$

and I forgot to mention you also have to assume that the body force $\mathbf{B} = -\nabla\Omega$ which of course works fine in gravity, where a lot of watery problems live. The first equation above and its special case the second equation are known as "**the Bernoulli's equations".**

Example 6.21.1. A generic flow (383) Here we are given a particular potential φ in Cartesians that solves Laplace. We compute the velocity vector $\mathbf{v}$. We know it has no curl since it came from φ ! We are instructed to write the $v^2/2 + p/\rho + \Omega$ = constant at two points, and at one point many things are 0 so we end up getting a simple expression for p. A major comment in this example is that an inviscid fluid does **not respect the no-slip rule** at a boundary! You can have a sudden tangential velocity discontinuity at a wall surface! So you sort of lose a boundary condition when working with a liquid like this. This problem adds a planar boundary in the last step and shows that you have slip on this boundary.

Example 6.21.2. Hole in Can Problem (383). Really we have a lock-like reservoir with a full-width but small-height slot opening at the bottom, and the lock is infinitely wide, so flow is uniform in the cross section (no curl among the **v** vectors you would draw). That is what makes this an irrotational problem and what makes Bernoulli apply.

What is the speed of the water flowing out the slot? That is the question. You just write the second equation above at two points: the top of the reservoir ($v^2 = 0$), and in the bottom hole. At each place we have atmospheric pressure p_0 so the p/ρ terms cancel out between the two points, so that

$$[v^2/2 + gy]^{top} = [v^2/2 + gy]^{hole}$$

$$0 + gh = v^2/2 \quad \Rightarrow \quad v^2 = 2gh. \quad \text{// } \textbf{Torricelli's formula}$$

If you just dropped a particle of water by this height, you would get

$$1/2\ m\ v^2 = mgh$$

which gives the same answer, but Lai does not mention that right here.

I think I could do this problem using "regular physics", maybe a pipe with p_1 and p_2 at the two ends with planar flow **v** field (since no wall drag). The pressure is a linear function in this pipe and it accelerates a slice of fluid as it moves along the path. (just an idea)

Of course we are assuming water has no viscosity. I noted earlier from the heat made that it seems pretty low, and here is a great little table. I think acetone is maybe 0.3 cP.

MATERIAL	APPROXIMATE VISCOSITY (in centipoise)
Water @ 70 F	1 to 5
Blood or Kerosene	10
Anti-Freeze or Ethylene Glycol	15
Motor Oil SAE10 or Mazola Corn Oil	50 to 100
Motor Oil SAE30 or Maple Syrup	150 to 200
Motor Oil SAE40 or Castor Oil	250 to 500
Motor Oil SAE60 or Glycerin	1,000 to 2,000
Karo Corn Syrup or Honey	2,000 to 3,000
Blackstrap Molasses	5,000 to 10,000
Hershey Chocolate Syrup	10,000 to 25,000
Heinz Ketchup or French's Mustard	50,000 to 70,000
Tomato Paste or Peanut Butter	150,000 to 250,000
Crisco Shortening or Lard	1,000,000 to 2,000,000
Caulking Compound	5,000,000 to 10,000,000
Window Putty	100,000,000

So water being inviscid is often a great approximation. And incompressible as well.

Now when you do this thing with a hole in a coffee can, there will no doubt be some curl in **v** as you get near the hole, but it is probably a small amount and so this method still works pretty well. No comments on this.

6.22 Irrotational Flow and Navier-Stokes (384).

We already know that NS becomes Euler when you set $\mu = 0$. But now look at that vector Laplacian term which is present when $\mu \neq 0$:

$$\nabla^2(\mathbf{v}_i) = \partial_j{}^2 v_i = \partial_j{}^2[\, -\partial_i \varphi] = -\partial_i\, \partial_j{}^2 \varphi \;= -\partial_i\, \nabla^2 \varphi \;=\; 0\,.$$

Thus, for irrotational flow, this nasty NS term vanishes even if $\mu \neq 0$! So even for a viscous fluid, an irrotational flow solution to some problem is at least feasible. But there is a problem. If you have a boundary at which **v** = 0 (non-slip), then $v_{\mathbf{perp}}$ and $v_{\mathbf{tan}}$ must both vanish there. That means

$\partial\varphi/dt_i = 0$, tangential everywhere on boundary => φ = constant on boundary

$\partial\varphi/dn = 0$, normal to boundary

that means you have Dirichlet and Neumann BC's at the same time on the boundary, and we know this is overspecified and there won't be a solution except in a contrived case. Somehow the boundary is going to generate vorticity (well, you can see that just from the fact that it drags on the tangential velocity in the neighborhood of a boundary). So a real problem is not going to have irrotational flow everywhere. But maybe the vorticity will all be in some thin "boundary layer" and outside that you can have irrotational flow.

Example 6.22.1. Couette flow. Back on p 381 in Example 6.19.2 we wrote an expression for the vorticity inside the cylindrical fluid shell. Based on the radii, you can set the rotation rates of the inner and outer cylinder such that there is zero curl !!! You speed up the inside cylinder to be faster than the outside and that sort of offsets the arrow pattern shown on page 374 making all the arrows equal and then no curl. Wow, that is pretty cool. You can turn off the curl with a rheostat. You cannot do this unless both cylinders are turning, one cannot be at rest!

6.23 The Vorticity Transport Equation (385).

This is a PDE in the ζ vorticity vector, shown in 6.23.4. Lai derives this equation in 2 long pages of detail which I could easily follow, but I decided to roll my own derivation below.

Example 6.23.1. Lai's derivation of the ζ equation.

My derivation. Start with (everything you need is stated right here)

$\rho D_t\mathbf{v} = \rho\mathbf{B} - \nabla p + \mu\nabla^2\mathbf{v}$ Navier Stokes p 361

$\zeta = \nabla x\mathbf{v}$ definition of ζ p 380

$D_t\zeta = (\nabla\mathbf{v})\zeta + (\mu/\rho)\nabla^2\zeta$ desired equation for ζ p 386

$\nabla\bullet\mathbf{v} = 0$ because we are going to assume incompressible .

Now suppose I just start this way,

$D_t\zeta = \partial_t\zeta + (\nabla\zeta)\mathbf{v}$ // see Lai Ch 3 notes and comment in tensor doc

Then I have to show that

$$\partial_t\zeta + (\nabla\zeta)\mathbf{v} = (\nabla\mathbf{v})\zeta + (\mu/\rho)\nabla^2\zeta$$

I can insert $\zeta = \nabla x\mathbf{v}$ everywhere to get (the brute force approach!)

$$\partial_t\nabla x\mathbf{v} + (\nabla(\nabla x\mathbf{v}))\mathbf{v} = (\nabla\mathbf{v})(\nabla x\mathbf{v}) + (\mu/\rho)\nabla^2(\nabla x\mathbf{v})$$

or

$$\rho\partial_t\nabla x\mathbf{v} + \rho(\nabla(\nabla x\mathbf{v}))\mathbf{v} = \rho(\nabla\mathbf{v})(\nabla x\mathbf{v}) + \mu\nabla^2(\nabla x\mathbf{v})$$

or

$$\rho\nabla x(\partial_t \mathbf{v}) + \rho(\nabla(\nabla x \mathbf{v}))\mathbf{v} \quad = \quad \rho(\nabla \mathbf{v})(\nabla x \mathbf{v}) + (\nabla \times [\mu\nabla^2 \mathbf{v}]) \ . \qquad // \ \mu = \text{constant}$$

We can replace $\mu\nabla^2\mathbf{v}$ using the NS, but first process NS a bit:

$$\rho D_t \mathbf{v} = \rho \mathbf{B} - \nabla p + \mu\nabla^2 \mathbf{v}$$

$$\rho D_t \mathbf{v} = -\rho\nabla\Omega - \nabla p + \mu\nabla^2 \mathbf{v}$$

$$\mu\nabla^2 \mathbf{v} = \rho D_t \mathbf{v} + \nabla(\rho\Omega + p) \ .$$

Our problem child to show true is then this :

$$\rho\nabla x(\partial_t \mathbf{v}) + \rho(\nabla(\nabla x \mathbf{v}))\mathbf{v} \quad = \quad \rho(\nabla \mathbf{v})(\nabla x \mathbf{v}) + (\nabla \times [\rho D_t \mathbf{v} + \nabla(\rho\Omega + p)]).$$

The very last term has curl of grad which vanishes, so we now have to show

$$\rho\nabla x(\partial_t \mathbf{v}) + \rho(\nabla(\nabla x \mathbf{v}))\mathbf{v} \quad = \quad \rho(\nabla \mathbf{v})(\nabla x \mathbf{v}) + (\nabla x[\rho D_t \mathbf{v}]) \ .$$

We are assuming ρ = constant, so cancel it to get

$$\nabla x(\partial_t \mathbf{v}) + (\nabla(\nabla x \mathbf{v}))\mathbf{v} \quad = \quad (\nabla \mathbf{v})(\nabla x \mathbf{v}) + (\nabla x[D_t \mathbf{v}])$$

Now write

$$D_t v = \partial_t \mathbf{v} + (\nabla \mathbf{v})\mathbf{v}$$

and then we have to show

$$\nabla x(\partial_t \mathbf{v}) + (\nabla(\nabla x \mathbf{v}))\mathbf{v} \quad = \quad (\nabla \mathbf{v})(\nabla x \mathbf{v}) + (\nabla x[\partial_t \mathbf{v}]) + (\nabla x[(\nabla \mathbf{v})\mathbf{v}])$$

and so the ∂_t terms cancel, and *we then have to show*

$$(\nabla(\nabla x \mathbf{v}))\mathbf{v} \quad = \quad (\nabla \mathbf{v})(\nabla x \mathbf{v}) + (\nabla x[(\nabla \mathbf{v})\mathbf{v}]) \ . \qquad \text{given } \nabla \bullet \mathbf{v} = 0 \qquad (*)$$

Aside: We *could* write this in a more standard notation be noting that

$$(\nabla \mathbf{a})\,\mathbf{b} = \mathbf{b} \bullet \nabla \mathbf{a} \qquad \text{meaning} \qquad [(\nabla \mathbf{a})\,\mathbf{b}]_i = \mathbf{b} \bullet \nabla a_i$$

Proof: $[(\nabla \mathbf{a})\,\mathbf{b}]_i = (\nabla \mathbf{a})_{ij} b_j = (\partial_j a_i) b_j = \mathbf{b} \bullet \nabla a_i$

Thus, in more standard notation (*) says

$$\mathbf{v} \bullet \nabla(\nabla x \mathbf{v})_i = (\nabla x \mathbf{v}) \bullet \nabla v_i + (\nabla \times [v_k \partial_k \mathbf{v}])_i \qquad \text{scalar} = \text{scalar} + \text{scalar}$$

which is not very enlightening.

Comment: We will next show that (*) is true provided div **v** = 0, and that will conclude our derivation of the vorticity equation! Notice how Ω does not matter in this proof, nor does the value of ρ or μ, both assumed to be constants.

Proof that (*) is valid. What a monster! Time to go to components. Take the ith component of each term

$$[(\nabla(\nabla x \mathbf{v}))\mathbf{v}]_i = [(\nabla \mathbf{v})(\nabla x \mathbf{v})]_i + [(\nabla x[(\nabla \mathbf{v})\mathbf{v}])]_i$$

Look at each term in turn:

$$[(\nabla(\nabla x \mathbf{v}))\mathbf{v}]_i = [\nabla(\nabla x \mathbf{v})]_{ij} v_j = (\partial_j [\varepsilon_{imn} \partial_m v_n]) v_j = \varepsilon_{imn} (\partial_j \partial_m v_n) v_j$$

$$[(\nabla \mathbf{v})(\nabla x \mathbf{v})]_i = (\nabla \mathbf{v})_{ij} (\nabla x \mathbf{v})_j = (\partial_j v_i) \varepsilon_{jmn} \partial_m v_n = \varepsilon_{jmn} (\partial_j v_i)(\partial_m v_n)$$

$$[(\nabla x[(\nabla \mathbf{v})\mathbf{v}])]_i = \varepsilon_{ijk} \partial_j [(\nabla \mathbf{v})\mathbf{v}]_k = \varepsilon_{ijk} \partial_j [(\nabla \mathbf{v})_{km} v_m] = \varepsilon_{ijk} \partial_j [(\partial_m v_k) v_m]$$

$$= \varepsilon_{ijk} (\partial_j v_m)(\partial_m v_k) + \varepsilon_{ijk} (\partial_j \partial_m v_k) v_m$$

Then I need to show that

$$\varepsilon_{imn} (\partial_j \partial_m v_n) v_j = \varepsilon_{jmn} (\partial_j v_i)(\partial_m v_n) + \varepsilon_{ijk} (\partial_j v_m)(\partial_m v_k) + \varepsilon_{ijk} (\partial_j \partial_m v_k) v_m \ .$$

Now in the last term, write

$$\varepsilon_{ijk} (\partial_j \partial_m v_k) v_m = \varepsilon_{iJK} (\partial_J \partial_M v_K) v_M = \varepsilon_{imn} (\partial_m \partial_j v_n) v_j$$

and this matches the first term, so now all we have to show is this,

$$0 = \varepsilon_{jmn} (\partial_j v_i)(\partial_m v_n) + \varepsilon_{ijk} (\partial_j v_m)(\partial_m v_k)$$

Define $S_{ij} = (\partial_j v_i)$. Then what we want to show is this fact (all implied sums)

$$0 = \varepsilon_{jmn} S_{ij} S_{nm} + \varepsilon_{ijk} S_{mj} S_{km} \qquad \text{for } i = 1,2,3 \qquad \text{where } S_{kk} = 0 \ .$$

But this is an obscure *matrix* theorem which I prove using Maple in my Matrix Theorems addendum doc. So we are done. Go backwards and we have proven (*) and thus we have verified that the ζ equation is indeed valid.

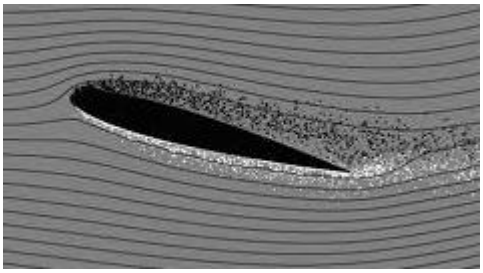
Example 6.23.2. Vorticity Equation for 2D flow

Well, 2D flow means that there IS a third dimension, but $v_3 = 0$ and the other v_i don't depend on x_3. So you can think of this as a "plane of flow" which plane is extruded into a 3D solid flow in the 3 direction such that things are the same in every plane x_3 = constant. This is a toy problem but OK. The ζ vector

points in the 3 direction and we get 6.23.5 on page 388. If we think of ζ_3 as a scalar, like temperature, then this sure looks like the heat diffusion equation where $\nu = \mu/\rho$ = kinematic viscosity p 386.

6.24 Concept of a Boundary Layer (388).

There is a whole theory that goes with this subject, some of which is in my Schaum book. He points out that "vorticity", as the scalar in the previous example, is a quantity like temperature which varies in a continuous blob and which "diffuses" like temperature. Of course it is really "heat" that diffuses and "carries" temperature with it. So look at the "wing" picture on page 389. You have a huge uniform flow and the wing is an obstacle. For low speed flow, the "vorticity" is created at the wing surface and it "diffuses" away from the wing and fills a large space around the wing. I could not find a good picture, but here is one anyway



You can "see" that the non-uniformity of the flow due to the obstacle (the streamlines are no longer uniform near the wing) will have some curl pointing in or out of paper. This is probably in fact a high speed example, so the flow although disturbed, is pretty uniform away from the wing.

So I guess I am a little disappointed. I was looking forward to the theory of lift from wings, but I don't think Lai is going to produce that theory. But maybe it will show up later.

The boundary (wing surface) generates vorticity because *you cannot help* having a curl **v** there since **v** has to have no tangential component at the boundary (viscous air). So it is like heat being injected from a boundary, and it diffuses into the **v** pattern away from the wing. The claim is that most of the serious curl **v** action occurs close to the wing surface, in a "**boundary layer**", for fast flow as in an airplane wing, and so outside this layer we have irrotational flow basically. Lai does point out that this "concept" makes dealing with problems a lot simpler than doing it all by brute force.

6.25 Compressible Newtonian fluid (389).

Here we learn just how complicated fluid dynamics really is! Lai summarizes the equations which we have to worry about when doing fluid dynamics in general.

1,2,3. First three equations are the three Navier-Stokes equations, 6 25.7. Here Lai has broken the simple $\lambda\Delta$ term in T into two terms so he can use that bulk viscosity k. The only reason for doing this is that sometimes you have k = 0, but really that simplifies nothing IMHO. The NS involve **v**, p, B, μ, k, ρ. You can regard **v**,p,ρ as 5 variables. (this is all just F = ma)

4. Continuity 6.25.8 involves **v**, ρ, 4 variables, so this is the 4th equation. (Just mass conservation within a moving particle)

5. The energy equation we found earlier is in 6.25.9 and has u(energy density),**v**,ρ,Θ(temp) as unknowns, so there are 6 variables here. When I say "has" I mean that derivatives are included. This equation implements energy conservation.

6. Some kind of pressure law (equation of state) says p = p (ρ,Θ), Boyle's Law being p = CρΘ as a simple example. So this links p,ρ,Θ together, 3 variables.

7. The last equation is u = u(ρ,Θ), a simple example being $u = c_v\Theta$. It is not made clear why we should expect energy density to depend on these two variables only. Maybe this is potential energy only, since kinetic ought to involve v^2. He calls it "internal energy".

So, you are staring at 7 scalar equations in 7 unknowns: **v**,p,ρ,u,Θ. Each of the 5 main scalar equations is a partial differential equation with 3 spatial dimensions and 1 time dimension. There are no second order time derivatives, but there are first and second order spatial derivatives. The two "equations of state" are just ingredients you have to know up front, you don't solve for them. Each PDE has boundary conditions to worry about. I expect that entropy somehow has to enter this whole discussion, but not mentioned here, though was mentioned earlier in the book.

6.26. Energy equation in terms of enthalpy (390).

In my entire "my interests" area, the word enthalpy appears in only four files. This is not a subject I have spent much time on, but have encountered in in various places.

Name	In Folder	Size	Type	Date Modified	Date Created
PSTD HS chem 1994.doc	D:\Work\My Interests\Chemistry\Chem Book Notes	99 KB	Microsoft Word Doc...	4/8/2008 7:04 AM	11/8/2003 8:51 PM
more campbell notes.doc	D:\Work\My Interests\Biology-Medicine\Campbell	28 KB	Microsoft Word Doc...	12/7/2003 10:16 PM	12/7/2003 7:36 PM
Baby Reif Level 2 Notes.doc	D:\Work\My Interests\Physics\thermodynamics_electrochemisty	73 KB	Microsoft Word Doc...	3/19/2012 10:11 AM	4/17/2007 10:31 AM
daddy reif.doc	D:\Work\My Interests\Physics\thermodynamics_electrochemisty	188 KB	Microsoft Word Doc...	4/19/2007 1:08 PM	4/18/2007 12:19 PM

Two are Reif, one is chemistry, one is biology, all are 2003 or 2007. The Chem notes just mention enthalpy in passing. The Campbell bio mention is minimal. I just reviewed the Baby Reif notes and there is an explanation of the **Gibb's Free energy,**

$G = U - TS + pV$ so $\Delta G = \Delta U - T\Delta S + p\Delta V$ then $\mathbf{\Delta S^* = -\Delta G/T}$

$dG = Vdp - SdT + \mu_i dN_i$

and how this applies to situations with your p,T system in contact with a reservoir of same p,T and how therefore this applies to all chemical reaction situations. I think the **Helmholtz Free Energy** has to do with constant temperature and volume instead.

$A = U - TS$ $dA = -SdT-pdV = 0$ if constant temp and volume (explosives)

Meanwhile, **enthalpy** is yet another combination

$H = U + pV$
$dH = TdS - Vdp$

All three of these "potentials" are described here: http://en.wikipedia.org/wiki/Helmholtz_free_energy.

So OK, this is a long subject. Lai makes this definition

$$h = u + p/\rho = u + p(\delta V/\delta m)$$

so it looks a bit like that above. Lai's u is energy per unit mass, so think $\delta m = 1$ and then we have

$$h = u + p\delta V$$

and this agrees exactly with the general enthalpy above. The kinetic energy per unit mass is going to be $(1/2)v^2$ so he then defines some kind of more total enthalpy

$$h_0 = u + p\delta V + v^2/2 = h + v^2/2 = \text{stagnation enthalpy}$$

Example 6.26.1. Here Lai starts with the "energy equation" involving $D_t u$ shown in 6.25.9 and he restates the equation with u replaced by the "stagnation enthalpy" h_0 which was defined p 390.The resulting equation is 6.26.3. He has put the sources q_i back into the equation, and we note that only the part T'_{ij} appears, which was the full T_{ij} minus the hydrostatic $-p\delta_{ij}$ part. So whatever it IS, the quantity h_o is "driven by" T'_{ij}. In this example Lai just does the math to derive this equation. I did not check it all because I don't think this is going to be used for much, but we shall see.

Example 6.26.2. Note that "inviscid" means $\mu = 0$ so the $2\mu D_{ij}$ term in T_{ij} goes away (see p 358). But Lai wants to argue that $\Delta = 0$ as well, because only then can he say $T'_{ij} = 0$. So he is really talking about an *incompressible* inviscid Newtonian fluid here, and I guess air would have to be ruled out. Water would be OK. With both of these in force, we end up with $T_{ij} = -p\delta_{ij}$ only.

This example involves some phrases that are new to me:

"non-heat-conducting fluid" -- this means that there is not motion of heat in such a fluid. So if you had some sources q_i , they could not feed heat into the material. So a source is I guess a contradiction in this case, so the implication is that $q_i = 0$. Probably air is a poor heat conductor.

http://en.wikipedia.org/wiki/List_of_thermal_conductivities units are W/m-K

water	0.6
air	0.024
CO2	0.015
concrete	1.0
steel	18
silver	430

"flow originates from a homogeneous state" -- I have no idea what this means or how it is related to this example. I agree that $T'_{ij} = 0$ since inviscid + incompressible. I agree that $q_i = 0$ since either not conducting or no sources. I agree that $\partial_t p = 0$ if we have a steady flow. So I agree that $D_t h_0 = 0$ and therefore h_0 = constant throughout the flow and I agree this relates p,ρ,u,v at different points in the flow.

Now the next item here is

$p = \rho R\Theta \qquad R = c_p - c_v \qquad u = c_v\Theta \qquad \gamma = c_p/c_v$

$\Rightarrow\ p = \rho(c_p - c_v)\Theta \quad \Rightarrow \qquad p/\rho = (c_p - c_v)\Theta = c_v(c_p/c_v - 1)\Theta = (\gamma - 1)u$

$\Rightarrow \qquad u = (p/\rho)\,(\gamma - 1)^{-1}$

We can then install this into our h_0 = constant rule to get

$$p/\rho + (p/\rho)\,(\gamma-1)^{-1} + v^2/2 = \text{constant}$$

or

$$(p/\rho)[\gamma/(\gamma-1)] + v^2/2 = \text{constant}$$

So this is a relation between p,ρ,v at different points in an inviscid, incompressible Newtonian fluid. It seems unlikely you could apply this rule then to air which seems very compressible.

6.27. Waves in a compressible inviscid fluid (392).

Here Lai develops the wave equation for waves in a compressible fluid with $\mu = 0$, starting from Navier Stokes. He puts primes on quantities which are small perturbations from the rest state. He then throws out terms quadratic in small quantities and ends up with 6.27.5,7,8 the last which says

$$\nabla^2 p' = \partial_t^2 \rho'$$

Now he has to assume that $p = p(\rho)$ in functional form, which he calls **barotropic**, with no explanation. I think this is an approximation that is well justified and "allows" for compressibility. With this assumption, we find that $p' = c_0^2 \rho'$ where $c_0^2 = (dp/d\rho)_0$ and then we get the wave equation for either p' or ρ' as in 6.27.11.

Recall that adiabatic means dQ = 0 and that means TdS = 0 for reversible I think, and that means dS = 0 and that means isentropic. Wiki http://en.wikipedia.org/wiki/Isentropic_process shows that in this case, the relationship between p and ρ is given by $p = \beta\rho^\gamma$. I think this is exactly the "adiabatic gas law" which is usually stated PV^γ = constant.

So for this gas law you can compute the derivative and get an expression for c

$$c^2 = \gamma p/\rho$$

I think the idea is that heat has no time to diffuse in sound waves so dQ = 0 for a volume.

He defines a local and a stagnation speed of sound.

Example 6.27.1. Here he writes p for a sound wave going in the x_1 direction. The wave amplitude is not a vector, it is just the pressure, so this is a compression wave and I think I would call that longitudinal because that is the direction in which you have p and ρ changing. He uses ℓ for wavelength since λ is already used as a constant for a fluid. He evaluates the size of "dropped quadratic terms" and we learn that

dropping such terms are justified if the fluid v_1 velocity is << c_0. If the pressure wave amplitude is ε, then we find $v_1 = \varepsilon/(\rho_0 c_0)$, so really this is a *low-amplitude* condition: $\varepsilon << \rho_0 c_o^2 = \gamma p_o$.

Example 6.27.2. This little set piece considers *head-on* reflection and refraction at a planar boundary between media 1 and 2. The conditions on the boundary are that p and v_{normal} be continuous coming in from either side. It turns out that the thing like index of refraction is ρc = **fluid impedance**. If this product is the same on both sides, there is no reflection at all. He computes the coefficients of reflection and refraction. Refracted is just called "transmitted" here.

6.28. Irrotational inviscid + (barotropic compressible) flow (395).

Here we are going to slightly generalize an earlier Bernoulli irrotational inviscid *in*compressible result. Now we can be compressible, but we must also be barotropic.

A potential φ is allowed here of course. Lai derives an equation analogous to p 383 6.21.10 (Bernoulli's equation) but generalized to allow for barotropic p = p(ρ) compressible flow. The difference is that p/ρ is replaced by $\int dp/\rho$ .Obviously if ρ = constant, then you recover the p/ρ. Note that adiabatic gas law = isentropic flow, which is a special case of barotropic flow.

Example 6.28.1 . If our barotropic equation is the adiabatic gas law $p(\rho) = \beta\rho^\gamma$, then we can compute $\int dp/\rho$ and put that into the new Bernoulli equation and get 6.28.10.

Example 6.28.2 If we combine our new Bernoulli 6.28.10 with the adiabatic gas law, we can solve for (p_0/p) in terms of v, having eliminated ρ. The math result is 6.28.11. For the first time (maybe the second time) we see that ratio v/c appearing, sort of like special relativity but for sound. This ratio is the Mach Number M = v/c.

Example 6.28.3 Here when M << 1, we can simplify 6.28.11 to get 6.28.12 which relates the three variables p, ρ, v. Again, all this is for inviscid, compressible, adiabatic (no body force). The surprising result is that this relation is the same that we obtained earlier in the *in*compressible case.

6.29. One dimensional compressible flow (398).

We have a pipe or duct and assume things are constant *across any given cross section* (which sort of conflicts with the viscosity idea, but assume inviscid). The idea here is that we are going to try varying the cross sectional area A by dA. At any slice of the duct, we have ρAv = constant which just says that dm/dt = constant in any cross section slice dz wide. That is, dm = ρAdz so dm/dt = ρAv. This is the mass passing through a cross section plane per second. We follow some slice as it goes along in z: A might vary, ρ might vary, v might vary, but the product ρAv = constant in z.

This fact can be written as 6.29.3 in terms of ratios. By differentiating our new Bernoulli with the $\int dp/\rho$ term Lai derives the **Hugoniot Equation** which says

$(dA/A) = (dv/v)(M^2-1)$ M = Mach = v/c

For M < 1, if you increase area by dA, dv *de*creases which seems logical and which must happen for incompressible flow. But for supersonic flow, the opposite somehow happens! I see the simple derivation of this equation, but I don't have any intuition at all.

6.30. Compressible fluid through a nozzle at bottom of a big tank (399).

A. The convergent nozzle. The first confusion here is that there is a typo and the name of this section is wrong. It is about a convergent nozzle, as shown p 400, NOT a divergent nozzle. We make our usual adiabatic, irrotational, inviscid, compressible assumptions about the gas so we can use 6.28.10. Somewhere in the convergent nozzle, we select a plane called "section 2". This is *not* at the final outlet, where the pressure is p_R which is the ambient pressure outside the tank, called the Receiver pressure.

So, we apply our fancied-up Bernoulli 6.28.10 to two points of interest: point 1 is somewhere in the interior of the large tank, while point 2 is somewhere in this section 2. We are assuming as well that we have v_2 = constant across section 2, maybe OK for inviscid. We throw in our usual adiabatic law which says things happen so fast there is no time for dQ to transfer in or out of a little volume. We find first an expression for v_2 in terms of (p_2/p_1) which is going to be the main ratio of interest here. We have our γ in several places from the gas law. [I don't see where "ideal gas" has been used, maybe that is part of the derivation of the adiabatic law.] It certainly seems that we need $p_2 < p_1$ for 6.30.4 to make sense. That is to say, we must have $(p_2/p_1) \leq 1$.

Next we look at dm/dt across our section 2 as $A_2\rho_2 v_2$. Inserting our v_2 expression, we finally come up with dm/dt = as shown 6.30.6. We like the fact that flow goes to 0 if $(p_2/p_1) = 1$. By the way, a typical value for air $\gamma \sim 1.3$, positive and not strange. But something fishy happens if we try to take $p_2 \rightarrow 0$. Mass flow seems to stop when this is true

$$(p_2/p_1)^{2/\gamma} = (p_2/p_1)^{(\gamma+1)/\gamma} \qquad \text{or} \qquad 1 = (p_2/p_1)^{(\gamma+1)/\gamma-2/\gamma} = (p_2/p_1)^{(\gamma-1)/\gamma}$$

I see only $p_2 = p_1$ for dm/dt = 0. But what if $p_2 \rightarrow 0$? This was not accounted for since we did a divide by I guess. So I think $p_2 = 0$ is non-physical somehow, it does not happen. Yes, he will now show this fact

Now let's accept the calculation of the point (p_2/p_1) where dm/dt is max, and accept that there is a max there and not a min.

Side calculation: what is v_2 at this special "critical" point? Start with:

$$v_2^2 = 2\gamma\,(\gamma-1)^{-1}(p_1/\rho_1)\,[\,1 - (p_2/p_1)^{(\gamma-1)/\gamma}\,]$$

Now replace (p_2/p_1) by the critical value shown in 6.30.8 to get

$$(p_2/p_1)^{(\gamma-1)/\gamma} = [\,(2/(\gamma+1))^{\gamma/(\gamma-1)}\,]^{(\gamma-1)/\gamma} = 2/(\gamma+1)$$

so that

$$v_2^2 = 2\gamma\,(\gamma-1)^{-1}(p_1/\rho_1)\,[\,1 - 2/(\gamma+1)\,] = 2\gamma\,(\gamma-1)^{-1}(p_1/\rho_1)\,[(\gamma-1)/(\gamma+1)\,]$$

$$= 2(p_1/\rho_1)\,[\gamma/(\gamma+1)\,]$$

Now the speed of sound at section 2 is given by

$$c_2^2 = \gamma\,(p_2/\rho_2)$$ // see page 393

Now the claim that Lai makes on p 400 is this

$$v_2 = c_2$$

which says

$$2(p_1/\rho_1)\,[\gamma/(\gamma+1)] = \gamma\,(p_2/\rho_2)$$

or

$$2(p_1/\rho_1) = (\gamma+1)\,(p_2/\rho_2)$$

Multiply both sides by ρ_2/p_1 to get

$$2(\rho_2/\rho_1) = (\gamma+1)\,(p_2/p_1)$$

Now from page 396 bottom we have

$$(\rho_2/\rho_1) = (p_2/p_1)^{1/\gamma}$$

So then we need to show that

$$2(p_2/p_1)^{1/\gamma} = (\gamma+1)\,(p_2/p_1)$$

or

$$(p_2/p_1)^{1/\gamma-1} = (\gamma+1)/2$$

or

$$(p_2/p_1)^{(1-\gamma)/\gamma} = (\gamma+1)/2$$

or

$$(p_2/p_1) = [(\gamma+1)/2]^{\gamma/(1-\gamma)} = [2/(\gamma+1)]^{\gamma/(\gamma-1)}$$

but miraculously this is again that critical value shown in 6.30.7. So his claim is correct, but there must be a faster way to show it. // end of derivation.

So I agree, if p_2 gets to the critical value, $v_c \rightarrow$ speed of sound at that point. Now look at

$$v_2^2 = 2\gamma\,(\gamma-1)^{-1}(p_1/\rho_1)\,[\,1 - (p_2/p_1)^{(\gamma-1)/\gamma}\,]$$

We see that as p_2 decreases, v_2 increases. So this says you cannot decrease $p_2 < p_c$ without breaking the sound barrier at section 2.

Now finally we apply the Hugoniot equation p 398. If we try to go to M>1 at section 2, we know that dv > 1 because v is increasing by our formula above as we try to decrease p_2 to this end, but that says that

we need dA > 1. But since we have a convergent nozzle by assumption, we have dA < 1 in the direction of the flow.

The conclusion then is this: you cannot have $p_2 < p_c$. So that fixes up our problem with dm/dt going to zero at $p_2 = 0$.

Now suddenly Lai has a non sequitor. He starts talking suddenly about that ambient p_R . Well, I guess it must be true that $p_2 > p_R$, otherwise no gas can emerge from the nozzle. So we have this situation

$p_2 \geq p_R$ => $p_2 \geq \max(p_R, p_c)$
$p_2 \geq p_c$

Case 1: if max = p_R the dm/dt is given by 6.30.9
Case 2: if max = p_c the dm/dt is given by 6.30.10 which result is independent of p_R

In the second case, as you further lower p_R, you don't alter dm/dt outflow.
In the first case, as you increase p_R, you affect outflow, probably reducing it.

A. The convergent-then-divergent nozzle. Picture is shown here on p 401. We just studied the left convergent part and we know that at the throat and to its left we have to be subsonic. If it happens that p_2 at the throat produces exactly sonic, the flow is "choked". A certain value of p_R might do this, but we don't really have a model for what is going on here, so I think Lai is just giving a description of what does happen without deriving it. In one regime of p_R, flow is subsonic in the entire divergent part of the nozzle and this is the a,b type curve. But a certain p_R hits the choke point, and if you go beyond that, you get supersonic in a section of the divergent piece followed by a p discontinuity which he calls a "shock". I guess this is a static shock front of some sort, like a static hole in a running river. To the right of the shock, you are subsonic again. Other strange things can happen as well.

So this was a very specialized topic that Lai must have worked on somewhere. Probably this subject concerns rocket engines or jet engines, but he gives no clue. It is not clear why all this flow is irrotational by the way, which was assumed.

6.31. Steady laminar flow in an elastic tube (blood vessel) (401).

This is a simple but interesting problem which combines elastic statics with fluid dynamics. Assume a blood vessel has p on the inside and p=0 on the outside. In practice, $p = \Delta p = p_{in} - p_{out}$ but that adds nothing, so just use p and p=0. As p increases, the tube expands since it is made of an elastic material with Young E_Y. Each equation needs comments here!

6.31.2 We found this earlier, Q is pushed by dp/dz going down the tube in the z direction. So the right equation here is pretty clear. Think of Q as some number, the flow rate.

6.31.3. Way back on p 202 we talked about axial strain being $\Delta\ell/\ell$, so think of the thin ring being straightened out so $E_{\theta\theta} = \Delta C/C$ where C is the linearized circumference. So this is $\Delta r/r$ as well.

6.31.4 Draw a picture of a sugar cube part of the vessel wall. The outward force is

$F_{out} = p\, dA_2 = p\, r d\theta\, dz$

$F_{in} = T_{\theta\theta} \sin(d\theta/2) * 2 * dA_1 = T_{\theta\theta}\, d\theta * t\, dz$

Since this particle is at rest, the forces balance so

$p\, r d\theta\, dz = T_{\theta\theta}\, d\theta * t\, dz \quad \Rightarrow \quad p\, r = T_{\theta\theta}\, t \quad \Rightarrow \quad T_{\theta\theta} = pr/t$

and he calls this the "hoop stress".

6.31.5 We know $T_{zz} = 0$ since nothing pushes our element in that direction. What about T_{rr}? On the inside surface of our section we have $T_{rr} = p$. But $T_{\theta\theta} = (r/t)p >> p$, so $T_{\theta\theta} >> T_{rr}$ so we just ignore T_{rr}. Then to good approx we have $T_{\theta\theta} = E_{\theta\theta}/E_Y$ from p 210 5.4.10 (all in cylindricals of course, another subject I have dealt with and is all clear). So then just install $T_{\theta\theta}$ from previous equation.

6.31.6,7 Set the two $E_{\theta\theta}$ expressions equal and solve for r. This then tells how the radius of the vessel increases with pressure p. Of course vessel is thin, $t << r_0$.

6.31.8 Install expression for r into 6.31.2 to get

$r^4\, dp = - (8\mu/\pi)\, Q\, dz$

$r_o^4 (1 - \alpha p)^{-4}\, dp = - (8\mu/\pi)\, Q\, dz \qquad \alpha = r_o/(tE_Y)$

Now as we move dz along the vessel, p changes. So integrate 0 to L

$$\int_{p(0)}^{p(L)} dp\, (1 - \alpha p)^{-4} = - Q\, (8\mu/\pi\, r_o^4) \int dz = - QL\, (8\mu/\pi\, r_o^4)$$

Change variable to $x = 1-\alpha p$ so that $dx = -\alpha dp$. Endpoints are then $(1-\alpha p(0))$ and $(1-\alpha p(L))$.

$$I = \int_{p(0)}^{p(L)} dp\, (1 - \alpha p)^{-4} = \int_{(1-\alpha p(0))}^{(1-\alpha p(L))} [-dx/\alpha]\, x^{-4} = (-1/\alpha) \int_{(1-\alpha p(0))}^{(1-\alpha p(L))} dx\, x^{-4}$$

$$= (-1/\alpha)\, (-1/3)\, \{ (1-\alpha p(L))^3 - (1-\alpha p(0))^3 \}$$

Therefore

$$(1/3\alpha)\, \{ (1-\alpha p(L))^3 - (1-\alpha p(0))^3 \} = - Q\, (8\mu L/\pi\, r_o^4)$$

$$Q = - (\pi r_0^4/24\alpha\mu L)\, \{ (1-\alpha p(L))^3 - (1-\alpha p(0))^3 \}$$

$$= - (\pi r_0^3 t E_Y/24\mu L)\, \{ (1-p(L)\, r_o/(tE_Y))^3 - (1-p(0)\, r_o/(tE_Y))^3 \}$$

$$= (\pi r_0^3 t E_Y/24\mu L)\, \{ (1- r_o\, p(0)\, /(tE_Y))^3 - (1- r_o\, p(L)\, /(tE_Y))^3 \} \qquad \text{// agrees 6.31.9}$$

So what does this tell us? The vessel has properties r_o, t, μ, L, E_Y. If we put p(0) at one end and p(L) at the far end, Q as shown is the rate of flow in the Hagen-Poiseuille model. For the rigid tube p 371, the answer came out being

$$Q = (\pi r_o^4/8\mu)[(p(0)-p(L))/L]$$

which is linear in the Δp on the ends of the tube.

Example 6.31.1. (402) If we do a linear approx to 6.31.7, we get 6.31.10 with $\alpha = 2r_0^2/(tE_Y)$. In any event, 6.31.10 is an empirical model with some compliance α. This gives a certain p = p(r). We then redo the analysis with this different model for p(r) and the result is (6.31.12) which is a certain 5th power difference formula for blood flow in a pulmonary blood vessel. I think this might be the first medical example in the entire book, 403 pages. You can see this example was tacked on at the last minute because two equation references are wrong (and it is at the end of a chapter). But the result is right, and that is the important thing.

Problems: We now have 58 problems for this long Chapter 6 on the Newtonian Viscous Fluid. I have done 4 of them as marked in the course of reading the chapter. 410-353 = 57 pages including the problems. Remember that Newtonian means a certain form for T ~ p + QD where D is the rate of deformation tensor, and this tensor-simplifies to $T_{ij} = -p\delta_{ij} + \lambda\Delta\delta_{ij} + 2\mu Dij$ where λ,μ are properties of the fluid (no connection with elastic properties of the same name) and where μ = viscosity = constant.