Lai Chapter 7 Meta Notes: Reynolds Transport Theorem 10.20.12

7.1 Green's Theorem (411) 1
7.2 Divergence Theorem (414) 2
7.3 Control and Material Volumes (417) 2
7.4 The Reynolds Transport Theorem (418). (RTT) 2
7.5 The Principle of Mass Conservation (420). 3
7.6 The Principle of Linear Momentum Conservation (422). 3
7.7 Moving Frames (427). 3
7.8 Moving Frames Continued (430). 4
7.9 Principle of Conservation of Angular Momentum (430). 4
7.10 Principle of Conservation of Energy (432) 4
7.11 Principle of Entropy (2nd law thermo) (436). 4

Overview. Lai introduces the material and control volume concept with V_m and V_c. Time derivatives of the V_m integral of something can be replaced by V_c integrals using the Reynolds Transport Theorem RTT. That "something" can be taken to be momentum, angular momentum, energy, mass, entropy for example. Our 5 "principles" are each first stated in some obvious way involving D_t of a V_m integral, and then that integral is replaced by V_c integrals and each principle then has a nice interpretation in terms of rate of change of something inside V_c, plus change due to flow of something in or out of V_c

A side topic in this chapter concerns volumes V_c and V_m which are "rotating". In that case, your obvious statement of each principle must include fictional forces and fictional torques. For linear and angular momentum principles, I have shown in detail how all this works in frames doc.

Control volume consideration provides a practical way to solve fluid problems that perhaps would be difficult to solve some other way. For the hanging rope problem, we consider linear momentum flowing out of our control volume and this leads to a curious solution of this simple problem. The spray against a vane was another problem.

The rotating sprinkler is the other main problem treated, and you then have to compute the rate at which the V_c tube is losing angular momentum due to the water spraying out. In this prototype problem, the control volume is rotating, so all the fancy stuff below gets used.

I have not really *done* any problems, just read his solutions.

7.1 Green's Theorem (411)

This is really the "integral of a gradient theorem", but Lai calls it "Green's Theorem". I just wrote a note talking about all the different things people call "Green's theorem". Calculus file. From it I confirm that

$$\int_A dA\,\partial_x\varphi = \int_C ds\, n_x\, \varphi = \int_C dy\, \varphi \qquad \text{// agrees with (7.1.1)}$$

$$\int_A dA\,\partial_y\varphi = \int_C ds\, n_y\, \varphi = -\int_C dx\, \varphi \qquad \text{// agrees with (7.1.2)}$$

I don't think any result from this section is used anywhere.

7.2 Divergence Theorem (414)

Old hat to me. One application is to convert surface force type integrals into volume stress integrals:

$$\mathbf{f} = \int_S \mathbf{t}\, dS = \int_S (T\mathbf{n})\, dS = \int_S T\, d\mathbf{S} \qquad \text{// force}$$

$$\mathbf{m} = \int_S \mathbf{r} \times \mathbf{t}\, dS = \int_V \mathbf{r} \times (\mathbf{div}\mathbf{T})\, dV \qquad \text{// torque}$$

The divergence theorem IS used in this chapter for sure.

7.3 Control and Material Volumes (417)

I think I understand the meaning of V_c and V_m. The V_m moves with the fluid blob, V_c does not.

7.4 The Reynolds Transport Theorem (418). (RTT)

It turns out I already derived both forms of the RTT in "Time derivatives of differentials and Squirmy Integrals.doc" (Lai support) and I tuned it up a bit just now. The basic forms of the RTT are these

$$D_t\left(\int_{V_m(t)} T(\mathbf{x}(t),t)\, dV\right) = \int_{V_c} (\partial_t T)\, dV + \int_{S_c} T\,(\mathbf{v}\bullet\mathbf{n})\, dS = \int_{V_c} (D_t T + T\, \mathrm{div}\, \mathbf{v})\, dV$$

(7.4.1) (7.4.2)

note: $\int_{V_c} (\partial_t T)\, dV = \partial_t\left(\int_{V_c} T\, dV\right)$ $dV = d\forall$

where T is an arbitrary function (could be component of a tensor). The unspoken idea is that the LHS integral applies to a "blob" of particles which moves along with the moving (material) volume V_m . No particles of matter ever enter or leave V_m during the motion, but of course V_m can change shape, so it is really $V_m(t)$. You use the RTT whenever you encounter a structure like $D_t(\int_{V_m} T\, dV)$. There are three contributions to this $D_t()$ object: (1) explicit ∂_t dependence of $T(\mathbf{x}(t),t)$ on t; (2) dependence through T's $\mathbf{x}(t)$ argument; (3) dependence because the boundary $V_m(t)$ is moving. The second term in (7.4.1) handles (2) + (3). The divergence theorem on this second term would say

$$\int_{S_c} T\,(\mathbf{v}\bullet\mathbf{n})\, dS = \int_{S_c} (T\mathbf{v}) \bullet d\mathbf{S} = \int_{V_c} \mathrm{div}(T\mathbf{v})\, dV$$

So it is must be true that (where T is any function)

$$\underset{2+3}{\mathrm{div}(T\mathbf{v})} + \underset{1}{\partial_t T} = \underset{1+2}{D_t T} + \underset{3}{T\, \mathrm{div}\, \mathbf{v}}$$

or

$$\partial_i (Tv)_i + \partial_t T = [\, \partial_t T + (\partial_i T)\, v_i] + T\, (\partial_i v_i)$$

$$\underset{2+3}{\partial_i(Tv)_i} = \underset{1}{(\partial_i T)}\,\underset{1}{v_i} + \underset{2}{T}\,\underset{3}{(\partial_i v_i)}$$

or

$\partial_i(Tv)_i = (\partial_i T)\, v_i + T\,(\partial_i v_i)$ // which is obviously true.

and you can see how the three sources of D_t are distributed into the terms.

A key fact: the object ($\int_{vm} T\, dV$) is what corresponds in regular mechanics to an "object of mass M". so this integral tells how much "T" that object has. Second term in (7.4.1) is quantity of T flowing out through the boundary, first term is time change explicitly within the boundary.

I have a doc talking about various web forms of this RTT theorem.

7.5 The Principle of Mass Conservation (420).

By saying that $D_t(\int_{vm}\rho dV) = 0$, we obtain the continuity equation using the Reynolds Transport Theorem. This is very similar to the derivation used in Chapter 4. No questions.

7.6 The Principle of Linear Momentum Conservation (422).

Here we have a pattern which we will see repeated in each of the sections to follow. The *first* equation of each section states "the principle" applied to a moving blob. Here we have $\mathbf{F} = d\mathbf{p}/dt$ where **F** is the usual surface traction plus body force integrals, and $d\mathbf{p}/dt = D_t(\int_{vm} \mathbf{v}\, \rho dV)$. These principles always involve a "change" in something, and here it is a change in linear momentum. After the RTT is applied, we get 7.6.2 (in this case) with an italics interpretation. As noted above, ($\int_{vm} dV$ ...) is the "object" to which we apply rules of regular mechanics like Newton's Law in this case.

Example 7.6.1 The Rope Hanging on a Table. Using control volumes to solve a simple problem.

Example 7.6.2 The Hose flowing onto a curved "vane" (426). Use control volume to get force on the vane.

7.7 Moving Frames (427).

What if frame S' = F_2 is a non-inertial rotating frame and so your V_m , glued in frame S', is rotating relative to some frame S = F_1? As in frames doc, we can include translating in the term rotating. It is a non-inertial frame. In this case, what does the "principle of linear momentum conservation" look like?

When Lai rolled out the G Rule (my Goldstein bugagoo), I was forced to write frames doc.

The big result of this section is equation (7.7.14) which is basically $m\mathbf{a} = \mathbf{F} + \mathbf{F}'_{fict}$ for the situation where the volume V_m is glue to a *rotating* frame of reference S'= S_2. Here $\mathbf{F} = \int_{sc2} \mathbf{t} dS + \int_{vc2} \rho \mathbf{B} dV$ is the regular force, and all the other terms are the fictitious forces,

$$\mathbf{F'}_{\mathtt{fict}} = -\ddot{\mathbf{b}}_{\mathtt{S}} \int_{\mathtt{Vc2}} \rho(\mathbf{x})dV - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \int_{\mathtt{Vc2}} \mathbf{r'}\, \rho(\mathbf{x})dV)$$

$$- 2\,\boldsymbol{\omega} \times \int_{\mathtt{Vc2}} \mathbf{v}_{\mathtt{F2}}(\mathbf{x})\rho(\mathbf{x})dV - \dot{\boldsymbol{\omega}} \times \int_{\mathtt{Vc2}} \mathbf{x}\, \rho(\mathbf{x})dV$$

analogous to

$$\mathbf{F'}_{\mathtt{fict}} = -m\ddot{\mathbf{b}}_{\mathtt{S}} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}) - 2m\,\boldsymbol{\omega} \times \mathbf{v'} - m\dot{\boldsymbol{\omega}} \times \mathbf{r'}$$

I just installed this into frames doc (10.21.12), it was missing.

7.8 Moving Frames Continued (430).

Here we write the RTT in both frame S (fixed) and S' (rotating) for integrand T = $\rho\mathbf{v}$ so the thing inside d/d() is still linear momentum. Then the LHS of this RTT in frame S' = S_2 is the same as the LHS of equation (7.7.14) of the last section which applies to frame S' = S_2 . If we then equate the RHS's of these two equations, we get (7.8.1) which is the only result of this section, and which is an equation Lai never uses in the rest of the book.

7.9 Principle of Conservation of Angular Momentum (430).

I worked through all details of this section through the second last equation (7.9.8) and its elegant physical interpretation as stated in italics: the total torque on a control volume V_c equals the change in the angular momentum contained in the control volume plus the *out*flow from same.

This entire section is now written up in frames doc and I even mention the **sprinkler** there.

7.10 Principle of Conservation of Energy (432).

The very first equation says that

$$D_t \int_{\mathtt{vm}} (\mathrm{KE+PE}) = \text{work done by t and B + heat flux in + heat sources inside}$$

7.11 Principle of Entropy (2nd law thermo) (436).

This inequality thing is the final principle.