**Chapter 7 Lai Notes: Reynolds Transport Theorem etc.** PhL 6.30.12


**7.1 Green's Theorem (411) ..... 1**
**7.2 Divergence Theorem (414) ..... 2**
**7.3 Control and Material Volumes (417) ..... 3**
**7.4 The Reynolds Transport Theorem (418). (RTT) ..... 3**
**7.5 Mass Conservation (420). ..... 4**
**7.6 Linear Momentum Conservation (422). ..... 4**
**7.7 Moving Frames (427). ..... 7**
**7.8 Moving Frames Continued (427). ..... 9**
**7.9 Principle of Conservation of Angular Momentum (430). ..... 10**
**7.10 Principle of Conservation of energy (432). ..... 12**
**7.11 Principle of Entropy (2nd law thermo) (436). ..... 13**


## 7.1 Green's Theorem (411)

This is really the "integral of a gradient theorem", but Lai calls it "Green's Theorem". I just wrote a note talking about all the different things people call "Green's theorem". Calculus file.

**Derivation of 7.1.1 and 7.1.2. 3.** My integral theorems sheet has (including proof from div theorem)

$$\int_V dV\, \nabla\varphi = \int_S d\mathbf{S}\, \varphi = \int_S dS\, \mathbf{n}\, \varphi \qquad \text{// "integral of a gradient theorem"}$$

If we go to volume as a 2D thing so the surface is then a closed curve C and dV→dA, we can say

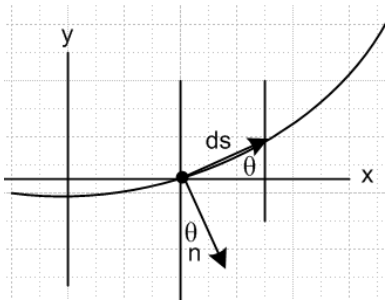
$$\int_A dA\, \nabla\varphi = \int_C d\mathbf{S}\, \varphi = \int_C ds\, \mathbf{n}\, \varphi \qquad \text{// note that } d\mathbf{S} = ds\, \mathbf{n}$$

where $\mathbf{n} = n_x \hat{\mathbf{x}} + n_y \hat{\mathbf{y}}$ . Taking x and y components we get

$$\int_A dA\, \partial_x\varphi = \int_C ds\, n_x\, \varphi$$

$$\int_A dA\, \partial_y\varphi = \int_C ds\, n_y\, \varphi$$

Now consider this picture




Then

$d\mathbf{s} \bullet \hat{\mathbf{x}} = dx = ds\cos\theta$ $\quad$ $\mathbf{n} \bullet \hat{\mathbf{x}} = n_{\mathbf{x}} = \cos(\pi/2-\theta) = \sin\theta$

$d\mathbf{s} \bullet \hat{\mathbf{y}} = dy = ds\sin\theta$ $\quad$ $\mathbf{n} \bullet \hat{\mathbf{y}} = n_{\mathbf{y}} = -\cos\theta$

from which we conclude that

$dx/ds = -n_{\mathbf{y}}$ $\quad$ and $dy/ds = n_{\mathbf{x}}$ $\quad$ // Lai top of page 413

Then we can restate our two equations above as

$\int_{\mathbf{A}} dA\partial_{\mathbf{x}}\varphi = \int_{\mathbf{C}} ds\, n_{\mathbf{x}}\, \varphi = \int_{\mathbf{C}} dy\, \varphi$ $\quad$ // agrees with (7.1.1)

$\int_{\mathbf{A}} dA\partial_{\mathbf{y}}\varphi = \int_{\mathbf{C}} ds\, n_{\mathbf{y}}\, \varphi = -\int_{\mathbf{C}} dx\, \varphi$ $\quad$ // agrees with (7.1.2)

where notice things are reversed on the right x ↔ y.

**7.2 Divergence Theorem (414)**

This section is old hat for me.

Example 7.2.1 considers $\mathbf{f} = \int_{\mathbf{S}} \mathbf{t}\, dS$ as the total force acting on some blob due to surface tractions, But we know that $\mathbf{t} = T\mathbf{n}$, so $\mathbf{f} = \int_{\mathbf{S}} (T\mathbf{n})\, dS = \int_{\mathbf{S}} T\, d\mathbf{S}$ where T is a *matrix*. I show on page 16 of Lai Ch 4 notes (apply regular divergence theorem to a row of T) this is $= \int_{\mathbf{V}} \mathbf{divT}\, dV$. This is an interesting result, saying that in some magic way, doing this volume integral over the blob also gives you the total force on the blob from those surface tractions.

Example 7.2.2 writes the total torque acting on a boundary as a surface integral, then converts that to a volume integral, but this time an extra term appears if $T_{\mathbf{ij}}$ happens to have an antisymmetric part. Then the thing $\mathbf{t}^{\mathbf{A}}$ is associated with $T^{(\mathbf{A})}$ in the usual manner. If there is no $T^{(\mathbf{A})}$, you get the very simple result that $\mathbf{m} = \int_{\mathbf{S}} \mathbf{r} \times \mathbf{t}\, dS = \int_{\mathbf{V}} \mathbf{r} \times (\mathbf{divT})\, dV$ which is similar to the **f** result above.

Example 7.2.3 considers the total work being done on a boundary as in 7.2.17 with the usual idea that work power is $\mathbf{F} \bullet \mathbf{v}$ and again this starts as a surface integral. When converted to a volume integral, we have two terms as shown. I am not quite sure how to interpret this volume integral. It is true that the surface mediates the work into the volume, and there is action in the volume, so I guess $\mathrm{div}(T^{\mathbf{T}}\mathbf{v})$ is the

density of this work being done in the volume. At least it is equivalent to this, even if that is not physically how things happen.

**7.3 Control and Material Volumes (417)**

Lai fails to make it totally clear at first shot, but his try is good. $V_c$ means a control volume which is fixed in space and through which fluid flows. $V_m$ means either "material" or "moving" volume (I like moving) which means it tracks the pathlines of all the particles in the volume, which means the boundary moves with velocity **v**. The population of $V_m$ never changes, even after a lot of flow, it is always the exact same particles. Thus, the mass M in $V_m$ given by 7.3.4 is a constant in time. What is a little confusing is that Lai uses the letter m on $V_m$ , but associates mass M with $V_m$ and then mass m with $V_c$, so the sort of mismatch is a little confusing to the reader. Lai should have drawn a nice Schaum picture. On page 418 the $V_c$ integral can be written with either $D_t$ or $\partial_t$ since it just acts on the t variable of the integrand. However, the $V_m$ integral must be written $D_t$ since $V_m$ as well as the integrand depend on t. Basically, I already knew everything in this section, a rare event.

**7.4 The Reynolds Transport Theorem (418). (RTT)**

It turns out I already derived both forms of the RTT in "Time derivatives of differentials and Squirmy Integrals.doc" (Lai support) and I tuned it up a bit just now. The basic forms of the RTT are these

$$D_t\left(\int_{V_m} T\, dV\right) = \int_{V_c} (\partial_t T)\, dV + \int_{S_c} T\,(\mathbf{v}\bullet\mathbf{n})\, dS = \int_{V_c} (D_t T + T \operatorname{div} \mathbf{v})\, dV$$

(7.4.1) (7.4.2)

where T is an arbitrary function (could be component of a tensor). The unspoken idea is that the LHS integral applies to a "blob" of particles which moves along with the moving (material) volume $V_m$ . No particles of matter ever enter or leave $V_m$ during the motion, but of course $V_m$ can change shape, so it is really $V_m(t)$. You use the RTT whenever you encounter a structure like $D_t(\int_{V_m} T\, dV)$.

Lai gives two separate derivations of the RTT which are basically what I did in the doc.

A key fact: the object $(\int_{V_m} T\, dV)$ is what corresponds in regular mechanics to an "object of mass M". so this integral tells how much "T" that object has. T could be velocity or momentum or energy etc. So later when we want to apply F = Ma, we will apply it to this entity $(\int_{V_m} T\, dV)$ . That is why we are doing to have to deal with such $V_m$ integrals. This $v_m$ integral, then, is the generalization of a "rigid body".

**Osborne Reynolds** FRS (23 August 1842 – 21 February 1912) was a prominent innovator in the understanding of fluid dynamics. This work was perhaps 1905.

Now Lai is going to apply this transport theorem to re-work all our integral forms of conservation rules. To some extent, as the intro notes, this is a repetition.

**Question**: The discussion of sections 7.3 and 7.4 takes place in some frame of reference in which the various quantities are defined. As I stare at page 417, I don't see anything that requires this frame of reference be an inertial frame of reference. There is no **F** = m**a** being used yet. On page 418 maybe the

same is true, so maybe Reynolds Transport is valid in any frame. I will resume this topic in the notes for section 7.8 below. It is true that RTT applies in any Frame, inertial or not!

Comments:

(1) The first term on the right in 7.4.1 can be written $\partial_t [\int_{V_c} T\, dV]$ since the $V_c$ has no time dependence and is a fixed volume in space. Often this is how this term appears on the web.

(2) Authors sometimes use ∀ with a horizontal slash to indicate volume so that V can be velocity.

(3) It is the this first term on the right of (7.4.1) which is $\partial_t [\int_{V_c} T\, dV]$ and which tells the rate at which something is changing within the control volume. So you can write

$$\partial_t [\int_{V_c} T\, dV] = D_t(\int_{V_m} T\, dV) - \int_{S_c} T\, (\mathbf{v}\bullet\mathbf{n})\, dS$$

so then the sign of the last term makes sense, since here it represents the flow coming into the control volume and thus adding to its contents of T. If you apply this to T = mass density ρ you get

$$\partial_t [\int_{V_c} \rho\, dV] = D_t(\int_{V_m} \rho\, dV) - \int_{S_c} \rho\, (\mathbf{v}\bullet\mathbf{n})\, dS$$
$$\partial_t [\int_{V_c} \rho\, dV] = \quad 0 \quad - \int_{S_c} \rho\, (\mathbf{v}\bullet\mathbf{n})\, dS$$

and so the rate at which $V_c$ mass increases is given entirely by the mass influx rate. The $D_t(\int_{V_m} T\, dV)$ term is not easy to interpret for arbitrary T. Typically as $V_m$ moves, some T leaks out or in.

(4) the sign of the last term in 7.4.1 RTT "is what it is", and it is *out*flux of T.

**7.5 The Principle of Mass Conservation (420).**

By saying that $D_t(\int_{V_m}\rho dV) = 0$, we obtain the continuity equation using the Reynolds Transport Theorem. This is very similar to the derivation used in Chapter 4. No questions.

**Example 7.5.1** assumes a certain simple 1D flow **x**(**X**,t) and proposes a certain time dependent ρ(t) function. If you compute **v** from **x**, you find that **v** and this ρ are compatible relative to continuity. Probably given the x flow, I could compute ρ using the compatibility equation, but that is not part of this example. Lai shows that for a fixed control volume, $D_t m$ varies with time, whereas $D_t M = 0$, all by direct calculation for the example. As an adder, he computes total momentum in a certain portion of the flow. This then is a time-dependent flow example.

**7.6 The Principle of Linear Momentum Conservation (422).**

Here we have a pattern which we will see repeated in each of the sections to follow. The *first* equation of each section states "the principle" applied to a moving blob. Here we have **F** = d**p**/dt where **F** is the usual

surface traction plus body force integrals, and d**p**/dt = $D_t(\int_{vm} \mathbf{v}\ \rho dV)$. These principles always involve a "change" in something, and here it is a change in linear momentum. After the RTT is applied, we get 7.6.2 with an italics interpretation. As noted above, ($\int_{vm} dV$ ...) is the "object" to which we apply rules of regular mechanics like Newton's Law in this case.

Total force equals $D_t P_m$ results (via Reynolds TT) in the Cauchy Equation of Motion, same as in Chapter 4, all very clear.

The adder here is to derive the PK $T_0$ tensor version of the Cauchy EOM and this too agrees with what we found in Chap 4.

**Example 7.6.1 The Rope Hanging on a Table.** This is a rather surprising example. First of all, the rope itself is treated as a fluid which is magically constrained to hold its shape. There is no friction with the table either on the top or at the corner where the rope does a 90 degree bend. The rope just flows around this corner magically. Total mass of rope is m. The problem of course is to find the motion of the rope and also to find the tension T in the rope "at the corner". This is a simple problem, but here it is treated very strangely as an example of using control volumes!

First let's do the simple solution. For the top piece of rope we have T as the only force, pulling that section to the right. So T = $\{m[(\ell\text{-}x)/\ell]\}a = m[(\ell\text{-}x)/\ell]\ddot{x}$. For the vertical piece of rope we have gravity pulling down and T pulling up so $F = -T + m[x/\ell]g = m[x/\ell]\ \ddot{x}$ . Then we can eliminate T to get

$$m[x/\ell]\ \ddot{x} = -\,m[(\ell\text{-}x)/\ell]\ddot{x} + m[x/\ell]g$$

or

$$[x]\ \ddot{x} = -\,[(\ell\text{-}x)]\ddot{x} + [x]g$$

or

$$x\ \ddot{x} + [(\ell\text{-}x)]\ddot{x} = [x]g$$

or

$$\ell\ \ddot{x} - gx = 0 \qquad \text{// which agrees with (vi)}$$

Let's just solve this while we are here. Define $\alpha^2 = g/\ell > 0$ so ODE says

$$\ddot{x} - \alpha^2 x = 0$$

Solutions are $e^{\pm\alpha t}$ so try $x(t) = A\ e^{\alpha t} + Be^{-\alpha t}$. Then $x(0) = (A+B) = x_0$.
Also have $v(0) = (\alpha A - \alpha B) = v_0 = 0$ say. Then A = B and $x_0 = 2A$ so solution is

$$x(t) = (x_0/2)\ [\ e^{\alpha t} + e^{-\alpha t}\ ] = x_o ch(\alpha t).$$

Corner tension is $T(t) = m[(\ell\text{-}x)/\ell]\ddot{x} = m[(\ell\text{-}x)/\ell]\ x_o\alpha^2 ch(\alpha t)$. Can also write this as

$$T = m[(\ell\text{-}x)/\ell]\ddot{x} = m[(\ell\text{-}x)/\ell]\ (g/\ell)x \qquad \text{// agrees with (ix).}$$

The tension goes to zero when $x = \ell$ and so all the rope is in vertical free fall.

Now let's try to reconstruct the "control volume method".

Look at control volume $V_{c1}$ which is a box which includes just the horizontal part of the rope. The total momentum in this box at some instant is $P = m[(\ell\text{-}x)/\ell]\dot{x}$ . But momentum is exhausting out the right end, and nothing is coming in, so we are losing momentum in this control volume. Consider little volume dx just before the corner where the loss is occurring. This volume contains momentum $m[dx/\ell]\dot{x}$ and the momentum outflow rate is therefore $(m/\ell)\,\dot{x}^2$, so we have $dP_{loss}/dt = (m/\ell)\,\dot{x}^2$. So, the rope in the control volume $V_{c1}$ is losing momentum because momentum is flowing out the right end, but it is gaining momentum because T is pulling on it. Then the $D_tP$ for the top rope section is

$$dP/dt = T - dP_{loss}/dt \quad \Rightarrow \quad T = dP/dt + dP_{loss}/dt = d/dt\ \{\ [m[(\ell\text{-}x)/\ell]\dot{x}\} + (m/\ell)\,\dot{x}^2$$

$$= [m[(\ell\text{-}x)/\ell]\,\ddot{x} - [m/\ell]\,\dot{x}^2 + (m/\ell)\,\dot{x}^2 = [m[(\ell\text{-}x)/\ell]\,\ddot{x}$$

Since $T = [m[(\ell\text{-}x)/\ell]\,\ddot{x}$ we can interpret this as Newton $F = \Delta m\, a$ and all is well.

Look at control volume $V_{c2}$ which is a box which includes just the vertical part of the rope. This box (as drawn) is gaining momentum at the same rate the upper box is losing it, which is $(m/\ell)\,\dot{x}^2$ .Of course this gain is now in the down direction. It is losing momentum due to T pulling back, but it is gaining momentum from gravity. So I guess we write

$$dP/dt = dP_{gain}/dt - T + m[x/\ell]g \qquad \text{and } P = m[x/\ell]\,\dot{x}$$

So our equation here is then

$$d/dt\ \{\ m[x/\ell]\,\dot{x}\ \} = (m/\ell)\,\dot{x}^2 - T + m[x/\ell]g$$

or

$$-T + m[x/\ell]g = d/dt\ \{\ m[x/\ell]\,\dot{x}\ \} - (m/\ell)\,\dot{x}^2 \qquad \text{// agrees with (iii)}$$

The RHS here is

$$\text{RHS} = m[x/\ell]\,\ddot{x} + m[\dot{x}/\ell]\,\dot{x} - (m/\ell)\,\dot{x}^2 = m[x/\ell]\,\ddot{x}$$

so therefore

$$-T + m[x/\ell]g = m[x/\ell]\,\ddot{x} = (m/\ell)\ x\ \ddot{x} \qquad \text{// agrees with (iv)}$$

Now we have obtained these two results,

$$T = [m[(\ell\text{-}x)/\ell]\,\ddot{x} \qquad \text{// from control volume 1}$$

$$-T + m[x/\ell]g = m[x/\ell]\,\ddot{x} = (m/\ell)\ x\ \ddot{x} \qquad \text{// from control volume 2}$$

Add them to get

$m[x/\ell]g = [m[(\ell\text{-}x)/\ell]\ddot{x} + (m/\ell)\, x\, \ddot{x} = m\ddot{x}$

or

$(g/\ell)x = \ddot{x}$

or

$\ddot{x} - (g/\ell)x = 0$ // agrees with (vi)

Summary: We considered here the "goings on" in two separate control volumes, and for each one we obtained a certain fact. We then combined these facts to get a solution to the problem. I think Lai is just trying to get the reader to "think in terms of control volumes".

**Example 7.6.2 The Hose flowing onto a curved "vane" (426)**

This is another exercise in thinking about a control volume and momentum going in and out. In this case, Q is the same in and out. Due to direction, there is a change in momentum for the control volume. Momentum going in at the left is ρQ is the mass/sec going in and then $\rho Q v_0$ is the momentum going in so

$$d\mathbf{P}/dt_{in} = \rho Q v_0\, \hat{\mathbf{x}}$$

The momentum coming out is

$$d\mathbf{P}/dt_{out} = \rho Q v_0\, \hat{\mathbf{s}} \qquad \text{where } \hat{\mathbf{s}} = \cos\theta\, \hat{\mathbf{x}} + \sin\theta\, \hat{\mathbf{y}}$$

Then combining this gain and this loss we get the total gain of momentum in the control volume to be

$$d\mathbf{P}/dt = \rho Q v_0[\hat{\mathbf{x}}\text{-}\hat{\mathbf{s}}] = \rho Q v_0\, [\hat{\mathbf{x}} - \cos\theta\, \hat{\mathbf{x}} - \sin\theta\, \hat{\mathbf{y}}\,] = \rho Q v_0\{\, (1\text{-}\cos\theta)\, \hat{\mathbf{x}} - \sin\theta\, \hat{\mathbf{y}}\,\}$$

This is *a* force on the control volume. But the control volume is at rest, so there must be some equal and opposite force that cancels it, that coming from the vane. Therefore

$$\text{force of vane on jet} = -\{\, \rho Q v_0\{\, (1\text{-}\cos\theta)\, \hat{\mathbf{x}} - \sin\theta\, \hat{\mathbf{y}}\,\} = \rho Q v_0\{\, (\cos\theta\text{ -}1)\, \hat{\mathbf{x}} + \sin\theta\, \hat{\mathbf{y}}\,\}$$

and this agrees with the result stated. The force of the jet on the vane is the negative.

Now the picture is a little fishy. If the incoming stream is really horizontal, then the vane really should come all the way down to the bottom (I have modified the picture), and secondly the water coming out the top does not have a "curved" path.

Notice that the result of this problem does not depend on the shape of the curved vane! It depends only on the water exit angle, or in general on the difference between the entry and exit angles. For some complicated curve, it might take a lot of work to solve this problem in a differential manner by adding up all the little forces on the sections. So control volume method is a win. [ But I could probably solve this problem fast using a simple static picture for vane of any shape.]

**7.7 Moving Frames (427).**

I arrived at this point June 30, 2012 and things ground to a halt. My log refers to this as "the old Goldstein bugaboo". I was never comfortable with what I now call the G Rule, so I ended up spending 9 calendar weeks (2 Cod) getting that matter cleared up, which resulted in an 148 page web document Rotating Frames of Reference. I remember guessing that it might take 6 weeks. (Other guesses were a week or two...).

So now armed with this new doc, maybe I can understand this section!

The idea is that a control volume $V_c$ can be in a rotating frame. Translation:

| Lai | me | |
|---|---|---|
| $F_1$ | S | fixed frame |
| $F_2$ | S' | rotating frame (moving frame) |
| $\mathbf{r}$ | $\mathbf{r}$ | |
| $\mathbf{x}$ | $\mathbf{r}'$ | |
| $(d\mathbf{r}/dt)_{F1} = \mathbf{v}_{F1}$ | $(d\mathbf{r}/dt)_S = \mathbf{v}_S = \mathbf{v}$ | |
| $(d\mathbf{x}/dt)_{F2} = \mathbf{v}_{F2}$ | $(d\mathbf{r}'/dt)_{S'} = \mathbf{v}'_{S'} = \mathbf{v}'$ | |
| $\mathbf{r} = \mathbf{R}_0 + \mathbf{x}$ | $\mathbf{r} = \mathbf{b} + \mathbf{r}'$ | |
| $\mathbf{R}_0$ | $\mathbf{b}$ | |
| $\mathbf{v}_0$ | $\dot{\mathbf{b}}_S$ | // interpret as velocity of S' origin relative to S origin |
| $(D/Dt)_{F1}$ | $(d/dt)_S$ | |
| $(\mathbf{a}_0)_{F1} = (\mathbf{a}_0)$ | $\ddot{\mathbf{b}}_S$ | |
| $\boldsymbol{\omega}$ | $\boldsymbol{\omega}$ | ang vel of $F_2$ rel to $F_1$ (of S' rel to S) |

OK, I have verified all equations on pages 427 through 429. We are just redoing our famous Particle acceleration equation, even down to the 2 in the Coriolis-like term. So in 7.7.14 all those terms in the square brackets are the fictitious forces.

Now I want to make the direct tie-in between frames doc and Lai 7.7.14. This 7.7.14 seems to say, based on the above translation table, and ignoring the mass integrals (think m = dm)

$$m\mathbf{a}' = \text{other forces} - m\ddot{\mathbf{b}}_S - m\boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r}' - 2m\boldsymbol{\omega} \times \mathbf{v}' - m\dot{\boldsymbol{\omega}} \times \mathbf{r}'$$

Now I look at my frames doc Section 12 (a) summary and I see these equations:

$$m\mathbf{a}' = \mathbf{F}'_{eff} = \mathbf{F} + \mathbf{F}'_{fict} = \mathbf{F} - m\ddot{\mathbf{b}}_S - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - 2m\,\boldsymbol{\omega} \times \mathbf{v}' - m\dot{\boldsymbol{\omega}} \times \mathbf{r}'$$

and we are in exact 100% agreement. We identify $\mathbf{F}$ = other forces = $\int \mathbf{t}dS + \int \rho\mathbf{B}dV$ = surface traction force on the blob plus body force on the blob. Very good! This is Newton's Law or Cauchy Equation of Motion in rotating frame S' = $F_2$.

But a little more work is needed. We get 100% agreement in terms of m = dm as just discussed, but what is the nature of those integrals in 7.7.14? Are they control volume integrals, or are they moving volume integrals? Based on this discussion, I would regard the integral to which $d_t$ is applied on the LHS

as being a $V_m$ integral, while all the other integrals (since no $d_t$ of an integral) can be $V_c$ integrals. This is because we are interested in the (bogus) Newton's Law physics of the particles which form a moving blob and which stay with that blob, which means they are in $V_m$ which we can regard as $V_m(t)$.

So here is my reading of (7.4.1)

$$m\mathbf{a} = d_t\mathbf{p} = (d/dt)_{F2}\int_{vm2} \mathbf{v}_{F2}(\mathbf{x})\rho(\mathbf{x})dV = \int_{Sc2} \mathbf{t}dS + \int_{vc2} \rho\mathbf{B}dV$$

$$- \ddot{\mathbf{b}}_S \int_{vc2} \rho(\mathbf{x})dV - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \int_{vc2} \mathbf{r}' \rho(\mathbf{x})dV )$$

$$- 2\,\boldsymbol{\omega} \times \int_{vc2} \mathbf{v}_{F2}(\mathbf{x})\rho(\mathbf{x})dV - \dot{\boldsymbol{\omega}} \times \int_{vc2} \mathbf{x}\,\rho(\mathbf{x})dV \qquad (7.7.14)$$

**7.8 Moving Frames Continued (430).**

As discussed in section 7.4 notes above, if the RTT is valid in any frame, inertial or not, then it should have these two forms for our frames $F_1$ and $F_2$ :

$$(d/dt)_{F1}\int_{vm1} \mathbf{T}\,dV = \int_{vc1} \partial_t(\mathbf{T})\,dV + \int_{Sc1} \mathbf{T}\,(\mathbf{v}_{F1}\bullet\, \mathbf{n}) \qquad \text{// RTT (7.4.1) in Frame } F_1$$

$$(d/dt)_{F2}\int_{vm2} \mathbf{T}\,dV = \int_{vc2} \partial_t(\mathbf{T})\,dV + \int_{Sc2} \mathbf{T}\,(\mathbf{v}_{F2}\bullet\, \mathbf{n}) \qquad \text{// RTT (7.4.1) in Frame } F_2$$

Here, $V_{c2}$ would be a control volume that is fixed within Frame $F_2$,= S' which is our non-inertial moving frame. We can apply this second line above to $\mathbf{T} = \rho\mathbf{v}_{F2}$ if we want and it then will say

$$(d/dt)_{F2}\int_{vm2} \rho\mathbf{v}_{F2}\,dV = \int_{vc2} \partial_t(\rho\mathbf{v}_{F2})\,dV + \int_{Sc2} \rho\mathbf{v}_{F2}\,(\mathbf{v}_{F2}\bullet\, \mathbf{n}) \qquad \text{// in Frame } F_2$$

But the LHS of this equation is the same as the LHS of (7.7.14) above, so we can equate the RHS's to get

$$\int_{vc2} \partial_t(\rho\mathbf{v}_{F2})\,dV + \int_{Sc2} \rho\mathbf{v}_{F2}\,(\mathbf{v}_{F2}\bullet\, \mathbf{n}) = \int_{Sc2}\mathbf{t}dS + \int_{vc2} \rho\mathbf{B}dV$$

$$- \ddot{\mathbf{b}}_S \int_{vc2} \rho(\mathbf{x})dV - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times [\int_{vc2} \mathbf{r}' \rho(\mathbf{x})dV ] ) \qquad \text{// } \int_{vc2} \rho(\mathbf{x})dV = \text{"m" in Lai}$$

$$- 2\,\boldsymbol{\omega} \times \int_{vc2} \mathbf{v}_{F2}(\mathbf{x})\rho(\mathbf{x})dV - \dot{\boldsymbol{\omega}} \times [\int_{vc2} \mathbf{x}\,\rho(\mathbf{x})dV] \qquad \text{// (7.8.1)}$$

As a special case, if $\dot{\boldsymbol{\omega}} = 0$ and $\boldsymbol{\omega} = 0$ and we only have translation as the relation between the two frames $F_1$ and $F_2$ (possibly accelerating) we get the simpler result (7.8.2).

The control volume here has label "2" so it is a volume in frame $F_2$ which might be a rotating frame. So within frame $F_2$ this control volume $V_{c2}$ is "fixed", I agree, and that is the title of this little section as

stated in the book. As viewed from Frame $F_1$, the control volume $V_{c2}$ appears to be moving, but we don't even think of calling it $V_{m2}$ which is a completely different animal.

So *finally* on date 9.15.12 I am happy with this and the previous sections.

**7.9 Principle of Conservation of Angular Momentum (430).**

Note added 10.24.12 regarding (7.9.9). In my 10/21 version of frames doc, I forced my frame S and frame S' observers to have the same physical reference point so that **c** and **c**' were connected by b. In the 10/24 version of frames doc I decoupled them, which allows one to have **c** = **c**' = 0 which is the way Lai does things. Doing that, I concluded that

$$\mathbf{N'}^{(0)}_{fict} = -\mathbf{b} \times m\mathbf{a'} + (\mathbf{r'} + \mathbf{b}) \times \mathbf{F'}_{fict} \qquad // \ \mathbf{c} = \mathbf{c'} = 0 \qquad (11.14)$$

where

$$\mathbf{F'}_{fict} = -m\ddot{\mathbf{b}}_S - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}) - 2m\,\boldsymbol{\omega} \times \mathbf{v'} - m\dot{\boldsymbol{\omega}} \times \mathbf{r'} \qquad (8.6) \qquad (11.12)$$

Lai always has **c** = **c**' = 0 just from the **x** x (..) appearance of all his integrals. Now what he states basically in (7.9.9) is this

$$\mathbf{N'}^{(0)}_{fict} = \mathbf{r'} \times \mathbf{F'}_{fict} \qquad // \text{ integrated}$$

so he really is assuming that **b** = 0 as well. So in my note to Lai, I will comment that $\mathbf{R_0} = 0$ is assumed, otherwise there are other terms.

Note: Lai refers to a torque as a moment, and to angular momentum as "moment of momentum". I see this is a commonly used phrase, and I can't find which term was used first. One quote claims the moment of momentum was used first, perhaps by Kelvin, OED sees first angular momentum in 1870. A "couple" as wiki points out is a pair of opposite forces which result in a torque and no total force. For this reason, the effect of the couple is independent of reference point, something not true for a torque.

I worked through all details of this section through the second last equation (7.9.8) and its elegant physical interpretation as stated in italics: the total torque on a control volume $V_c$ equals the change in the angular momentum contained in the control volume plus the *out*flow from same.

The first part of this section is a sort of proof that T must be symmetric and this is still in the context of ordinary body forces and there is still an exception to this idea as we learned long ago which is not mentioned here.

Let's examine the steps that Lai takes in this section:

(1) he just states the law $d\mathbf{L}/dt = \boldsymbol{\tau}$ in the very first equation (7.9.1). The torque $\boldsymbol{\tau}$ is a combination of surface traction moment and body force moment contributions as integrals over surface and volume. Then L is an integral, and we have $D_t$ applied to it. Whenever this is the case in fluid mechanics where our "system of interest" is a moving blob, the integral to which $D_t$ is applied is over $V_m$ . Since no time derivatives, the torque integrals are just $V_c$ integrals.

(2) he then uses the RRT to rewrite the $D_t$ integral thing so you end up only with $V_c$ integrals and eventually this leads to (7.9.8) which is the big result in which only $V_c$ integrals appear and no $D_t$. The purpose of using the RTT is to get rid of any object of the form $D_t \int_{V_m}$ in favor of $\int_{V_c}$ integrals.

(3) If $V_c$ are in a rotating frame, then you have to add (7.9.9).

Now a question arises which I did not address in frames doc: How do you state the "bogus *angular* form of **F** = m**a**." ? This question is probably going to require more additions to frames doc. The angular form in general could be stated as $\boldsymbol{\tau} = d\mathbf{L}/dt$ and maybe there will be some fictitious torques in the bogus version!

OK, I have done this in a preliminary way and will add to frames doc. I conclude that if you refer angular momentum to the origin of frame S for both your Frame S and Frame S' stuff, then (7.9.9) is true. Alternatively, if you always relate torque to origin of frame you are using, then (7.9.9) applies when frames have the same origin.

This is the case for the wonderful "sprinkler example" on page 431. I have seen this type of sprinkler in my childhood and in fact they still exist. The main feature here is that the control volume is in a rotating frame of reference and therefore we are going to need the adders of (7.9.9).

**Example 7.9.1. Rotating Sprinkler.** The control volume is the entire horizontal sprinkler tube including its little ends which are each bent at angle θ. The first task is to compute $d\mathbf{L}/dt$ for the control volume which is in a rotating frame (so you cannot say things like $\boldsymbol{\tau} = d\mathbf{L}/dt$ or $\mathbf{F} = d\mathbf{P}/dt$). But you can compute $d\mathbf{L}/dt$ using the face that $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ which is true in any frame. Since $\mathbf{r} = r_o \mathbf{e}_1$ for the right spigot, this is a constant, and therefore $d\mathbf{L}/dt = \mathbf{r} \times d\mathbf{p}/dt = r_o \sin\theta \, dp/dt$ . Now dp/dt = v dm/dt. We are given Q at the start, so we can compute that Q = vA ($m^3$/sec) so then dp/dt = v dm/dt = (Q/A) dm/dt. Now in time dt, we know that the amount of ejected mass is just dm = ρdzA where z is a temp direction along the spout. But then we have dm/dt = ρA dz/dt = ρAv. Therefore, dp/dt = (Q/A) dm/dt = (Q/A) ρAv = (Q/A) ρA(Q/A) = (Q/A)ρQ. You see that Q appears twice, one for the mass rate, and once in the momentum formula. Therefore we find that dp/dt = (Q/A)ρQ. Then the $d\mathbf{L}/dt$ from one end is $dL/dt = r_0 \sin\theta (Q/A)\rho Q = \rho Q (Q/A) \sin\theta r_0$. Each end makes the same torque, so then $d\mathbf{L}/dt = 2\rho Q (Q/A) \sin\theta r_0 \mathbf{e}_3$ and we obtain equation (ii). Again, since all this is in a rotating frame, we cannot just say $\boldsymbol{\tau} = d\mathbf{L}/dt$ since that is Newton's Law, and since Newton's Law is invalid in a rotating frame.

How does this calculation I just did relate to (7.9.8) ? I think we just computed the last term of (7.9.8) which is the rate at which angular momentum flows out of the control volume. I was not thinking of this term while doing the calculation, but I think that is it, modulo a sign. The first term on the RHS is zero because there is no internal change of L in the control volume.

What about the LHS of (7.9.8) ? Is there a surface traction on $V_c$? Is there a moment associated with this t? Lai is vague on this. He says that the pressure is "zero" in some sense at the spigot and I guess that makes $\mathbf{t} = 0$ . I might instead argue that since the flow is straight out the spigot, you might have $\mathbf{t} = -p\mathbf{n}$ as a normal surface traction where **n** is at angle θ. On the other hand, there is no external agency that can possibly be applying a surface traction to $V_c$ at the flat spigot ends. These are "free", just as we dealt with free surfaces in elastic mechanics. So that then is my argument of why no surface traction contribution to the LHS of equation (7.9.8). As for the second term, the only body force is gravity, but its moment is not in our $e_3$ direction of interest, so it gives 0.

So here is the situation regarding (7.9.8) : (1) The LHS = 0; (2) the RHS is the dL/dt thing we calculated. (3) we have to add (7.9.9) fictitious torques to the LHS of (7.9.8). So, the next step is to stare at (7.9.9) which has four terms. The first gives 0 due to the integration of x from $-r_0$ to $r_0$ with x being an odd integrand. The Euler term is 0, the quad cross product term is also 0 as you can show on scratch, so only the last term of (7.9.9) survives. We then show that

$$\text{last term in (7.9.9)} = -2 \int dm\, \mathbf{x} \times (\boldsymbol{\omega} \times \mathbf{v}) = -2 \int dm\, (\omega Q/A)x\, \mathbf{e}_3 = -2\,(\omega Q/A)\, \mathbf{e}_3 \int (A\rho dx)\, x$$

$$= -\,(\omega\rho Q)\, r_0^2\, \mathbf{e}_3$$

where this is for the right arm only, so double it for both arms.

Now we are prepared to apply (7.9.8) including (7.9.9)

$$\underset{\text{(7.9.8) LHS}}{0} + \underset{\text{(7.9.9) last term}}{[-2(\omega\rho Q)\, r_0^2\, \mathbf{e}_3]} = \underset{\text{(7.9.8) RHS second term}}{d\mathbf{L}/dt = 2\rho Q(Q/A)\sin\theta r_0 \mathbf{e}_3}$$

or

$$-\,(\omega)\, r_0 = (Q/A)\sin\theta$$

$$\Rightarrow\ \omega = -(Q/A)\sin\theta/r_0$$

and this then is how fast your frictionless sprinkler rotates. Perhaps if spigots were both pointed up a bit, result would be the same if θ is taken as the angle seen from above.

Can this problem be solved by elementary means? One way is to use Newton's Angular Law with fictitious torques added which would be a frames doc application. I am not sure how you would solve this in the non-rotating from but no doubt you can. Perhaps once the tube reaches the above ω, there is no longer any angular acceleration.

Comment: this is an excellent example problem. It demonstrates several aspects of fluid dynamics problem solving (1) the use of a control volume; (2) use of conservation of angular momentum. (3) the complication of a rotating frame of reference.

**7.10 Principle of Conservation of Energy (432).**

Again, the very first equation says that

$$D_t \int_{V_m} (\text{KE+PE}) = \text{work done by t and B} + \text{heat flux in} + \text{heat sources inside}$$

As in the previous section, we use the RTT to get rod of the $D_t \int_{V_m}$ object in favor of $\int_{V_c}$ integrals and we end up with (7.10.5) which indeed has only $V_c$ type integrals. As expected, we eventually end up with the differential form of energy conservation which is (7.10.9) which says $D_u$ = work + heat + heat. The body force **B** vanishes from sight in this math and only tr(TD) remains. The EOM is what got rid of B.

The **example** here is a bit beyond me right now. I follow all the steps, but I don't understand the physical situation. The control volume is a thin cross section of ideal gas in a duct with different parameters on the two faces. Eventually he finds that $p_2$ can be related to $p_1$ in two different ways. One is just the usual $p_2 = p_1$ but the other implies a p discontinuity at the cross section and this is called a "normal compression shock". But we have really now learned about such things much in this book.

**7.11 Principle of Entropy (2nd law thermo) (436).**

The entropy topic keeps recurring in the book. The idea is that entropy can never decrease, and here this is being written in the integral form. As usual, the very first equation is the statement of the fact and now we have the famous ≥ operator in place of the = of the previous sections. We have a $D_t \int_{vm}$ situation where the integral gathers up the entropy in the blob, we integrate ρηdV. As usual, we use the RTT to unload this integral, and we end up with all control volume integrals as in (7.11.11) with its usual italicized interpretation which makes sense to me. I did not verify the equations of this 2 page section, putting that off for some future time.

And this, my friends, brings us to the end of Lai Chapter 7! He has a set of 27 problems I could do if I were energetic. I started this Chapter on July 1, 2012 and today is Sept 16, 2012, so about 10 calendar weeks played out while I did this relatively short Chapter.

This was a very good chapter, though I think Lai failed to stress the notion I state above that it is the $V_m$ integral which is the "object" subject to Newton's Law.