Grad of Tensor by the Lai method PhL 10.13.12

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How does Lai do it? Start with his (8A.12) which, converted to my notation, says

$$d\hat{\mathbf{e}}_i = K_{ijk}\, dx_j\, \hat{\mathbf{e}}_k \qquad \text{His } \Gamma \text{ I write as K}$$

so he is saying

$$\partial_j \hat{\mathbf{e}}_i = K_{ijk}\hat{\mathbf{e}}_k \qquad \text{or} \qquad \partial_j \hat{\mathbf{e}}_n = K_{njk}\hat{\mathbf{e}}_k$$

In tensor doc App F I have

$$(\partial'_j \mathbf{e}_n) = \Gamma'^{k}_{jn}\, \mathbf{e}_k$$

So write this last as

$$\partial'_j(h'_n \hat{\mathbf{e}}_n) = \Gamma'^{k}_{jn} h'_k\, \hat{\mathbf{e}}_k$$

$$\partial'_j(h'_n)\, \hat{\mathbf{e}}_n + h'_n \partial'_j \hat{\mathbf{e}}_n = \Gamma'^{k}_{jn} h'_k\, \hat{\mathbf{e}}_k$$

$$h'_n \partial'_j \hat{\mathbf{e}}_n = \Gamma'^{k}_{jn} h'_k\, \hat{\mathbf{e}}_k - \partial'_j(h'_n)\, \hat{\mathbf{e}}_n$$

$$\partial'_j \hat{\mathbf{e}}_n = [\Gamma'^{k}_{jn} h'_k / h'_n]\, \hat{\mathbf{e}}_k - \partial'_j(h'_n)/ h'_n \hat{\mathbf{e}}_n$$

$$= [\Gamma'^{k}_{jn} h'_k / h'_n - \delta_{n,k}\, \partial'_j(h'_n)/ h'_n]\, \hat{\mathbf{e}}_k$$

Comparison then shows that

$$K_{njk} = h'^{-1}_n [h'_k \Gamma'^{k}_{jn} - \partial'_j(h'_n)\, \delta_{n,k}]$$

$$K_{ijk} = h'^{-1}_i [h'_k \Gamma'^{k}_{ji} - \partial'_j(h'_i)\, \delta_{i,k}]$$

so this is the gamma object he is using. Let's just see what the values come out to be. I did it in Maple, and the result is that my K_{ijk} agrees with his Γ_{ijk} for sphericals on page 503 top.

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for i from 1 to 3 do
  for j from 1 to 3 do
    for k from 1 to 3 do
      f := (1/hp[i])*(hp[k]*G(k,j,i)- delta(i,k)*Diff(hp[i],xp[j]));
      print(i,j,k,"     ",simplify(value(f)));
        od
  od
od;
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1, 1, 1, " ", 0	2, 1, 1, " ", 0	3, 1, 1, " ", 0
1, 1, 2, " ", 0	2, 1, 2, " ", 0	3, 1, 2, " ", 0
1, 1, 3, " ", 0	2, 1, 3, " ", 0	3, 1, 3, " ", 0
1, 2, 1, " ", 0	2, 2, 1, " ", -1	3, 2, 1, " ", 0
1, 2, 2, " ", 1	2, 2, 2, " ", 0	3, 2, 2, " ", 0
1, 2, 3, " ", 0	2, 2, 3, " ", 0	3, 2, 3, " ", 0
1, 3, 1, " ", 0	2, 3, 1, " ", 0	3, 3, 1, " ", −sin(θ)
1, 3, 2, " ", 0	2, 3, 2, " ", 0	3, 3, 2, " ", −cos(θ)
1, 3, 3, " ", sin(θ)	2, 3, 3, " ", cos(θ)	3, 3, 3, " ", 0

Next, looking at page 501, I can associate

$M_{ijk} = (\nabla\mathcal{J})'_{ijk}$ M = ∇T Lai page 503 (8A.15)

which is good. How his *claim* on top of page 504 is that

$M_{ijm}h_m = \partial_m T_{ij} + T_{qj}\Gamma_{qmi} + T_{iq}\Gamma_{gmj}$

Translated to my notation, this claim says

$(\nabla\mathcal{J})'_{ijm}h'_m = \partial_m\mathcal{J}'_{ij} + \mathcal{J}'_{qj}K_{qmi} + \mathcal{J}'_{iq}K_{gmj}$ // no sum on m

$(\nabla\mathcal{J})'_{ijm} = h'^{-1}_m\partial_m\mathcal{J}'_{ij} + h'^{-1}_m\mathcal{J}'_{qj}K_{qmi} + h'^{-1}_m\mathcal{J}'_{iq}K_{gmj}$

$(\nabla\mathcal{J})'_{ijk} = h'^{-1}_k\partial_k\mathcal{J}'_{ij} + h'^{-1}_k\mathcal{J}'_{qj}K_{qki} + h'^{-1}_k\mathcal{J}'_{iq}K_{gkj}$

$= (h'_i\, h'_j\, h'_k)^{-1}\,[h'_i\, h'_j\,(\partial_k\mathcal{J}'_{ij}) + h'_i\, h'_j\,\mathcal{J}'_{qj}K_{qki} + h'_i\, h'_j\,\mathcal{J}'_{iq}K_{gkj}]$

My formula for this same object is:

$(\nabla\mathcal{J})'_{ijk} = (h'_i\, h'_j\, h'_k)^{-1}\,(\nabla T)'_{ijk}$

where $(\nabla T)'_{ijk} = h'_i\, h'_j(\partial'_k\mathcal{J}'_{ij}) + \partial'_k(h'_i\, h'_j)\,\mathcal{J}'_{ij} - \Gamma'^{n}_{ik}(h'_n\, h'_j\mathcal{J}'_{nj}) - \Gamma'^{n}_{jk}(h'_i\, h'_n\mathcal{J}'_{in})$

So we are happy if we can show that

$[h'_i\, h'_j\,(\partial_k\mathcal{J}'_{ij}) + h'_i\, h'_j\,\mathcal{J}'_{qj}K_{qki} + h'_i\, h'_j\,\mathcal{J}'_{iq}K_{gkj}]$

$= h'_i\, h'_j(\partial'_k\mathcal{J}'_{ij}) + \partial'_k(h'_i\, h'_j)\,\mathcal{J}'_{ij} - \Gamma'^{n}_{ik}(h'_n\, h'_j\mathcal{J}'_{nj}) - \Gamma'^{n}_{jk}(h'_i\, h'_n\mathcal{J}'_{in})$

The first terms on each side match, so I have to show that

$$h'_i\, h'_j\, \mathcal{T}'_{qj} K_{qki} + h'_i\, h'_j\, \mathcal{T}'_{iq} K_{qkj}$$

$$= \partial'_k(h'_i\, h'_j)\, \mathcal{T}'_{ij} - \Gamma'^{n}_{ik}(h'_n\, h'_j \mathcal{T}'_{nj}) - \Gamma'^{n}_{jk}(h'_i\, h'_n \mathcal{T}'_{in}) \qquad (*)$$

We know that

$$K_{ijk} = h'^{-1}_i\, [h'_k \Gamma'^{k}_{ji} - \partial'_j(h'_i)\, \delta_{i,k}],$$

$$K_{IJK} = h'^{-1}_I\, [h'_K \Gamma'^{K}_{JI} - \partial'_J(h'_I)\, \delta_{I,K}],$$

$$K_{qki} = h'^{-1}_q\, [h'_i \Gamma'^{i}_{kq} - \partial'_k(h'_q)\, \delta_{q,i}],$$

$$K_{qkj} = h'^{-1}_q\, [h'_j \Gamma'^{j}_{kq} - \partial'_k(h'_q)\, \delta_{q,j}],$$

We can then evaluate the LHS of (*) above

$$\text{LHS} = h'_i\, h'_j\, \mathcal{T}'_{qj} K_{qki} + h'_i\, h'_j\, \mathcal{T}'_{iq} K_{gkj}$$

$$= h'_i\, h'_j\, \mathcal{T}'_{qj}\, h'^{-1}_q\, [h'_i \Gamma'^{i}_{kq} - \partial_k(h'_q)\, \delta_{q,i}] + h'_i\, h'_j\, \mathcal{T}'_{iq}\, h'^{-1}_q\, [h'_j \Gamma'^{j}_{kq} - \partial_k(h'_q)\, \delta_{q,j}]$$

$$= h'_i\, h'_j\, \mathcal{T}'_{nj}\, h'^{-1}_n\, [h'_i \Gamma'^{i}_{kn} - \partial_k(h'_n)\, \delta_{n,i}] + h'_i\, h'_j\, \mathcal{T}'_{in}\, h'^{-1}_n\, [h'_j \Gamma'^{j}_{kn} - \partial_k(h'_n)\, \delta_{n,j}]$$

$$= h'^{2}_i\, h'_j\, \mathcal{T}'_{nj}\, h'^{-1}_n\, \Gamma'^{i}_{kn} + h'_i\, h'^{2}_j\, \mathcal{T}'_{in}\, h'^{-1}_n \Gamma'^{j}_{kn}$$

$$- h'_i\, h'_j\, \mathcal{T}'_{nj}\, h'^{-1}_n \partial_k(h'_n)\, \delta_{n,i} - h'_i\, h'_j\, \mathcal{T}'_{in}\, h'^{-1}_n \partial_k(h'_n)\, \delta_{n,j}$$

$$= h'^{2}_i\, h'_j\, h'^{-1}_n\, \Gamma'^{i}_{kn}\, \mathcal{T}'_{nj} + h'_i\, h'^{2}_j\, h'^{-1}_n \Gamma'^{j}_{kn}\, \mathcal{T}'_{in}$$
$$- h'_i\, h'_j\, \mathcal{T}'_{ij}\, h'^{-1}_i \partial_k(h'_i) - h'_i\, h'_j\, \mathcal{T}'_{ij}\, h'^{-1}_j \partial_k(h'_j)$$

$$= h'^{2}_i\, h'_j\, h'^{-1}_n\, \Gamma'^{i}_{kn}\, \mathcal{T}'_{nj} + h'_i\, h'^{2}_j\, h'^{-1}_n \Gamma'^{j}_{kn}\, \mathcal{T}'_{in}$$
$$- h'_j\, \partial_k(h'_i)\, \mathcal{T}'_{ij} - h'_i \partial_k(h'_j)\, \mathcal{T}'_{ij}$$

$$= h'^{2}_i\, h'_j\, h'^{-1}_n\, \Gamma'^{i}_{kn}\, \mathcal{T}'_{nj} + h'_i\, h'^{2}_j\, h'^{-1}_n \Gamma'^{j}_{kn}\, \mathcal{T}'_{in}$$
$$- [\, h'_j\, \partial_k(h'_i) + h'_i \partial_k h'_j)]\, \mathcal{T}'_{ij}$$

$$= h'^{2}_i\, h'_j\, h'^{-1}_n\, \Gamma'^{i}_{kn}\, \mathcal{T}'_{nj} + h'_i\, h'^{2}_j\, h'^{-1}_n \Gamma'^{j}_{kn}\, \mathcal{T}'_{in} - \partial'_k(h'_i\, h'_j)\, \mathcal{T}'_{ij}$$

So I now want to show that

$$\partial'_k(h'_i\, h'_j)\, \mathcal{T}'_{ij} - \Gamma'^{n}_{ik}(h'_n\, h'_j \mathcal{T}'_{nj}) - \Gamma'^{n}_{jk}(h'_i\, h'_n \mathcal{T}'_{in})$$

$$= h'^{2}_i\, h'_j\, h'^{-1}_n\, \Gamma'^{i}_{kn}\, \mathcal{T}'_{nj} + h'_i\, h'^{2}_j\, h'^{-1}_n \Gamma'^{j}_{kn}\, \mathcal{T}'_{in} - \partial'_k(h'_i\, h'_j)\, \mathcal{T}'_{ij} \qquad (**)$$

Unfortunately, the Γ sum index is not in the same place on the two sides, so I have to do something about that. I do have this theorem:

$$(\partial'_c g'^{ab}) = -[g'^{an}\,\Gamma'^{b}{}_{cn} + g'^{bn}\,\Gamma'^{a}{}_{cn}]$$

For orthogonals, we have $g'^{ab} = \delta_{a,b} h'^{-2}_a$

$$(\partial'_c g'^{ab}) = \partial'_c[\delta_{a,b} h'^{-2}_a] = \delta_{a,b}\partial'_c\,(h'^{-2}_a) = -2\,h'^{-3}_a \delta_{a,b}(\partial'_c h'_a)$$

The theorem then reads

$$+2\,h'^{-3}_a \delta_{a,b}(\partial'_c h'_a) = g'^{an}\,\Gamma'^{b}{}_{cn} + g'^{bn}\,\Gamma'^{a}{}_{cn} = h'^{-2}_a\,\Gamma'^{b}{}_{ca} + h'^{-2}_b\,\Gamma'^{a}{}_{cb}$$

or

$$+2\,h'^{-3}_a \delta_{a,b}(\partial'_c h'_a) = h'^{-2}_a\,\Gamma'^{b}{}_{ca} + h'^{-2}_b\,\Gamma'^{a}{}_{cb}$$

or

$$+2\,h'^{-1}_a \delta_{a,b}(\partial'_c h'_a) = \Gamma'^{b}{}_{ca} + h'^{2}_a h'^{-2}_b\,\Gamma'^{a}{}_{cb}$$

or

$$\Gamma'^{b}{}_{ca} = -h'^{2}_a h'^{-2}_b\,\Gamma'^{a}{}_{cb} + 2\,h'^{-1}_a \delta_{a,b}(\partial'_c h'_a)$$

or

$$\Gamma'^{i}{}_{kn} = -h'^{2}_n h'^{-2}_i\,\Gamma'^{n}{}_{ki} + 2\,h'^{-1}_n \delta_{n,i}(\partial'_k h'_n)$$
$$\Gamma'^{j}{}_{kn} = -h'^{2}_n h'^{-2}_j\,\Gamma'^{n}{}_{kj} + 2\,h'^{-1}_n \delta_{n,j}(\partial'_k h'_n)$$

If I use these in the RHS of (**), I get

$$h'^{2}_i\,h'_j\;h'^{-1}_n\,\Gamma'^{i}{}_{kn}\,\mathcal{T}'_{nj} + h'_i\,h'^{2}_j\;h'^{-1}_n\Gamma'^{j}{}_{kn}\,\mathcal{T}'_{in} - \partial'_k(h'_i\,h'_j)\,\mathcal{T}'_{ij}$$

$$= h'^{2}_i\,h'_j\;h'^{-1}_n$$

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Go back here and try to rewrite:

$$K_{ijk} = h'^{-1}_i\,[h'_k \Gamma'^{k}{}_{ji} - \partial'_j(h'_i)\,\delta_{i,k}]$$

Theorem said

$$\Gamma'^{b}{}_{ca} = -h'^{2}_a h'^{-2}_b\,\Gamma'^{a}{}_{cb} + 2\,h'^{-1}_a \delta_{a,b}(\partial'_c h'_a)$$
$$\Gamma'^{k}{}_{ji} = -h'^{2}_i h'^{-2}_k\,\Gamma'^{i}{}_{jk} + 2\,h'^{-1}_i \delta_{i,k}(\partial'_j h'_i)$$

Then

$$K_{ijk} = h'^{-1}_i h'_k \Gamma'^{k}{}_{ji} - h'^{-1}_i \partial'_j(h'_i)\,\delta_{i,k}$$

$$= h'^{-1}_i h'_k[-h'^{2}_i h'^{-2}_k\,\Gamma'^{i}{}_{jk} + 2\,h'^{-1}_i \delta_{i,k}(\partial'_j h'_i)] - h'^{-1}_i \partial'_j(h'_i)\,\delta_{i,k}$$

$$= -h'_i h'^{-1}_k\,\Gamma'^{i}{}_{jk} + 2\,h'^{-2}_i h'_k\,\delta_{i,k}(\partial'_j h'_i) - h'^{-1}_i\,\delta_{i,k}\,(\partial'_j h'_i)$$

$$= - h'_i h'^{-1}_k \Gamma'^{i}_{jk} + 2\, h'^{-1}_i \delta_{i,k}(\partial'_j h'_i) - h'^{-1}_i \delta_{i,k}(\partial'_j h'_i)$$

$$= - h'_i h'^{-1}_k \Gamma'^{i}_{jk} + h'^{-1}_i \delta_{i,k}(\partial'_j h'_i)$$

$$= - h'_i h'^{-1}_k \Gamma'^{i}_{jk} + h'^{-1}_k \delta_{i,k}(\partial'_j h'_i)$$

$$= h'^{-1}_k [- h'_i \Gamma'^{i}_{jk} + \delta_{i,k}(\partial'_j h'_i)]$$

So I now have an alternate form

$$K_{ijk} = h'^{-1}_k [- h'_i \Gamma'^{i}_{jk} + \delta_{i,k}(\partial'_j h'_i)]$$

Now go back to the thing we wanted to show

$$h'_i h'_j \mathcal{T}'_{qj} K_{qki} + h'_i h'_j \mathcal{T}'_{iq} K_{qkj}$$

$$= \partial'_k(h'_i h'_j)\, \mathcal{T}'_{ij} - \Gamma'^{n}_{ik}(h'_n h'_j \mathcal{T}'_{nj}) - \Gamma'^{n}_{jk}(h'_i h'_n \mathcal{T}'_{in}) \qquad (*)$$

Now prepare to appropriate K objects

$$K_{ijk} = h'^{-1}_k [- h'_i \Gamma'^{i}_{jk} + \delta_{i,k}(\partial'_j h'_i)]$$
$$K_{IJK} = h'^{-1}_K [- h'_I \Gamma'^{I}_{JK} + \delta_{I,K}(\partial'_J h'_I)]$$
$$K_{qki} = h'^{-1}_i [- h'_q \Gamma'^{q}_{ki} + \delta_{q,i}(\partial'_k h'_q)]$$
$$K_{qkj} = h'^{-1}_j [- h'_q \Gamma'^{q}_{kj} + \delta_{q,j}(\partial'_k h'_q)]$$

Then for LHS(*) we get

$$h'_i h'_j \mathcal{T}'_{qj} K_{qki} + h'_i h'_j \mathcal{T}'_{iq} K_{qkj}$$

$$= h'_i h'_j \mathcal{T}'_{qj} h'^{-1}_i [- h'_q \Gamma'^{q}_{ki} + \delta_{q,i}(\partial'_k h'_q)] + h'_i h'_j \mathcal{T}'_{iq} h'^{-1}_j [- h'_q \Gamma'^{q}_{kj} + \delta_{q,j}(\partial'_k h'_q)]$$

$$= h'_j \mathcal{T}'_{qj} [- h'_q \Gamma'^{q}_{ki} + \delta_{q,i}(\partial'_k h'_q)] + h'_i \mathcal{T}'_{iq} [- h'_q \Gamma'^{q}_{kj} + \delta_{q,j}(\partial'_k h'_q)]$$

$$= h'_j \mathcal{T}'_{nj} [- h'_n \Gamma'^{n}_{ki} + \delta_{n,i}(\partial'_k h'_n)] + h'_i \mathcal{T}'_{in} [- h'_n \Gamma'^{n}_{kj} + \delta_{n,j}(\partial'_k h'_n)]$$

$$= h'_j \mathcal{T}'_{nj} [- h'_n \Gamma'^{n}_{ki} + \delta_{n,i}(\partial'_k h'_i)] + h'_i \mathcal{T}'_{in} [- h'_n \Gamma'^{n}_{kj} + \delta_{n,j}(\partial'_k h'_j)]$$

$$= - h'_n h'_j \Gamma'^{n}_{ki} \mathcal{T}'_{nj} - h'_i h'_n \Gamma'^{n}_{kj} \mathcal{T}'_{in} + h'_j \mathcal{T}'_{nj} \delta_{n,i}(\partial'_k h'_i) + h'_i \mathcal{T}'_{in} \delta_{n,j}(\partial'_k h'_j)$$

$$= - h'_n h'_j \Gamma'^{n}_{ki} \mathcal{T}'_{nj} - h'_i h'_n \Gamma'^{n}_{kj} \mathcal{T}'_{in} + h'_j \mathcal{T}'_{ij}(\partial'_k h'_i) + h'_i \mathcal{T}'_{ij} (\partial'_k h'_j)$$

$$= - h'_n h'_j \Gamma'^{n}_{ki} \mathcal{T}'_{nj} - h'_i h'_n \Gamma'^{n}_{kj} \mathcal{T}'_{in} + [\, h'_j (\partial'_k h'_i) + h'_i (\partial'_k h'_j)]\, \mathcal{T}'_{ij}$$

$$= - h'_n\, h'_j\, \Gamma'^{n}_{ki}\, \mathcal{T}'_{nj} - h'_i\, h'_n\, \Gamma'^{n}_{kj}\, \mathcal{T}'_{in} + \partial'_k(h'_i\, h'_j)\, \mathcal{T}'_{ij}$$

and FINALLY we get agreement. Let's recheck the maple with this new form. Yes! So this is the way it works best

$$K_{ijk} = h'^{-1}_k[- h'_i\, \Gamma'^{i}_{jk} + \delta_{i,k}(\partial'_j h'_i)\ ]$$

or

$$\Gamma^{(Lai)}{}_{ijk} = h'^{-1}_k[- h'_i\, \Gamma'^{i}_{jk} + \delta_{i,k}(\partial'_j h'_i)\ ]$$