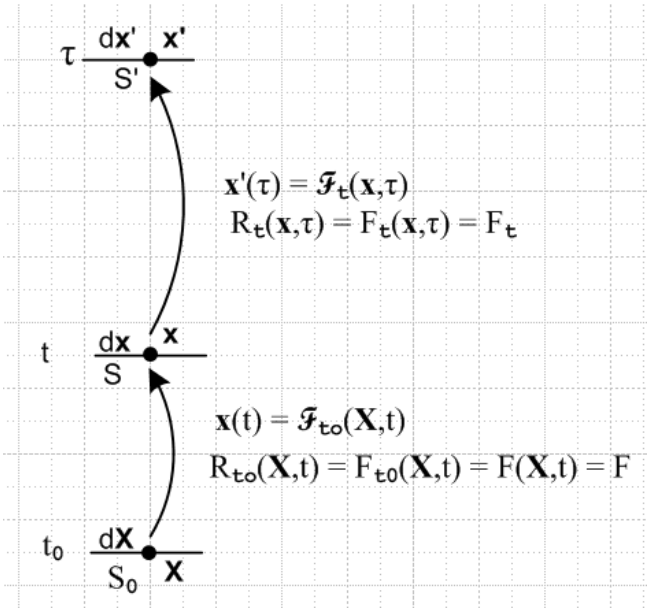
The Rivlin-Eriksen tensors PhL 12.15.12

Things here are much more complicated that Lai lets on. Below I had to do 6 Lemmas and 6 Theorems to be happy with the derivation of (8.11.1) and (8.11.3) !!

This is a complicated subject because we are dealing with two "pictures" and one must be clear which is which at any given time. Recall this from Appendix J,



Picture 1, where **X** is a FIXED reference: (d**X** is a dumbbell near **X**)

$d\mathbf{x} = F(\mathbf{X},t)\, d\mathbf{X}$ $\quad dx_i = F_{ij}dX_j$

$dx_i = (\partial x_i/\partial X_j)dX_j$,

$F_{ij} = (\partial x_i/\partial X_j)$ or $\mathbf{F} = (\nabla \mathbf{x})$ // Lai p 105 (3.18.4)

Picture 2, where **x** is a FIXED reference: (d**x** is a dumbbell near **x**)

$d\mathbf{x}' = F_t(\mathbf{x},\tau)\, d\mathbf{x}$ $\quad dx'_i = (F_t)_{ij}dx_j$ // Lai p 457 (8.7.2)

$dx'_i = (\partial x'_i/\partial x_j)dx_j$,

$(F_t)_{ij} = (\partial x'_i/\partial x_j)$ or $\mathbf{F}_t = (\nabla \mathbf{x}')$ // Lai p 457 (8.7.3)

Lai's Derivation Path.

Part A. We start off with a batch of Lemmas, some collected from earlier in the book:

Lemma 1. $\mathbf{v} = d\mathbf{x}/dt = D\mathbf{x}/Dt = \hat{\mathbf{v}}(\mathbf{X},t) = \tilde{\mathbf{v}}(\mathbf{x},t)$

You can parameterize the velocity of a fluid either way. These are the Lagrangian and Eulerian pictures. The two functions are different, and we normally assume that **v** means $\tilde{\mathbf{v}}(\mathbf{x},t)$ which write as $\mathbf{v}(\mathbf{x},t)$.

Lemma 2. $d\mathbf{v} = d\hat{\mathbf{v}}(\mathbf{X},t) = \hat{\mathbf{v}}(\mathbf{X}+d\mathbf{X},t) - \hat{\mathbf{v}}(\mathbf{X},t) = (\nabla^{(\mathbf{X})}\hat{\mathbf{v}}(\mathbf{X},t))d\mathbf{X}$

$d\mathbf{v} = d\tilde{\mathbf{v}}(\mathbf{x},t) = \tilde{\mathbf{v}}(\mathbf{x}+d\mathbf{x},t) - \tilde{\mathbf{v}}(\mathbf{x},t) = (\nabla^{(\mathbf{x})}\tilde{\mathbf{v}}(\mathbf{x},t))d\mathbf{x}$

Here d**v** is the velocity differential between two close locations in a fluid at time t. You can write d**v** in the two ways shown, based on Lemma 1. This is a "dumbbell based concept", where the dumbbells marks the two points in space at which you measure your two velocities to get d**v**. They are d**X** and d**x**. The expressions on the far right arise from writing d**v** using normal calculus of several variables and realizing that dt = 0 so there are no $(\partial_t \mathbf{v})dt$ contributions. The dt=0 idea is because the two ends of the dumbbell exist at the same time t.

Corollary 2. $(\nabla^{(\mathbf{X})}\hat{\mathbf{v}}(\mathbf{X},t))d\mathbf{X} = (\nabla^{(\mathbf{x})}\tilde{\mathbf{v}}(\mathbf{x},t))d\mathbf{x}$

Lemma 3. $d\mathbf{v} = d_t(d\mathbf{x})$

Proof: $(d\mathbf{x}) = \mathbf{x}(\mathbf{X}+d\mathbf{X},t) - \mathbf{x}(\mathbf{X},t)$ = dumbbell at time t, so then

$d_t(d\mathbf{x}) = d_t\mathbf{x}(\mathbf{X}+d\mathbf{X},t) - d_t\mathbf{x}(\mathbf{X},t) = \hat{\mathbf{v}}(\mathbf{X}+d\mathbf{X},t) - \hat{\mathbf{v}}(\mathbf{X},t) = d\mathbf{v}$ by Lemma 2.

Lemma 4. $d_t(d\mathbf{x}) = (\nabla^{(\mathbf{X})}\hat{\mathbf{v}}(\mathbf{X},t))d\mathbf{X} = (\nabla^{(\mathbf{x})}\tilde{\mathbf{v}}(\mathbf{x},t))d\mathbf{x}$ // Lai p 95 (3.12.5)

This follows at once from Lemma 3 and Lemma 2.

Corollary 4. $d_t(d\mathbf{x}) = (\nabla\mathbf{v})dx$ // Lai p 95 (3.12.6)

This is just a compact statement of the Lemma 4, where $(\nabla\mathbf{v}) = (\nabla^{(\mathbf{x})}\tilde{\mathbf{v}}(\mathbf{x},t))$.

Lemma 5. $(1/2)\, d_t[(ds)^2] = d\mathbf{x} \bullet D\, d\mathbf{x}$ // page 97, 3.13.11

Proof: Watch carefully:

$d\mathbf{x} \bullet D\, d\mathbf{x} = d\mathbf{x} \bullet (D+W)\, d\mathbf{x}$ // because W is antisymmetric

because: $d\mathbf{x} \bullet W\, d\mathbf{x} = dx_i W_{ij} dx_j = 0$

Therefore, according to the p 95 3.13.1 definitions of D and W,

$d\mathbf{x} \bullet D\, d\mathbf{x} = d\mathbf{x} \bullet (\nabla\mathbf{v})\, d\mathbf{x}$

But Corollary 4 says $d_t(d\mathbf{x}) = (\nabla\mathbf{v})dx$ so we have

$d\mathbf{x} \bullet D\, d\mathbf{x} = d\mathbf{x} \bullet d_t(d\mathbf{x})$

Now we can write

$(ds)^2 = dx \bullet dx$ so $d_t(ds)^2 = 2\, dx \bullet d_t(dx)$

Therefore

$dx \bullet D\, dx = dx \bullet d_t(dx) = (1/2)\, d_t(ds)^2$ QED.

Lemma 6: If T is symmetric, and if $x^T T x = 0$ for all x, then T = 0.

Let D be the diagonalized version of T

$T = S^T D S$ $D = STS^T$

Then

$x^T T x = 0 \Rightarrow x^T(S^T D S)x = (Sx)^T D (Sx) = y^T D y = 0$. $y = Sx$

Since x is an arbitrary vector, we can regard y as an arbitrary vector for fixed S. We then have

$y^T D y = 0$ for all y, where D is diagonal.

This says

$\Sigma_i y_i D_{ii} y_i = 0 \Rightarrow \Sigma_i\, y_i^2 D_{ii} = 0$.

If we take $y_i = \delta_{i,n}$ so that $y = u_n$, we get

$\Sigma_i\, \delta_{i,n}\, \delta_{i,n}\, D_{ii} = 0 \Rightarrow D_{nn} = 0$

Thus, D = 0 as a matrix, and it follows that $T = S^T D S = 0$ as well. QED

If T is not symmetric, the conclusion does not follow.

Part B. We have here a tricky series of steps.

Theorem 1. $dx \bullet \partial_\tau^n C_t(x,\tau)\, dx = \partial_\tau^n ([ds']^2)$

Here we are in Picture 2 described above, the upper half of the picture where x, dx = fixed.

Proof:

$dx \bullet \partial_\tau^n C_t(x,\tau)\, dx = \partial_\tau^n (dx \bullet C_t(x,\tau)\, dx) = \partial_\tau^n (dx \bullet F_t^T F_t\, dx)$

$= \partial_\tau^n [F_t(x,\tau) dx \bullet F_t(x,\tau) dx] = \partial_\tau^n [dx' \bullet dx'] = \partial_\tau^n ([ds']^2)$ QED

Notice that **x** and d**x** are not functions of τ, they are fixed like **X** and d**X** are fixed in Picture 1. That is why the ∂_τ operator treats the d**x** dumbbells as constants.

Theorem 2. One can replace ∂_t by d_t anywhere and everywhere in Theorem 1.

In Picture 2, **x** is fixed just the way **X** is fixed in Picture 1. This means $d_t\, f(\mathbf{x},t) = \partial_t f(\mathbf{x},t)$ for any function f in the Picture 2 context. We can then repeat the entire proof of Theorem 1

$$d\mathbf{x} \bullet d_\tau^{\,n} C_t(\mathbf{x},\tau)\, d\mathbf{x} \;=\; d_\tau^{\,n}\,(d\mathbf{x} \bullet C_t(\mathbf{x},\tau)\, d\mathbf{x}\,) = d_\tau^{\,n}\,(d\mathbf{x} \bullet F_t^{\mathrm{T}} F_t\, d\mathbf{x}\,)$$

$$= d_\tau^{\,n}\,[\, F_t(\mathbf{x},\tau) d\mathbf{x} \bullet F_t(\mathbf{x},\tau) d\mathbf{x}] \;=\; d_\tau^{\,n}\,[\, d\mathbf{x}' \bullet d\mathbf{x}'\,] \;= d_\tau^{\,n}\,(\,[ds']^2) \qquad \text{QED}$$

Summary of Theorem 1 and Theorem 2:

$$d\mathbf{x} \bullet \partial_\tau^{\,n} C_t(\mathbf{x},\tau)\, d\mathbf{x} \;= d\mathbf{x} \bullet d_\tau^{\,n} C_t(\mathbf{x},\tau)\, d\mathbf{x} \;= \partial_\tau^{\,n}\,(\,[ds']^2) \;= d_\tau^{\,n}\,(\,[ds']^2) \qquad (*)$$

Theorem 3. Limit $[\; d_\tau^{\,n}\,(\,[ds']^2)\,]^{\tau=t} = d_t^{\,n}(\,[ds]^2)$.

In Picture 2, as we do this limit, we know that $F_t \to 1$ and $ds' \to ds$ and $d_\tau = d_t$ when $\tau = t$. The expression on the right involves $[ds]^2(\mathbf{x},t) = d\mathbf{x}(\mathbf{x},t) \bullet d\mathbf{x}(\mathbf{x},t)$.

Theorem 3. $d_t^{\,n}(\,[ds]^2) = \partial_t^{\,n}(\,[ds]^2)$

In Picture 2, **x** is a fixed reference point, so we know that $d_t[d\mathbf{x}(\mathbf{x},t)] \;= \partial_t[d\mathbf{x}(\mathbf{x},t)]$, and it then follows that $d_t[ds]^2 \;= \partial_t[ds]^2$. Recall that **x** is fixed in Pic 2 the way **X** is fixed in Pic 1. This is the same idea as Theorem 2 and applies really to any statement in Picture 1.

Theorem 4: $d_t^{\,n}(\,[ds]^2) \;=\; d\mathbf{x} \bullet A_n(\mathbf{x},t)\, d\mathbf{x}$

Proof: From (*) we have

$$d\mathbf{x} \bullet \partial_\tau^{\,n} C_t(\mathbf{x},\tau)\, d\mathbf{x} \;= d_\tau^{\,n}\,(\,[ds']^2)$$

We take the limit of both sides as $\tau \to t$. The LHS is of course $d\mathbf{x} \bullet A_n(\mathbf{x},t)\, d\mathbf{x}$. The RHS according to Theorem 3 is $\partial_t^{\,n}(\,[ds]^2)$. QED

Theorem 5 (recursion) $A_{n+1} = (\nabla \mathbf{v})^{\mathrm{T}} A_n + D_t A_n \;+ A_n(\nabla \mathbf{v})$ // Lai p 468 (8.11.3)

Proof: Start with the result of Theorem 4,

$$d_t^{\,n}(\,[ds(\mathbf{x},t)]^2) \;=\; d\mathbf{x} \bullet A_n(\mathbf{x},t)\, d\mathbf{x}$$

We now switch to Picture 1 where there is no τ and where $\mathbf{X}$ and t_0 are the references. Now $d\mathbf{x}$ is no longer a constant as it was in Picture 2. We then apply d_t to the above equation

$$d_t^{n+1}([ds(\mathbf{x},t)]^2) = d_t[d\mathbf{x} \bullet A_n(\mathbf{x},t)\, d\mathbf{x}]$$

$$= d_t(d\mathbf{x}) \bullet \mathbf{A_n}\, d\mathbf{x} + d\mathbf{x} \bullet d_t\mathbf{A_n}\, d\mathbf{x} + d\mathbf{x} \bullet \mathbf{A_n}\, d_t(d\mathbf{x})$$

Next, we apply the result of Corollary 4 that $d_t(d\mathbf{x}) = (\nabla\mathbf{v})d\mathbf{x}$ to write this as

$$d_t^{n+1}([ds(\mathbf{x},t)]^2) = (\nabla\mathbf{v})d\mathbf{x} \bullet \mathbf{A_n}\, d\mathbf{x} + d\mathbf{x} \bullet d_t\mathbf{A_n}\, d\mathbf{x} + d\mathbf{x} \bullet \mathbf{A_n}\, (\nabla\mathbf{v})d\mathbf{x}$$

We then move $(\nabla\mathbf{v})$ in the first term to the right of the $\bullet$ where it becomes a transpose matrix,

$$d_t^{n+1}([ds(\mathbf{x},t)]^2) = d\mathbf{x} \bullet (\nabla\mathbf{v})\mathbf{A_n}\, d\mathbf{x} + d\mathbf{x} \bullet d_t\mathbf{A_n}\, d\mathbf{x} + d\mathbf{x} \bullet \mathbf{A_n}\, (\nabla\mathbf{v})d\mathbf{x}$$

$$= d\mathbf{x} \bullet [\, (\nabla\mathbf{v})^{\mathbf{T}}\mathbf{A_n} + d_t\mathbf{A_n} + \mathbf{A_n}(\nabla\mathbf{v})]\; d\mathbf{x}$$

But of course from theorem 4 we have

$$d_t^{n+1}([ds(\mathbf{x},t)]^2) = d\mathbf{x} \bullet A_{n+1}(\mathbf{x},t)\, d\mathbf{x} \qquad \textbf{(*)}$$

Comparison of the last two equations says

$$d\mathbf{x} \bullet [\, (\nabla\mathbf{v})^{\mathbf{T}}\mathbf{A_n} + d_t\mathbf{A_n} + \mathbf{A_n}(\nabla\mathbf{v})]\; d\mathbf{x} = d\mathbf{x} \bullet A_{n+1}(\mathbf{x},t)\, d\mathbf{x}$$

or

$$d\mathbf{x} \bullet [\, (\nabla\mathbf{v})^{\mathbf{T}}A_n + d_tA_n + A_n(\nabla\mathbf{v}) - A_{n+1}(\mathbf{x},t)]\; d\mathbf{x} = 0$$

The object in [...] is known to be symmetric. The two ∇ terms added are of course symmetric. We know that since C_t is symmetric, so is A_n and then so is d_tA_n. Therefore we can apply Lemma 6 to conclude that

$$A_{n+1}(\mathbf{x},t) = (\nabla\mathbf{v})^{\mathbf{T}}A_n + d_tA_n + A_n(\nabla\mathbf{v})$$

Lai makes no mention of Lemma 6.

Theorem 6: $A_1 = 2D$

Proof: From Lemma 5 we know that

$$(1/2)\, d_t[(ds)^2] = d\mathbf{x} \bullet D\, d\mathbf{x}$$

And from (*) in the last theorem we know that, with n = 0,

$d_t^n([ds(\mathbf{x},t)]^2) = d\mathbf{x} \bullet A_1(\mathbf{x},t)\, d\mathbf{x}$

Comparison shows that

$d\mathbf{x} \bullet A_1(\mathbf{x},t)\, d\mathbf{x} = d\mathbf{x} \bullet 2D\, d\mathbf{x}$

or

$dx \bullet [A_1 - 2D]\, d\mathbf{x} = 0$

Since D is the symmetric part of (∇v) and since A_1 is symmetric, we can apply our Lemma 6 to conclude that

$A_1 = 2D$ QED.